Solution 4

Section 6.4

3. It is trivial for $n = 2$. Assume it is true for $n = k$. Then for $n = k + 1$, we have

$(fg)^{(k+1)}(x)$

$= \left((fg)^{(k)}\right)'(x) = \frac{d}{dx}\left[\sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix} f^{(k-j)}(x)\, g^{(j)}(x)\right]$, by induction hypothesis

$= \sum_{j=0}^{k}\frac{d}{dx}\left[\begin{pmatrix} n \\ k \end{pmatrix} f^{(k-j)}(x)\, g^{(j)}(x)\right]$

$= \sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}[f^{(k+1-j)}(x)\, g^{(j)}(x) + f^{(k-j)}(x)\, g^{(j+1)}(x)]$

$= \sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix} f^{(k+1-j)}(x)\, g^{(j)}(x) + \sum_{j=1}^{k+1}\begin{pmatrix} k \\ j-1 \end{pmatrix} f^{(k+1-j)}(x)\, g^{(j)}(x)$

$= f^{(k+1)}(x)\, g(x) + \sum_{j=1}^{k}\left[\begin{pmatrix} k \\ j \end{pmatrix} + \begin{pmatrix} k \\ j-1 \end{pmatrix}\right] f^{(k+1-j)}(x)\, g^{(j)}(x) + f(x)\, g^{(k+1)}(x)$

$= \sum_{j=0}^{k+1}\begin{pmatrix} k+1 \\ j \end{pmatrix} f^{(k+1-j)}(x)\, g^{(j)}(x)$, since

$\begin{pmatrix} k \\ j \end{pmatrix} + \begin{pmatrix} k \\ j-1 \end{pmatrix} = \frac{k!}{j!(k-j)!} + \frac{k!}{(j-1)!(k+1-j)!} = \frac{k!(k+1-j) + k!j}{j!(k+1-j)!} = \begin{pmatrix} k+1 \\ j \end{pmatrix}$

By M.I., it is true for all n.

10. *Method 1*

By Taylor theorem on $x \mapsto e^x$, $x \geq 0$, $e^{1/x^2} \geq 1 + \frac{1}{x^2} + \cdots + \frac{1}{k!x^{2k}} \geq \frac{1}{k!x^{2k}}$, for $k \in \mathbb{N}$.

Hence $\left|\frac{h(x)}{x^k}\right| = \frac{e^{-1/x^2}}{|\,x\,|^k} \leq \frac{1}{|\,x\,|^k}(k!x^{2k}) = k!|\,x\,|^k \Rightarrow \lim_{x\to 0}\frac{h(x)}{x^k} = 0$, for $k \in \mathbb{N}$.

Method 2

$\lim_{x\to 0}\frac{h(x)}{x^k} = \lim_{x\to 0}\frac{e^{-1/x^2}}{x^k} = \lim_{y\to\infty}\frac{e^{-y^2}}{y^{-k}} = \lim_{y\to\infty}\frac{y^k}{e^{y^2}} = \lim_{y\to\infty}\frac{ky^{k-1}}{2ye^{y^2}} = \lim_{y\to\infty}\frac{ky^{k-2}}{2e^{y^2}} = \cdots$

$$= \begin{cases} \lim_{y\to\infty} C/ye^{y^2}, & \text{if } k \text{ is odd} \\ \lim_{y\to\infty} C/e^{y^2}, & \text{if } k \text{ is even} \end{cases} = 0, \text{ for some } C := C(k) \in \mathbb{R}.$$

Now $h'(0) = \lim_{x\to 0}\frac{h(x)}{x} = 0$, by L'Hôpital rule I.

Assume it is true for $n < k$. Then for $n = k$, by successive application of L'Hôpital rule I,

$0 = \lim_{x\to 0}\frac{h(x)}{x^k} = \lim_{x\to 0}\frac{h'(x)}{kx^{k-1}} = \cdots = \lim_{x\to 0}\frac{h^{(k-1)}(x)}{k!x} = \frac{h^{(k)}(0)}{k!} \Rightarrow h^{(k)}(0) = 0$

By M.I., $h^{(n)}(0) = 0$ for all $n \in \mathbb{N}$.

By Taylor theorem, $\exists\ \xi$ between x and 0 s.t.

$$h(x) = h(0) + h'(0)x + \cdots + \frac{h^{(n)}(0)}{n!}x^n + \frac{h^{(n+1)}(\xi)}{(n+1)!}x^{n+1} = \frac{h^{(n+1)}(\xi)}{(n+1)!}x^{n+1} =: R_n(x)$$

Hence, for $x \neq 0$, $\lim R_n(x) = h(x) = e^{-1/x^2} \neq 0$.

Remark It is very difficult here to derive the result, $\lim R_n(x) \neq 0$ for $x \neq 0$, from $R_n(x) := \dfrac{h^{(n+1)}(\xi)}{(n+1)!}x^{n+1}$ directly by Leibniz rule. If you don't believe, you may try.

12. Use Taylor Expansion of $\sin x$ at point $x_0 = 0$, if $|x| \leq 1$, there exists c with $|c| < 1$ such that
$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{c^7}{5040}.$$
Then we have $|\sin x - (x - \frac{x^3}{6} + \frac{x^5}{120})| = |\frac{c^7}{5040}| < \frac{1}{5040}$.

14. (a) By Theorem 6.4.4, $f'(0) = \cdots = f''(0) = 0$, but $f^{(3)}(0) \neq 0$. f has neither a relative minimum nor relative maximum at $x = 0$.
    (b) By Theorem 6.4.4, $f'(0) = \cdots = f''(0) = 0$, but $f^{(3)}(0) \neq 0$. f has neither a relative minimum nor relative maximum at $x = 0$.
    (c) $h'(0) = 1$. f has neither a relative minimum nor relative maximum at $x = 0$.
    (d) By Theorem 6.4.4, $f'(0) = \cdots = f^{(3)}(0) = 0$, but $f^{(4)}(0) > 0$. f has a relative minimum at $x = 0$.

Supplementary Exercises

1. Claim: Fix $n \in \mathbb{N},\ 0 \leq j \leq 2^n$, then $f\left(\frac{j}{2^n}x + \left(1 - \frac{j}{2^n}\right)y\right) \leq \frac{j}{2^n}f(x) + \left(1 - \frac{j}{2^n}\right)f(y)$

It is trivial for $n = 2$. Assume it is true for $n = k$. Then for $n = k + 1$.

Note that
$$\begin{aligned}\frac{j}{2^{k+1}}x + \left(1 - \frac{j}{2^{k+1}}\right)y &= \frac{j}{2^k}\frac{x}{2} + \left(1 - \frac{j}{2^k}\right)\frac{y}{2} + \left[\frac{j}{2^k} + \left(1 - \frac{j}{2^k}\right)\right]\frac{y}{2} \\ &= \frac{j}{2^k}\left(\frac{x+y}{2}\right) + \left(1 - \frac{j}{2^k}\right)y\end{aligned}$$

Hence, if $j \leq 2^k$, then by induction hypothesis and case $n = 2$, we have

$f\left(\frac{j}{2^{k+1}}x + \left(1 - \frac{j}{2^{k+1}}\right)y\right)$

$\leq \frac{j}{2^k}f\left(\frac{x+y}{2}\right) + \left(1 - \frac{j}{2^k}\right)f(y) \leq \frac{j}{2^k}\left(\frac{1}{2}f(x) + \frac{1}{2}f(y)\right) + \left(1 - \frac{j}{2^k}\right)f(y)$

$\leq \frac{j}{2^{k+1}}f(x) + \left(1 - \frac{j}{2^{k+1}}\right)f(y)$

Now, if $2^k < j \leq 2^{k+1}$, then $0 \leq 2^{k+1} - j < 2^k$, hence replace j by $2^{k+1} - j$, and x by y, we get the same result.

By M.I., the claim is true for all n.

Let $\lambda \in [0, 1]$, $\forall\, n \in \mathbb{N}$, define $j_n := [\lambda 2^n] \leq 2^n$. Hence $\lambda 2^n - 1 < j_n \leq \lambda 2^n$.

Then $\lambda - \dfrac{1}{2^n} < \dfrac{j_n}{2^n} \leq \lambda \ \Rightarrow\ \lim \dfrac{j_n}{2^n} = \lambda$, by Squeeze theorem.

By the claim, $\forall\ n \in \mathbb{N},\ f\left(\frac{j_n}{2^n}x + \left(1 - \frac{j_n}{2^n}\right)y\right) \leq \frac{j_n}{2^n}f(x) + \left(1 - \frac{j_n}{2^n}\right)f(y)$

Letting $n \to +\infty$, we get $f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)$, by continuity.

Remark The result is not in general true if the hypothesis of continuity is omitted, i.e. there exists a discontinuous function, which is thus not convex, satisfying the inequality stated in the question. However, it is not easy to construct such example.

2. Suppose f is convex, then by theorem 1.5 of Notes 1, we have f' is increasing function. Let $x \neq y \in [a, b]$. By mean value theorem, $\exists \xi$ in between x and y such that

$$f(y) - f(x) = f'(\xi)(y - x).$$

Hence

$$\frac{f(y) - f(x)}{y - x} = f'(\xi) = \begin{cases} \geq f(x), & \text{if } x < y. \\ \leq f(x), & \text{if } x > y. \end{cases}$$

Suppose $f(y) - f(x) \geq f'(x)(y - x), \quad \forall x, y \in [a, b]$. We attempt to show that f' is increasing. Let $y > x$, by our assumption, we have

$$f(y) - f(x) \geq f'(x)(y - x)$$

and

$$f(x) - f(y) \geq f'(y)(x - y)$$

which imply

$$f'(y) \geq \frac{f(x) - f(y)}{x - y} = \frac{f(y) - f(x)}{y - x} \geq f'(x).$$

Therefore, f' is increasing. Again by theorem 1.5 of Notes 1, f is concex on $[a, b]$.

3. The product of two convex functions is in general not convex. Take $f(x) := x,\ g(x) := \dfrac{1}{\sqrt{x}}$ on $(0, 1)$. Then $f''(x) = 0,\ g'(x) = -\frac{1}{2}x^{-3/2},\ g''(x) = \frac{3}{4}x^{-5/2} \geq 0$ on $(0, 1)$. Hence $f,\ g$ are convex. Now $(fg)(x) = \sqrt{x}$ on $(0, 1)$.

But $(fg)'(x) = \frac{1}{2}x^{-1/2} \ \Rightarrow\ (fg)''(x) = -\frac{1}{4}x^{-3/2} < 0$ on $(0, 1)$. Hence fg is not convex.

The composition of two convex functions is also in general not convex. Take $f(x) = -x$, $g(x) = x^2$, then f and g are both convex on $\mathbb{R}$, but $(f \circ g)(x) = f(x^2) = -x^2$ which is certainly not convex on $\mathbb{R}$.

4. (a) *Method 1 – Mathematical Induction*

It is trivial for $n = 2$. Assume it is true for $n = k$. Then for $n = k + 1$, we have

$$\begin{aligned}
&f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k + \lambda_{k+1} x_{k+1}) \\
&\leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots \\
&\qquad + (\lambda_k + \lambda_{k+1}) f\left(\frac{\lambda_k}{\lambda_k + \lambda_{k+1}} x_k + \frac{\lambda_k}{\lambda_{k+1} + \lambda_{k+1}} x_{k+1}\right), \text{ by induction hypothesis} \\
&\leq \lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots \\
&\qquad + (\lambda_k + \lambda_{k+1}) \left(\frac{\lambda_k}{\lambda_k + \lambda_{k+1}} f(x_k) + \frac{\lambda_k}{\lambda_{k+1} + \lambda_{k+1}} f(x_{k+1})\right), \text{ by case } n = 2
\end{aligned}$$

$= \lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_k f(x_k) + \lambda_{k+1} f(x_{k+1})$.
By M.I., it is true for all n.

Method 2 – Supporting Line: $y = m(x-\alpha) + f(\alpha)$, *$\alpha$ as defined below.*

Denote $\alpha := \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n \in (a,b)$, since $\lambda_i \in (0,1)$, $\sum_{i=1}^{n} \lambda_i = 1$.

Let $m \in [f'_-(\alpha), f'_+(\alpha)] \neq \varnothing$, since $f'_-(\alpha) \leq f'_+(\alpha)$ due to convexity of f.
(see Note 3 Theorem 2.2).

If $x > \alpha$, $\dfrac{f(x)-f(\alpha)}{x-\alpha} \geq \lim_{x\to\alpha^+} \dfrac{f(x)-f(\alpha)}{x-\alpha} = f'_+(\alpha) \geq m$.

If $x < \alpha$, $\dfrac{f(x)-f(\alpha)}{x-\alpha} \leq \lim_{x\to\alpha^-} \dfrac{f(x)-f(\alpha)}{x-\alpha} = f'_-(\alpha) \leq m$.

(see Note 3 Theorem 2.1 and Theorem 2.2).
Together, we have $f(x) \geq m(x-\alpha) + f(\alpha)$, $\forall\, x \in (a,b)$, since it is trivial if $x = \alpha$.
In particular, for each i, we have

$$f(x_i) \geq m(x_i - \alpha) + f(\alpha)$$

$$\sum_{i=1}^{n} \lambda_i f(x_i) \geq m\left(\sum_{i=1}^{n} \lambda_i x_i - \alpha \sum_{i=1}^{n} \lambda_i\right) + f(\alpha)\sum_{i=1}^{n} \lambda_i$$

Hence $\lambda_1 f(x_1) + \lambda_2 f(x_2) + \cdots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n)$.

(b) Let $a_1, a_2, \ldots, a_n > 0$. Note that $(e^x)'' = e^x > 0$, hence $x \mapsto e^x$ is convex.
By Jensen inequality, we have

$$\begin{aligned}\frac{a_1 + a_2 + \cdots + a_n}{n} &= \frac{1}{n}e^{\ln a_1} + \frac{1}{n}e^{\ln a_2} + \cdots + \frac{1}{n}e^{\ln a_n} \geq e^{\frac{1}{n}\ln a_1 + \frac{1}{n}\ln a_2 + \cdots + \frac{1}{n}\ln a_n} \\ &= e^{\frac{1}{n}\ln(a_1 a_2 \cdots a_n)} = \sqrt[n]{a_1 a_2 \cdots a_n},\end{aligned}$$

which is the AM-GM inequality.

5. Since $f(x) = e^x$ is strictly convex, for $x \in (-\infty, \infty)$. Therefore,

$$f\Big(\frac{1}{p}(p\log x) + \frac{1}{q}(q\log y)\Big) \leq \frac{1}{p}f(p\log x) + \frac{1}{q}f(q\log y)$$

$$\Leftrightarrow xy \leq \frac{x^p}{p} + \frac{y^q}{q}$$

with " = " holds iff $p\log x = q\log y$ i.e. $x^p = y^q$.

6. Let $A = \left(\sum_{k=1}^{n} |a_k|^p\right)^{\frac{1}{p}}$, $B = \left(\sum_{k=1}^{n} |b_k|^q\right)^{\frac{1}{q}}$, $x_k = \dfrac{|a_k|}{A}$ and $y_k = \dfrac{|b_k|}{B}$. Then

$$\sum_{k=1}^{n} x_k^p = 1 \text{ and } \sum_{k=1}^{n} y_k^q = 1.$$

It suffices to show $\sum_{k=1}^{n} x_k y_k \leq 1$. By Young's inequality,

$$x_k y_k \leq \frac{1}{p} x_k^p + \frac{1}{q} y_k^q.$$

Sum over k, we have

$$\sum_{k=1}^{n} x_k y_k \leq \frac{1}{p} + \frac{1}{q} = 1$$

with " = " holds iff $x_k^p = y_k^q, \quad \forall 1 \leq k \leq n$.