MATH2060B Exercise 10

Deadline: Apr 8, 2015 at noon.

The questions are from Bartle and Sherbert, *Introduction to Real Analysis*, Wiley, 4th edition, unless otherwise stated.

Section 9.1 Q.7, 8, 13
Section 9.2 Q.1(a)(c), 2(b)(c)(d), 4(b)(d), 6, 7(a), 8, 15
Section 9.3 Q.1(b)(d), 2, 8(a)(c)

For those who want more practice (no need to turn in the following, including the supplementary exercises):

Section 9.1 Q.2, 3, 9, 10, 15
Section 9.2 Q.3(b)(d), 16, 19, 20

Supplementary Exercises

1. Let $a_n, n \geq 1$, be positive numbers. The infinite product $\prod_{k=1}^{\infty} a_k$ is called *convergent* if the infinite series $\sum_{k=1}^{\infty} \log a_k$ is convergent, and we define

$$\prod_{k=1}^{\infty} a_k = \exp\left(\sum_{k=1}^{\infty} \log a_k\right).$$

   (a) If $\prod_{k=1}^{\infty} a_k$ is convergent, show that $\lim_{n\to\infty} a_n = 1$;

   (b) Suppose that $a_n = 1+p_n$ where $p_n \geq 0$ for all n. Show that $\prod_{k=1}^{\infty}(1+p_k)$ is convergent if and only if $\sum_{k=1}^{\infty} p_k$ is convergent. Hint: Establish the inequalities

$$\sum_{k=1}^{n} p_k \leq \prod_{k=1}^{n}(1+p_k) \leq \exp\left(\sum_{k=1}^{n} p_k\right).$$

2. This problem shows how infinite products could come up in an unexpected way.

   (a) Prove the identity

$$\sum_{k=1} \frac{1}{k^a} = \frac{1}{\prod_p \left(1 - \frac{1}{p^a}\right)}, \quad a > 1,$$

   where the product is taken over all prime numbers. (Hint: Use the unique factorization of positive integers into product of primes.)

   (b) Deduce that there are infinitely many prime numbers.