Solution 11

Section 9.3

3. Denote $z_n := \begin{cases} 2/n, & \text{if } n \text{ even;} \\ 1/(n+1), & \text{if } n \text{ odd.} \end{cases}$ Then $z_n \to 0$, $z_2 > z_1$, (z_n) is not decreasing.

   $s_{2k} := \sum_{n=1}^{2k} (-1)^{n+1} z_n = \frac{1}{2} - \frac{2}{2} + \frac{1}{4} - \frac{2}{4} + \cdots + \frac{1}{2k} - \frac{2}{2k} = -\frac{1}{2}\sum_{n=1}^{k} \frac{1}{n}$. Hence (s_{2k}) diverges.

   Thus, Leibniz Test (alternating series test) fails.

5. Denote $(x_n) := \left(\frac{1}{n}\right)$, which is decreasing and $\lim x_n = 0$, and $(y_n) := (1, -1, -1, 1, \dots)$.

   Then $s_n := \sum_{k=1}^{n} y_k = \begin{cases} (-1)^{\frac{n+1}{2}}, & \text{if } n \text{ odd} \\ 0, & \text{if } n \text{ even} \end{cases}$ is bounded. By Dirichlet's test, we have

   $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + + - - \cdots = \sum_{n=1}^{\infty} x_n y_n$ converges.

7. Let p, q be positive integer, we claim that

$$\lim_{n\to\infty} \frac{(\log n)^p}{n^q} = 0.$$

   By L Hospital Rule,

$$\lim_{x\to\infty} \frac{\log x}{x^{\frac{q}{p}}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{q}{p} x^{\frac{q}{p}-1}} = \lim_{x\to\infty} \frac{1}{\frac{q}{p} x^{\frac{q}{p}}} = 0.$$

   Moreover, for large $x > 0$,

$$\frac{d}{dx}\left(\frac{\log x}{x^{\frac{p}{q}}}\right) = \frac{p - q\log x}{p x^{\frac{q}{p}+1}} < 0.$$

   Therefore $\frac{(\log n)^p}{n^q}$ is decreasing for large n. By alternating test, $\sum (-1)^n \frac{(\log n)^p}{n^q}$ converges.

10. By Abel's lemma, we have

$$\sum_{n=1}^{N} \frac{a_n}{n} = \frac{s_N}{N} + \sum_{n=1}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) s_n = \frac{s_N}{N} + \sum_{n=1}^{N-1} \frac{s_n}{n(n+1)} \quad \forall\ N \in \mathbb{N}. \tag{1}$$

   Now by hypothesis, $\exists\ M > 0$ s.t. $|s_N| \le M \ \Rightarrow\ \left|\frac{s_N}{N}\right| \le \frac{M}{N} \to 0$ as $N \to \infty$.

   Moreover, $\left|\frac{s_n}{n(n+1)}\right| \le \frac{M}{n(n+1)} \le \frac{M}{n^2} \ \Rightarrow\ \sum \frac{s_n}{n(n+1)} < \infty$, by Comparison test.

   Letting $N \to \infty$ in (1), we have $\sum_{n=1}^{\infty} \frac{a_n}{n} = \sum_{n=1}^{\infty} \frac{s_n}{n(n+1)}$.

11. Define $(y_n) := (1, -1, -1, 1, 1, 1, \dots)$, and denote by s_n its n^{th} partial sum.
Claim $s_{\frac{n(n+1)}{2}} = (-1)^{n+1}\left[\!\left[\dfrac{n+1}{2}\right]\!\right]$. If the claim is true, (s_n) is unbounded.
Now $s_{\frac{1(1+1)}{2}} = 1 = (-1)^{1+1}\left[\!\left[\dfrac{1+1}{2}\right]\!\right]$. Assume it is true for $n = k$. Then for $n = k+1$,

$$s_{\frac{(k+1)(k+2)}{2}} = s_{\frac{k(k+1)}{2}+(k+1)} = s_{\frac{k(k+1)}{2}} + \underbrace{(-1)^{k+2} + \cdots + (-1)^{k+2}}_{k+1\text{ terms}}$$

$$= (-1)^{k+1}\left[\!\left[\frac{k+1}{2}\right]\!\right] + (-1)^{k+2}(k+1), \quad \text{by induction hypothesis}$$

$$= \begin{cases} \dfrac{k+1}{2} - (k+1), & \text{if } k \text{ odd} \\ -\dfrac{k}{2} + (k+1), & \text{if } k \text{ even} \end{cases} = \begin{cases} -\dfrac{k+1}{2}, & \text{if } k \text{ odd} \\ \dfrac{k+2}{2}, & \text{if } k \text{ even} \end{cases} = (-1)^{k+2}\left[\!\left[\frac{k+2}{2}\right]\!\right]$$

By M.I., (s_n) is unbounded. Hence Dirichlet's test cannot directly apply.
Now define $f(x) := \dfrac{x(x+1)}{2}$, $\forall\, x \geq 1$. Then $f'(x) = x + \dfrac{1}{2} > 0$, $\forall\, x \geq 1$. Let $n \in \mathbb{N}$.
If $f(x) = \dfrac{x(x+1)}{2} = n$, $x = \dfrac{-1+\sqrt{1+8n}}{2} \geq 1$. Denote $k := [x] \leq x$, and $k \geq 1$.
Now s_n lies between $s_{\frac{k(k+1)}{2}}$, $s_{\frac{(k+1)(k+2)}{2}}$, hence we have

$$|s_n| \leq \frac{k+2}{2} \leq \frac{2k+2k}{2} = 2k \leq -1 + \sqrt{1+8n} \leq \sqrt{n+8n} = 3n^{1/2}.$$

By result in Question 14, the series converges.

14. By Abel's lemma, we have

$$\sum_{n=1}^{N} \frac{a_n}{n} = \frac{s_N}{N} + \sum_{n=1}^{N-1}\left(\frac{1}{n} - \frac{1}{n+1}\right) s_n = \frac{s_N}{N} + \sum_{n=1}^{N-1} \frac{s_n}{n(n+1)} \quad \forall\, N \in \mathbb{N}. \tag{2}$$

By hypothesis, $\exists\, M > 0$ s.t. $|s_N| \leq MN^r \;\Rightarrow\; \left|\dfrac{s_N}{N}\right| \leq \dfrac{M}{N^{1-r}} \to 0$ as $N \to \infty$.
Moreover, $\left|\dfrac{s_n}{n(n+1)}\right| \leq \dfrac{Mn^r}{n(n+1)} \leq \dfrac{M}{n^{2-r}} \;\Rightarrow\; \sum \dfrac{s_n}{n(n+1)} < \infty$, by Comparison test,
since $\sum \dfrac{1}{n^{2-r}} < \infty$ as $2 - r > 1 \;\Leftrightarrow\; r < 1$, by integral test.
Letting $N \to \infty$ in (2), we have $\displaystyle\sum_{n=1}^{\infty} \frac{a_n}{n} = \sum_{n=1}^{\infty} \frac{s_n}{n(n+1)}$ converges.

15. (b) By Cauchy-Schwarz inequality, we have $\forall\, N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n} \leq \left(\sum_{n=1}^{N} a_n\right)^{1/2} \left(\sum_{n=1}^{N} \frac{1}{n^2}\right)^{1/2} < \left(\sum_{n=1}^{\infty} a_n\right)^{1/2} \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right)^{1/2}.$$

Hence as $N \mapsto \displaystyle\sum_{n=1}^{N} \frac{\sqrt{a_n}}{n}$ is increasing, $\displaystyle\sum_{n=1}^{\infty} b_n$ converges.

(d) Define $a_n := \dfrac{1}{n(\ln n)^2}$. Note $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^2} = \int_2^\infty \frac{d(\ln x)}{(\ln x)^2} = \frac{-1}{\ln x}\Big|_2^\infty = \frac{1}{\ln 2} < \infty$.

By integral test, $\sum a_n$ converges.

Now $b_n = \sqrt{\dfrac{a_n}{n}} = \dfrac{1}{n\ln n}$. Note $\displaystyle\int_3^\infty \frac{dx}{x\ln x} = \int_3^\infty \frac{d(\ln x)}{\ln x} = \ln\ln x\Big|_3^\infty = \infty$.

By integral test, $\sum b_n$ diverges.

Section 9.4

1. (a) Now $|\, f_n(x)\,| := \dfrac{1}{x^2+n^2} \le \dfrac{1}{n^2}$, $\forall\, x \in \mathbb{R}$. Since $\sum \dfrac{1}{n^2} < \infty$, by M-Test, $\sum f_n$ converges uniformly on $\mathbb{R}$.

   (c) Now $|\, f_n(x)\,| := \left|\sin\dfrac{x}{n^2}\right| \le \left|\dfrac{x}{n^2}\right|$, $\forall\, x \in \mathbb{R}$. Since $\sum \dfrac{|\,x\,|}{n^2} < \infty$, by Comparison test, $\sum f_n$ converges absolutely on $\mathbb{R}$. Let $M > 0$. Then $\forall\, |\,x\,| \le M$, $|\, f_n(x)\,| \le \dfrac{M}{n^2}$

   Hence, by M-Test, it converges uniformly on $[-M, M]$.

   (e) First we note when $x = 0$, the series converges. Then we rewrite $f_n(x)$ in following way,

   $\dfrac{x^n}{(x^n+1)} = \dfrac{1}{1+1/x^{-n}}$ $(x > 0)$, set $t = \dfrac{1}{x}$, we have,

$$f_n(t) = \frac{1}{1+t^n} \ (t > 0)$$

   For $t \le 1$, $f_n(t) \ge \dfrac{1}{2}$, hence the series diverges.

   For $t > 1$, $f_n(t) < \dfrac{1}{t^n}$, by M-test, $f_n(t)$ absolutely converges.

   However, since $\sup_{t>1} f_{n+1}(t) = \dfrac{1}{2}$, by Cauchy Criterion, $f_n(t)$ does not uniformly converges on $(1, \infty)$

   As a conclusion, the series $\sum f_n$ converges absolutely but not uniformly on $[0, 1)$.

2. Using Weierstrass M-test, we know

$$|a_n \sin nx| \le |a_n|,$$

   so the conclusion follows.

5. Let $L := \lim\left|\dfrac{a_n}{a_{n+1}}\right| \in (0, \infty)$. If $|\,x\,| < L$, $\lim\left|\dfrac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim\left|\dfrac{a_{n+1}}{a_n}\right| \cdot |\,x\,| < \dfrac{1}{L} \cdot L = 1$.

   By ratio test, $\sum a_n x^n$ converges absolutely if $|\,x\,| < L$.

   If $|\,x\,| > L$, $\lim\left|\dfrac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \lim\left|\dfrac{a_{n+1}}{a_n}\right| \cdot |\,x\,| > \dfrac{1}{L} \cdot L = 1$.

By ratio test, $\sum a_n x^n$ diverges if $|x| > L$. By Cauchy-Hadamard theorem, $R = L$.

If $L = 0$, then for $|x| > 0$, $\lim \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim \left| \frac{a_{n+1}}{a_n} \right| \cdot |x| = |x| \lim \left| \frac{a_{n+1}}{a_n} \right| = \infty.$

By ratio test, $\sum a_n x^n$ diverges if $|x| > 0$. By Cauchy-Hadamard theorem, $R = 0 = L$.

If $L = \infty$, then for $x \in \mathbb{R}$, $\lim \left| \frac{a_{n+1}x^{n+1}}{a_n x^n} \right| = \lim \left| \frac{a_{n+1}}{a_n} \right| \cdot |x| = |x| \lim \left| \frac{a_{n+1}}{a_n} \right| = 0.$

By ratio test, $\sum a_n x^n$ converges if $|x| < \infty$. By Cauchy-Hadamard theorem, $R = \infty = L$.

Example: Consider the power series $1 + x^2 + x^4 + \cdots$. Here $a_{2n} = 1$ but $a_{2n+1} = 0$, so $\lim_{n\to\infty} |a_n/a_{n+1}|$ does not exist but $\rho = \limsup_{n\to\infty}(|a_n|^{1/n}) = 1$ and $R = 1$.

6. (a) $\lim\limits_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{1}{n} = 0$. Hence the radius of convergence is ∞.

(b) $\lim \left| \frac{a_n}{a_{n+1}} \right| = \lim (n+1) \left(1 + \frac{1}{n}\right)^{-\alpha} = \infty$, which diverges properly.
Hence the radius of convergence is ∞.

(c) $\lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{n^n (n+1)!}{(n+1)^{n+1} n!} = \lim \left(1 + \frac{1}{n}\right)^{-n} = e^{-1}$ Hence the radius of convergence is e^{-1}.

(d) $\lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{(\ln n)^{-1}}{[\ln(n+1)]^{-1}} = \lim \frac{\ln(n+1)}{\ln n}.$

Now $\lim\limits_{x\to\infty} \frac{\ln(x+1)}{\ln x} = \lim\limits_{x\to\infty} \frac{1/(x+1)}{1/x} = \lim\limits_{x\to\infty} \frac{1}{1+1/x} = 1$, by L'Hôpital's rule.

By sequential criterion, we have $\lim \left| \frac{a_n}{a_{n+1}} \right| = \lim \frac{\ln(n+1)}{\ln n} = 1.$

Hence the radius of convergence is 1.

(f) Now $|a_n|^{1/n} = n^{-1/\sqrt{n}}$. Define $f(x) := x^{-2/x}$, $x \geq 1$. Then

$f'(x) = e^{\frac{-2\ln x}{x}} \cdot \frac{x(-2/x) - (-2\ln x)\cdot 1}{x^2} = 2x^{-2/x} \cdot \frac{\ln x - 1}{x^2} > 0$, for $x > e$.

By sequential criterion, $n \mapsto n^{-1/\sqrt{n}}$ is increasing for $n \geq 3$, we have

$\sup\limits_{n\geq k} |a_n| = \lim n^{-1/\sqrt{n}}, \ \forall\, k \geq 3.$

Now $\lim\limits_{x\to\infty} x^{-2/x} = \lim\limits_{x\to\infty} e^{\frac{-2\ln x}{x}} = e^{\lim_{x\to\infty} \frac{-2\ln x}{x}} = e^{\lim_{x\to\infty} \frac{-2}{x}} = e^0 = 1$, by L'Hôpital's rule.

By sequential criterion, we have $\rho := \limsup |a_n| = \lim n^{-1/\sqrt{n}} = 1$.

Hence the radius of convergence $= 1/\rho = 1$.

11. Use Taylor expansion at point $x = 0$, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(c)}{(n+1)!} x^(n+1)$$

for $|x| < r$ and $0 < |c| < |x|$.

$$|\frac{f^{(n+1)}(c)}{(n+1)!}x^{(n+1)}| < \frac{B}{(n+1)!}r^{n+1}$$

Since $\frac{r^{n+1}}{(n+1)!} \to 0$ as $n \to \infty$. So $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ converges to $f(x)$ for $|x| < r$.

12. $f'(0) = \lim_{h\to 0} \frac{e^{-1/h^2}}{h}$, set $t = 1/h$, we have,

$$f'(0) = \lim_{t\to\infty} \frac{t}{e^{t^2}}$$

Applying L'Hospital Rule we get $f'(0) = \lim_{t\to\infty} \frac{1}{2te^{t^2}} = 0$.

Assume that $f^{(k)}(0) = 0$, we want to show that $f^{(k+1)}(0) = 0$

We note that $f'(x) = \frac{2}{x^3}e^{-1/x^2} = P_3(\frac{1}{x})e^{-1/x^2}$ for $x \neq 0$, where P_3 is a polynomial with highest order 3.

We want to show that $f^{(n)}(x) = P_{3n}(\frac{1}{x})e^{-1/x^2}$ for $x \neq 0$. Prove it by induction:

Assume that $f^{(k)}(x) = P_{3k}(\frac{1}{x})e^{-1/x^2}$ for $x \neq 0$, then $f^{(k+1)}(x) = \frac{2}{x^3}P_{3k}(\frac{1}{x})e^{-1/x^2} + P_{3k-1}(\frac{1}{x})e^{-1/x^2} = P_{3(k+1)}(\frac{1}{x})e^{-1/x^2}$ for $x \neq 0$.

Now we consider $f^{(k+1)}(0) = \lim_{h\to 0} \frac{P_{3(k+1)}(\frac{1}{h})e^{-1/h^2}}{h}$, set $t = 1/h$, we have,

$$f^{(k+1)}(0) = \lim_{t\to\infty} \frac{t}{P_{3(k+1)}(t)e^{t^2}}$$

Applying L'Hospital Rule we get $f^{(k+1)}(0) = \lim_{t\to\infty} \frac{1}{P_{3(k+2)}(t)e^{t^2}} = 0$.

By M.I., we have $f^{(n)}(0) = 0 \; \forall n \in \mathbb{N}$. Hence this function is not given by its Taylor expansion about $x = 0$.

16. It is well-known that the expansion

$$\frac{1}{1+x} = \sum_{n=0}^{\infty}(-1)^n x^n,$$

holds for $x \in (-1, 1)$. The power series on the right hand side has radius of convergence equal to 1. Hence it converges uniformly on interval $[-R, R]$, $|R| < 1$. Hence we can integrate this formula from 0 to R to get

$$\ln(1+R) = \int_0^R \frac{1}{1+x} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}R^n.$$

The conclusion follows after replacing R by x.

17. We have

$$\sum_{n=0}^{\infty}(-1)^n t^{2n} = \frac{1}{1+t^2}$$

for $t \in (-1,1)$ (this is a geometric series), and for any $x \in (-1,1)$, the convergence is uniform on $[-|x|,|x|]$ (since the radius of convergence of the power series on the left-hand side is 1, as is easy to check). Thus integrating, we get

$$\sum_{n=0}^{\infty}\int_0^x (-1)^n t^{2n} dt = \int_0^x \frac{1}{1+t^2} dt,$$

i.e.

$$\arctan x = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}$$

for any $x \in (-1,1)$.

19. Define $f(x) := \displaystyle\int_0^x e^{-t^2}\, dt$, for $x \in \mathbb{R}$. Clearly f is smooth and $f(0) = 0$.

$f'(x) = e^{-x^2}$. Then $f''(x) = -2xe^{-x^2} = -2xf'(x)$. Hence $f'(0) = 1,\ f''(0) = 0$.
Then, by Leibniz rule, we have, for $n \in \mathbb{N}$, $[f^{(n+2)}(x) = -2xf^{(n+1)}(x) - 2nf^{(n)}(x) \Rightarrow f^{(n+2)}(0) = -2nf^{(n)}(0)]$ Clearly, $f^{(2n)}(0) = 0$, and

$$\begin{aligned} f^{(2n+1)}(0) &= -2(2n-1)f^{(2(n-1)+1)}(0) = (-2)^2(2n-1)(2n-3)f^{(2(n-2)+1)}(0) \\ &= \cdots = (-2)^n(2n-1)(2n-3)\cdots 1\cdot f'(0) = (-2)^n\frac{(2n)!}{2^n n!} \\ &= \frac{(-1)^n(2n)!}{n!} \end{aligned}$$

A Maclaurin series expansion (i.e. Taylor series centered at 0) for $\displaystyle\int_0^x e^{-t^2}\, dt$

$$= \sum_{n=0}^{\infty}\frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{n!(2n+1)}.$$

Supplementary Exercises

1. Now $\displaystyle\sup_{x\in(0,b)}\left|\frac{x^n}{n}\right| = \frac{b^n}{n}$. If $b < 1$, we have $0 < \dfrac{b^n}{n} \le b^n\ \forall\ n \in \mathbb{N}$, and

$\displaystyle\sum_{n=1}^{\infty} b^n = \frac{b}{1-b}$, hence by Comparison Test, $\displaystyle\sum_{n=1}^{\infty}\frac{b^n}{n}$ converges.

It follows from Weierstrass's M-test that $\sum_{n=0}^{\infty}\frac{x^n}{n}$ converges uniformly on $(0,b)$ if $b < 1$.
Now if $b > 1$, the series $\sum_{n=1}^{\infty}\frac{x^n}{n}$ does not even converge pointwisely for all $x \in (0,b)$ (it diverges whenever $x \ge 1$, by comparison to $\sum_{n=1}^{\infty}\frac{1}{n}$. Hence the series certainly does not converge uniformly on $(0,b)$ if $b > 1$.

The only case that remains is when $b = 1$. We claim that the series also fail to converge uniformly on $(0,1)$, because it is not Cauchy in sup-norm on $(0,1)$. Assume otherwise. Then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$\left\|\frac{x^m}{m} + \cdots + \frac{x^n}{n}\right\|_{(0,1)} < \frac{1}{2}$$

whenever $n \geq m \geq N$. But

$$\left\| \frac{x^m}{m} + \cdots + \frac{x^n}{n} \right\|_{(0,1)} = \frac{1}{n} + \cdots + \frac{1}{m},$$

so the above implies that $\{\sum_{k=1}^n \frac{1}{k}\}_{n \in \mathbb{N}}$ is a Cauchy sequence of real numbers, which in turn implies that $\sum_{k=1}^\infty \frac{1}{k}$ is convergent, a contradiction.

(The cases $b < 1$ and $b > 1$ follow also from the general discussion of power series, once we observe that the radius of convergence of the power series $\sum_{n=1}^\infty \frac{x^n}{n}$ is 1.)

2. Note that

$$\frac{1}{1+x} = \sum_{n=0}^\infty (-x)^n$$

whenever $|x| < 1$. Hence

$$\frac{1}{1+x^2} = \sum_{n=0}^\infty (-x^2)^n = \sum_{n=0}^\infty (-1)^n x^{2n} \tag{3}$$

whenever $|x| < 1$. The series on the right hand side of (3) diverges when $|x| \geq 1$. Hence (3) is the power series representation of $1/(1+x^2)$, and it is valid precisely when $|x| < 1$.

3. If the radius of convergence of $\sum_{n=0}^\infty a_n x^n$ is r, then the series converges whenever $|x| < r$, and diverges whenever $|x| > r$. Now this implies

$$\sum_{n=0}^\infty a_n x^{2n}$$

converges whenever $|x| < r^{1/2}$, and diverges whenever $|x| > r^{1/2}$. Hence the radius of convergence of this new power series must be $r^{1/2}$.