Solution 10

Section 9.1

2. Denote by $\sum_{n=1}^{\infty} a_n$ a conditionally convergent series, $a_n^+ := \max\{a_n, 0\}$, $a_n^- := \max\{-a_n, 0\}$.
Then we have $|\, a_n \,| = a_n^+ + a_n^-$, $\quad a_n = a_n^+ - a_n^-$.
Now $\sum_{a_n \geq 0} a_n = \sum_{n=1}^{\infty} a_n^+$, $\sum_{a_n < 0} a_n = -\sum_{n=1}^{\infty} a_n^-$. Then $\forall\ N$, $\sum_{n=1}^{N} a_n^+ + \sum_{n=1}^{N} a_n^- = \sum_{n=1}^{N} |\, a_n \,|$. Assume $\sum a_n^+$, $\sum a_n^- < \infty$, then $\sum_{n=1}^{\infty} |\, a_n \,| = \sum_{n=1}^{\infty} a_n^+ + \sum_{n=1}^{\infty} a_n^- < \infty$, which is contradiction.
Hence, if $\sum_{n=1}^{\infty} a_n^+ < \infty$, then $\sum_{n=1}^{\infty} a_n^- = \infty$, but since $\sum_{n=1}^{\infty} a_n < \infty$, then
$$\sum_{n=1}^{\infty} a_n^- = \lim \sum_{n=1}^{N} a_n^- = \lim \sum_{n=1}^{N} (a_n^+ - a_n) = \sum_{n=1}^{\infty} a_n^+ - \sum_{n=1}^{\infty} a_n < \infty,$$
which is contradiction. Similarly, it is impossible to have $\sum_{n=1}^{\infty} a_n^- < \infty$.
Together, $\sum_{a_n \geq 0} a_n = \sum_{n=1}^{\infty} a_n^+ = \infty$, $\sum_{a_n < 0} a_n = -\sum_{n=1}^{\infty} a_n^- = -\infty$.

3. Since $\sum a_n$ converges conditionally, $\sum_{a_n \geq 0} a_n = \infty$, $\sum_{a_n < 0} a_n = -\infty$ and $a_n \to 0$.
Hence $\exists\ K$ s.t. $|\, a_n \,| < 1/2$, $\forall\ n \geq K$.
¿From $\{\, a_n : a_n \geq 0 \,\}$, it is possible to pick b_1, b_2, $\ldots$, b_{n_1} s.t. $\sum_{n=1}^{n_1} b_n > 1$.
Then, from $\{\, a_n : a_n < 0 \,\}$, pick b_{n_1+1}, b_{n_1+2}, $\ldots$, b_{n_2} s.t. $0 < \sum_{n=1}^{n_2} b_n \leq 1$.
Next, from $\{\, a_n : a_n \geq 0 \,\} \backslash \{\, b_1,\ b_2,\ \ldots,\ b_{n_1} \,\}$, pick b_{n_2+1}, b_{n_2+2}, $\ldots$, b_{n_3} s.t. $\sum_{n=1}^{n_3} b_n > 2$.
Then, from $\{\, a_n : a_n < 0 \,\} \backslash \{\, b_1,\ b_2,\ \ldots,\ b_{n_2} \,\}$, pick b_{n_3+1}, b_{n_3+2}, $\ldots$, b_{n_4} s.t. $1 < \sum_{n=1}^{n_4} b_n \leq 2$.
Continuing this process, every terms in a_n's will eventually be picked and hence we obtain a rearrangement $(\, b_n \,)$ s.t. $\forall\ k$, $\exists\ N \in \mathbb{N}$ s.t. $\sum_{n=1}^{N} b_n > k$. Hence $\sum b_n$ diverges to ∞.

7. (a) $\sum_{n=1}^{N} |\, a_n b_n \,| \leq M \sum_{n=1}^{N} |\, a_n \,| \leq M \sum_{n=1}^{\infty} |\, a_n \,|$, where $|\, b_n \,| \leq M,\ \forall\ n.$

   Let $N \to \infty$, $\sum_{n=1}^{\infty} |\, a_n b_n \,| \leq M \sum_{n=1}^{\infty} |\, a_n \,| < \infty$, hence $\sum a_n b_n$ converges absolutely.

   (b) Take $a_n := \dfrac{(-1)^n}{n}$, $b_n := (-1)^n$, then $a_n b_n = \dfrac{1}{n} \;\Rightarrow\; \sum a_n b_n = \sum \dfrac{1}{n}$ diverges.

8. Take $a_n := \dfrac{(-1)^n}{\sqrt{n}} \;\Rightarrow\; \sum a_n = \sum \dfrac{(-1)^n}{\sqrt{n}}$ converges by alternating series test (Theorem 9.3.2). But $\sum a_n^2 = \sum \dfrac{1}{n}$ diverges.

   Remark It can also be used to answer the question: Give an example of two convergent serieses $\sum a_n$, $\sum b_n$ such that $\sum a_n b_n$ diverges.

9. Denote $s_n := \sum_{k=1}^{n} a_k$. Then, for $n \in \mathbb{N}$, $s_{2n} - s_n \geq n a_{2n} = \dfrac{1}{2}(2n) a_{2n} > 0$, and

   $s_{2n+1} - s_n \geq (n+1) a_{2n} \geq \dfrac{1}{2}(2n+1) a_{2n+1} > 0.$
   Let $n \to \infty$, $\lim (2n) a_{2n} = 0,\ \lim (2n+1) a_{2n+1} = 0 \;\Rightarrow\; \lim n a_n = 0.$

10. Take $a_n := \dfrac{1}{n \ln n}$, by Cauchy condensation test (by question 12 in Section 3.7 p.95),

    $\sum a_n = \sum \dfrac{1}{n \ln n}$ diverges. Hence $\lim n a_n = \lim n \left(\dfrac{1}{n \ln n} \right) = \lim \dfrac{1}{\ln n} = 0.$

13. (a) $\dfrac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}} = \dfrac{1}{\sqrt{n}(\sqrt{n+1} + \sqrt{n})} > \dfrac{1}{2(n+1)}$

    Since Harmonic series is divergent, so $\sum_{n=1}^{\infty} (\dfrac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n}})$ is also divergent.

    (b) $\dfrac{\sqrt{n+1} - \sqrt{n}}{n} = \dfrac{1}{n(\sqrt{n+1} + \sqrt{n})} < \dfrac{1}{2n^{1.5}}.$

    Since $\sum_{n=1}^{\infty} 1/over2n^{1.5}$ is convergent, so $\sum_{n=1}^{\infty} (\dfrac{\sqrt{n+1} - \sqrt{n}}{n})$ is also convergent.

15. First we assume (i) is correct. Note that $\forall M \in \mathbb{N}$, let $i' = \max\{i | a_{ij} = c_k, k = 1, \ldots, M\}$. Then we have,

    $$\sum_{k=1}^{M} c_k \leq \sum_{i=1}^{i'} A_i \leq B$$

    That means $\sum_{k=1}^{M} c_k$ is upper bounded, hence (ii) holds.

Then assuming (ii) is correct. We want to prove A_i exists first, then applying similar technique to $\sum_{i=1}^{\infty} A_i$. Because $\sum_{j=1}^{N} a_{ij}$ is sum of subset of enumeration, hence

$$\sum_{j=1}^{N} a_{ij} \leq \sum_{k=1}^{\infty} \leq C$$

That means A_i exists. So $\forall \epsilon$ and i, $\exists N(i,\epsilon)$ such that $\forall n > N(i,\epsilon), \sum_{j=1}^{n} a_{ij} > A_i - \frac{\epsilon}{2^i}$. So, $\forall M \in \mathbb{N}$,

$$\sum_{i=1}^{M} \sum_{j=1}^{N(i,\epsilon)} a_{ij} > \sum_{i=1}^{M} A_i - \epsilon$$

Note that the left part of the equation is less than C. Let $\epsilon \to 0$ we have

$$\sum_{i=1}^{M} A_i \leq C$$

So (i) holds.
Let $M \to \infty$ in the proof above, we have $B \geq C$ and $C \geq B$. That is $B = C$.

Section 9.2

1. (a) *Method 1*

Denote $x_n := \dfrac{1}{(n+1)(n+2)}$. Then $\left|\dfrac{x_{n+1}}{x_n}\right| = \dfrac{n+1}{n+3} = 1 - \dfrac{2}{n+3}$

Hence $\lim n\left(1 - \left|\dfrac{x_{n+1}}{x_n}\right|\right) = \lim \dfrac{2}{1+3/n} = 2 > 1$.

By Raabe's test, $\sum x_n$ converges absolutely.

Method 2

Now $x_n := \dfrac{1}{(n+1)(n+2)} \le \dfrac{1}{n^2}$. Since $\sum \dfrac{1}{n^2} < \infty$, by Comparison test, $\sum x_n$ converges absolutely since each x_n is positive.

(c) Since $\lim 2^{-1/n} = 2^0 = 1 \neq 0$. By n^{th} term test, $\sum 2^{-1/n}$ diverges.

2. (b) Now $x_n := \big(n^2(n+1)\big)^{-1/2} \le n^{-3/2}$. Since $\sum \dfrac{1}{n^{3/2}} < \infty$, by Comparison test, $\sum x_n$ converges.

(c) $x_n = \dfrac{n!}{n^n} = \dfrac{n(n-1)\cdots 21}{nn\cdots nn} \le \dfrac{2}{n^2}$. Therefore by comparison test, the series diverges.

(d) Denote $x_n := (-1)^n \dfrac{n}{n+1}$. Then $\lim x_{2n} = 1$ and $\lim x_{2n-1} = -1$. By nth term test, $\sum x_n$ diverges.

3. (b) Now $\lim\limits_{x\to\infty} \dfrac{x^{e^x}}{e^{2x}} = \lim\limits_{x\to\infty} e^{e^x \ln x - 2x} \ge \lim\limits_{x\to\infty} e^{(1+x)\ln x - 2x} \ge \lim\limits_{x\to\infty} e^{\ln x + x(\ln x - 2)} = \infty$.

By sequential criterion, $\lim\limits_{n\to\infty} \dfrac{(\ln n)^n}{n^2} = \infty \;\Rightarrow\; \exists\, K$ s.t. $\dfrac{(\ln n)^n}{n^2} > 1$, for $n \ge K$.

Hence, for $n \ge K$, $(\ln n)^{-n} < \dfrac{1}{n^2}$. Since $\sum\limits_{n=K}^{\infty} \dfrac{1}{n^2} < \infty$, by Comparison test, $\sum (\ln n)^{-n}$ converges (absolutely) (for sufficiently large n).

(d) Now $\lim\limits_{x\to\infty} \dfrac{(e^x)^x}{e^{e^x}} = \lim\limits_{x\to\infty} \dfrac{e^{x^2}}{e^{e^x}} = \lim\limits_{x\to\infty} e^{x^2 - e^x}$.

By Taylor theorem, for $x \ge 3$, $e^x \ge x + \dfrac{x^2}{2} + \dfrac{x^3}{3!} \ge x + \dfrac{x^2}{2} + \dfrac{x^2 \cdot 3}{3!} = x + x^2$.

Hence, $\lim\limits_{x\to\infty} \dfrac{(e^x)^x}{e^{e^x}} = \lim\limits_{x\to\infty} e^{x^2 - e^x} \le \lim\limits_{x\to\infty} e^{x^2 - (x+x^2)} = \lim\limits_{x\to\infty} e^{-x} = 0$.

By sequential criterion, $\lim\limits_{n\to\infty} \dfrac{(\ln n)^{\ln\ln n}}{n} = 0 \;\Rightarrow\; \exists\, K$ s.t. $\dfrac{(\ln n)^{\ln\ln n}}{n} < 1$, for $n \ge K$.

Hence, for $n \ge K$, $(\ln n)^{-\ln\ln n} > \dfrac{1}{n}$. Since $\sum\limits_{n=K}^{\infty} \dfrac{1}{n}$ diverges, by Comparison test, $\sum (\ln n)^{-\ln\ln n}$ diverges (for sufficiently large n).

Remark Basically what we did above is to try to compare the *tail of sequence* by $(1/n^2)$ if we want to show convergence, and by $(1/n)$ if we want to show divergence instead. However, it is difficult to prove the divergence of $\sum \frac{1}{n\ln n}$ by the method as mentioned above, because it is too ugly that $\frac{1}{n^2} \le \frac{1}{n\ln n} \le \frac{1}{n}$ for large n.

4. (b) Denote $x_n := n^n e^{-n}$. Then $\left|\frac{x_{n+1}}{x_n}\right| = \frac{n+1}{e}\left(1+\frac{1}{n}\right)^n \ge \frac{2(n+1)}{e} \to \infty$.

   By ratio test, $\sum x_n$ diverges.

   (d) Now $\lim_{x\to\infty} \frac{e^{e^{x/2}}/x}{e^{2x}} = \lim_{x\to\infty} \frac{e^{e^{x/2}-2x}}{x}$.

   By Taylor theorem, for $x \ge 10$, $e^{x/2} \ge (x/2) + \frac{(x/2)^2}{2!} + \cdots + \frac{(x/2)^5}{5!} \ge \frac{5x}{2}$.

   Hence, $\lim_{x\to\infty} \frac{e^{e^{x/2}}/x}{e^{2x}} = \lim_{x\to\infty} \frac{e^{e^{x/2}-2x}}{x} \ge \lim_{x\to\infty} \frac{e^{x/2}}{x} = \lim_{x\to\infty} \frac{e^{x/2}/2}{1} = \infty$.

   By sequential criterion, $\lim_{n\to\infty} \frac{e^{\sqrt{n}}/\ln n}{n^2} = \infty \Rightarrow \exists\, K$ s.t. $\frac{e^{\sqrt{n}}/\ln n}{n^2} > 1$, for $n \ge K$.

   Hence, for $n \ge K$, $(\ln n)e^{-\sqrt{n}} < \frac{1}{n^2}$. Since $\sum_{n=K}^{\infty} \frac{1}{n^2} < \infty$,

   by Comparison test, $\sum (\ln n)e^{-\sqrt{n}}$ converges (absolutely) (for sufficiently large n).

6. Define $f(x) := (ax+b)^{-p}$. Then $f'(x) := -ap(ax+b)^{-p-1} < 0$, for $x \ge 1$. Moreover,

$$\int_1^R f = \int_1^R \frac{dx}{(ax+b)^p} = \begin{cases} \left.\frac{(ax+b)^{1-p}}{a(1-p)}\right|_1^R, & \text{for } p \ne 1 \\ \ln(ax+b)\big|_1^R, & \text{for } p = 1 \end{cases}$$

$$= \begin{cases} \frac{1}{a(1-p)}\left(\frac{1}{(aR+b)^{p-1}} - \frac{1}{(a+b)^{p-1}}\right), & \text{for } p \ne 1 \\ \ln(aR+b) - \ln(a+b), & \text{for } p = 1 \end{cases}$$

If $p > 1$, then $\lim_{R\to\infty} \int_1^R f = \frac{(a+b)^{1-p}}{a(p-1)}$, by integral test, $\sum (an+b)^{-p} < \infty$.

If $p \le 1$, then $\int_1^R f$ diverges as $R \to \infty$, by integral test, $\sum (an+b)^{-p}$ diverges.

7. (a) Denote $x_n := \frac{n!}{3\cdot 5\cdot 7\cdots(2n+1)}$. Then $\left|\frac{x_{n+1}}{x_n}\right| = \frac{n+1}{2n+3} \to \frac{1}{2} < 1$.

   By ratio test, $\sum x_n$ converges absolutely.

8. Note that this series is a rearrangement of $a, a^2, \dots, a^{n-1}, a^n, \dots$, which we already know is absolutely convergent.
   Root test:
$$|x_n|^{1/n} = \begin{cases} a^{(n-1)/n}, & \text{n = 2k;} \\ a^{n/(n-1)}, & \text{n = 2k-1.} \end{cases}$$
   In both cases $|x_n|^{1/n} < 1$. By root test, the infinite series is convergent.
   Ratio test:
$$\frac{x_{n+1}}{x_n} = 1/a > 1 \quad \forall n = 2k+1, k \in \mathbb{N}$$
   and
$$\frac{x_{n+1}}{x_n} = a^2 < 1 \quad \forall n = 2k, k \in \mathbb{N}$$
   We can't use ratio test to judge if this series is convergent.

15. $c_{n+1} - c_n = \dfrac{1}{n+1} - (\ln(n+1) - \ln n) = \dfrac{1}{n+1} - \dfrac{1}{\xi} < 0$, for some $\xi \in (n, n+1)$, by MVT.

    Hence $(\, c_n \,)$ is a decreasing sequence. Now we know
$$\int_k^{k+1} \frac{dx}{x} < \frac{1}{k} \quad \Rightarrow \quad \ln n = \int_1^n \frac{dx}{x} < \sum_{k=1}^{n-1} \frac{1}{k} < 1 + \frac{1}{2} + \cdots + \frac{1}{n} \quad \Rightarrow \quad c_n > 0.$$
    Hence $(\, c_n \,)$ is bounded from below by 0, $C := \lim c_n$ exists. Now we have
    $b_n = c_{2n} - c_n + \ln 2 \quad \Rightarrow \quad \lim b_n = C - C + \ln 2 = \ln 2.$

16. Group the terms according to the # of digits in the denominators, hence the partial sum
$$s_n \le \underbrace{\left(\frac{1}{1} + \cdots + \frac{1}{9}\right)}_{\text{no } 1/6} + \underbrace{\left(\frac{1}{10} + \frac{1}{11} + \cdots + \frac{1}{99}\right)}_{\text{no terms with digit 6}} + \cdots + \underbrace{\left(\frac{1}{\underbrace{10\cdots0}_{N\text{ digits}}} + \cdots + \frac{1}{\underbrace{99\cdots9}_{N\text{ digits}}}\right)}_{\text{no terms with digit 6}}$$
    where n is an N-digit natural number.
    For a k-digit natural number without a digit 6, there are 8 choices for the 1st digit and 9 choices for the other $(k-1)$ digits.
    Hence # of k-digit natural numbers without a digit $6 = 8 \times 9^{k-1}$.
$$s_n \le \underbrace{\left(\frac{1}{1} + \cdots + \frac{1}{1}\right)}_{8\text{ terms}} + \underbrace{\left(\frac{1}{10} + \frac{1}{10} + \cdots + \frac{1}{10}\right)}_{8\times 9\text{ terms}} + \cdots + \underbrace{\left(\frac{1}{10^{N-1}} + \cdots + \frac{1}{10^{N-1}}\right)}_{8\times 9^{N-1}\text{ terms}}$$
$$= 8 + 8 \times \left(\frac{9}{10}\right) + \cdots + 8 \times \left(\frac{9}{10}\right)^{N-1} < \frac{8}{1 - 9/10} = 80.$$
    Since $(\, s_n \,)$ is increasing and bounded above by 80, then $\sum \dfrac{1}{n_k}$ converges to a limit < 80.†

    Now $m_k = 10k - 4$, for $k \in \mathbb{N}$. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{m_k} = \sum_{k=1}^{\infty} \frac{1}{10k-4} \ge \frac{1}{10} \sum_{k=1}^{\infty} \frac{1}{k}$ diverges.
    Now $\{\, p_k \,\}$ is the collection of numbers not ended in 6, hence it contains a subcollection of numbers ended in 1, namely $p_{k_j} := 10j + 1$. Then

$$\sum_{k=1}^{\infty}\frac{1}{p_k} \geq \sum_{j=1}^{\infty}\frac{1}{p_{k_j}} = \sum_{j=1}^{\infty}\frac{1}{10j+1} \geq \sum_{j=1}^{\infty}\frac{1}{10j+j} = \frac{1}{11}\sum_{j=1}^{\infty}\frac{1}{j} \text{ diverges.}$$

Remark We can get the same result if 6 is replaced by a fixed digit among $1, 2, \ldots, 9$. If 6 is replaced by 0, we can get the bound 90 instead of 80.

19. We adopt the notation in the question. Since $b_1 = \sqrt{A} - \sqrt{A_1}$ and $b_n = \sqrt{A - A_{n-1}} - \sqrt{A - A_n} > 0$,

$$\sum_{k=1}^{N} b_k = \sqrt{A} - \sqrt{A - A_N} \to \sqrt{A} \text{ as } N \to \infty.$$

Hence the series converges. Now, we check that $\lim_{n\to\infty}\frac{a_n}{b_n} = 0$. For $n > 1$,

$$b_n = \sqrt{A - A_{n-1}} - \sqrt{A - A_n} = \frac{A_n - A_{n-1}}{\sqrt{A - A_{n-1}} + \sqrt{A - A_n}} = \frac{a_n}{\sqrt{A - A_{n-1}} + \sqrt{A - A_n}}.$$

Using the fact that $\lim_{n\to\infty} A_n = A$, we conclude that

$$\frac{a_n}{b_n} = \sqrt{A - A_{n-1}} + \sqrt{A - A_n} \to 0 \text{ as } n \to \infty.$$

20. Let $b_n = a_n/\sqrt{A_n}$ where A_n is the nth partial sum of $\sum a_n$. It is clear that

$$\lim\,(b_n/a_n) = \lim 1/\sqrt{A_n} = 0$$

since $\sum a_n$ is divergent. Now we prove $\sum b_n$ is also divergent.

$$\sum b_n \geq \sum_{n=1}^{M} b_n \geq \sum_{n=1}^{M} a_n/\sqrt{A_M} = \sqrt{A_M} \quad \forall M \in \mathbb{N}$$

Letting $M \to \infty$, we have the desired conclusion.

Section 9.3

1. (b) Denote $x_n := \dfrac{1}{n+1} > 0$. Note $x_{n+1} - x_n = \dfrac{-1}{n(n+1)} < 0$, i.e. $(\,x_n\,)$ is decreasing and $\lim x_n = \lim \dfrac{1}{n+1} = 0$. By Leibniz Test (alternating series test), $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n+1} < \infty$. $\displaystyle\sum_{n=1}^{\infty}\left|\,(-1)^{n+1}x_n\,\right| = \sum_{n=1}^{\infty}\left|\frac{(-1)^{n+1}}{n+1}\right| = \sum_{n=1}^{\infty}\frac{1}{n+1} \geq \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.
   Hence it converges conditionally.

   (d) Denote $x_n := \dfrac{\ln n}{n}$. Define $f(x) := \dfrac{x}{e^x}$. Then $f'(x) = \dfrac{e^x - xe^x}{e^{2x}} = \dfrac{1-x}{e^x} < 0$ if $x > 1$.
   Hence f is decreasing if $x > 1$. By L'Hopital's rule, $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\frac{1}{e^x} = 0$.

By sequential criterion, $\lim x_n = \lim f(\ln n) = 0$ and $\forall\, n \geq 3$, $x_n > 0$ and $x_{n+1} = f(\ln(n+1)) < f(\ln n) = x_n$.

By Leibniz Test (alternating series test), $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\ln n}{n} < \infty$.

$\displaystyle\sum_{n=1}^{\infty}\left|\,(-1)^{n+1}x_n\,\right| = \sum_{n=1}^{\infty}\left|(-1)^{n+1}\frac{\ln n}{n}\right| = \sum_{n=1}^{\infty}\frac{\ln n}{n} \geq \sum_{n=3}^{\infty}\frac{1}{n}$ diverges.

Hence it converges conditionally.

2. Now $s_{2(n+1)} - s_{2n} = z_{2n+1} - z_{2n+2} \geq 0$, and $s_{2n+1} - s_{2n-1} = -(z_{2n} - z_{2n+1}) \leq 0$, $s_{2n} - s_{2n-1} = -z_{2n} \leq 0$, i.e. $s_{2n} \leq s_{2n-1}$, $\forall\, n \in \mathbb{N}$.
   Together, $s_2 \leq s_4 \leq s_6 \leq \cdots \leq s_{2n} \leq s_{2n-1} \leq \cdots \leq s_5 \leq s_3 \leq s_1$.
   Hence s lies between s_n and s_{n+1}, so $|\,s - s_n\,| \leq |\,s_{n+1} - s_n\,| \leq z_{n+1}$.

8. (a) Denote $x_n := \dfrac{n^n}{(n+1)^{n+1}} = \dfrac{1}{(1+1/n)^n}\cdot\dfrac{1}{n+1}$. Since $n \mapsto \left(1+\dfrac{1}{n}\right)^n$ is increasing,
   $x_{n+1} = \dfrac{1}{(1+1/(n+1))^{n+1}}\cdot\dfrac{1}{n+2} \leq \dfrac{1}{(1+1/n)^n}\cdot\dfrac{1}{n+1} = x_n$.
   Now $\lim x_n = \lim \dfrac{1}{(1+1/n)^n}\cdot\dfrac{1}{n+1} = \dfrac{1}{e}\cdot 0 = 0$.
   By Leibniz Test (alternating series test), $\displaystyle\sum(-1)^n\frac{n^n}{(n+1)^{n+1}}$ converges.

   (c) Now $\lim\left|(-1)^n\dfrac{(n+1)^n}{n^n}\right| = e \neq 0$. By n^{th} term test, $\displaystyle\sum(-1)^n\frac{(n+1)^n}{n^n}$ diverges.

Supplementary Exercises

1. (a) Since $\sum_{k=1}\log a_k$ converges, we have by n-th term test, $\lim_{k\to\infty}\log a_k = 0$. Therefore by continuity of e^x

$$1 = e^{\lim_{k\to\infty}\log a_k} = \lim_{k\to\infty} e^{\log a_k} = \lim_{k\to\infty} a_k.$$

   (b) The first inequality follows from

$$\prod_{k=1}^{n}(1+p_k) = 1 + \sum_{k=1}^{n} p_k + \sum_{m=2}^{n}\ \sum_{1\leq i_1<i_2<\cdots<i_m\leq n} p_{i_1}p_{i_2}\cdots p_{i_m} \geq \sum_{k=1}^{n} p_k.$$

   For the second one, we observe that

$$e^{p_k} = \sum_{j=2}^{\infty}\frac{(p_k)^j}{j!} + 1 + p_k \geq 1 + p_k.$$

   Hence, $\prod_{k=1}^{\infty}(1+p_k)$ is convergent implies that $\sum_{k=1}^{n} p_k$ is bounded above, so the series converges. Now if $\sum_{k=1}^{\infty} p_k$ converges, then

$$0 \leq \sum_{k=1}^{n}\log(1+p_k) \leq \sum_{k=1}^{n} p_k \leq \sum_{k=1}^{\infty} p_k < \infty,$$

   therefore $\sum_{k=1}^{\infty}\log(1+p_k)$ is convergent.

2. (a) Now We shall follow the hint given in the question. Let $a > 1$

$$\frac{1}{2^a}\sum_{k=1}\frac{1}{k^a} = \sum_{k=1}\frac{1}{(2k)^a}$$

and therefore

$$\big(1-\frac{1}{2^a}\big)\sum_{k=1}\frac{1}{k^a} = \sum_{k=1}\frac{1}{(2k-1)^a} = 1+\frac{1}{3^a}+\frac{1}{5^a}+\cdots.$$

Similarly

$$\big(1-\frac{1}{2^a}\big)\big(1-\frac{1}{3^a}\big)\sum_{k=1}\frac{1}{k^a} = \sum_{2\nmid k \text{ and } 3\nmid k}\frac{1}{k^a}$$

and

$$\big(1-\frac{1}{2^a}\big)\big(1-\frac{1}{3^a}\big)\big(1-\frac{1}{5^a}\big)\sum_{k=1}\frac{1}{k^a} = \sum_{2\nmid k, 3\nmid k \text{ and } 5\nmid k}\frac{1}{k^a}.$$

Since every integer ≥ 2 is either a prime or product of primes. We have let N be any large integer > 2

$$\prod_{p<N}\big(1-\frac{1}{p^a}\big)\sum_{k=1}\frac{1}{k^a} = 1+\sum_{k>1, p\nmid k, \forall p<N}\frac{1}{k^a}.$$

Hence

$$\Big|\prod_{p<N}\big(1-\frac{1}{p^a}\big)\sum_{k=1}\frac{1}{k^a}-1\Big| \leq \sum_{k=N}\frac{1}{k^a}$$

where RHS $\to 0$ as $N\to\infty$. Result follows.

(b) Method 1, suppose there are finitely many prime, say $p_1,\cdots,p_N \geq 2$, then RHS of the identity in 2a) defines a continuous function f on $[1,\infty)$, which is given by

$$f(a) = \frac{1}{\prod_p\big(1-\frac{1}{p^a}\big)}.$$

In particular, it is bounded on [1,2]. However, by integral test and 2a), we conclude that for $a < 2$

$$f(a) = \sum_{k=1}\frac{1}{k^a} \geq 1+\int_2^\infty \frac{1}{t^a}dt = 1+\frac{2^{1-a}}{a-1} \geq \frac{1}{2(a-1)},$$

which is unbounded on $(1,2]$. Contradiction.

Method 2, suppose there are finitely many prime, say $p_1,\cdots,p_N$. We consider the following integer

$$k := \prod_{n=1}^N p_n + 1.$$

Every integer ≥ 2 is either a prime or product of primes. Now $k\geq 2$ is not divisible by p_i for all $i\leq N$ and therefore is a prime $\neq p_i, \forall i$, which is impossible.