**Math 1010C Term 1 2015**
**Supplementary exercises 4**

The following exercises are not to be submitted, but they form an important part of the course, and you're advised to go through them carefully.

1. Solve Exercises 1-14 of Section 4.1 of Thomas' calculus (12th edition).

2. Let $f(x) = x^3 - 6x^2 + 9x - 2$.

   (a) Show that $f$ is continuous on $\mathbb{R}$.

   (b) What does the extreme value theorem tell us about $f$ on the interval $[-1, 2]$? (State carefully what you use about the interval $[-1, 2]$ when you apply the extreme value theorem.)

   (c) We know by part (b) that there exists (at least one) $x_0 \in [-1, 2]$ such that $f$ achieves its (absolute) maximum on $[-1, 2]$ at $x_0$. Show that if $x_0$ is not one of the boundary points, i.e. if $x_0 \neq -1$ nor 2, then $x_0$ is a critical point of $f$.

   (d) Hence show that $x_0$ cannot be anything other than $-1$, 2, or 1.

   (e) Show that actually $x_0$ cannot be $-1$ or 2.

   (f) What must $x_0$ be then?

   (g) Using the above, find the maximum value of $f$ on $[-1, 2]$.

   (h) We are going to repeat the above for the (absolute) minimum of $f$ on $[-1, 2]$. We know by part (b) that there exists (at least one) $y_0 \in [-1, 2]$ such that $f$ achieves its (absolute) minimum on $[-1, 2]$ at $y_0$. Show that if $y_0$ is not one of the boundary points, i.e. if $y_0 \neq -1$ nor 2, then $y_0$ is a critical point of $f$.

   (i) Hence show that $y_0$ cannot be anything other than $-1$, 2, or 1.

   (j) Show that actually $y_0$ cannot be 1 or 2.

   (k) What must $y_0$ be then?

   (l) Using the above, find the minimum value of $f$ on $[-1, 2]$.

   (m) Confirm your answer by plotting a graph of $f$, say using a graphing calculator or a computer software.

3. Let $f$ be as in Question 2.

   (a) Repeat the above question, to find the maximum and minimum value of $f$ on the interval $[-1, 5]$.

   (b) Explain why the above method would fail if one wants to find the maximum/minimum value of $f$ on the open interval $(-1, 5)$. Does $f$ have a maximum or a minimum on $(-1, 5)$? (Hint: Use the graph of $f$ to help you determine the answer.)

(c) What if now one wants to find the maximum of $f$ on $[-1, \infty)$? Does $f$ have a maximum on $[-1, \infty)$?

(d) What if now one wants to find the minimum of $f$ on $[-1, \infty)$? Does $f$ have a maximum on $[-1, \infty)$? (Hint: First show $f(x) \geq f(3)$ for all $x \in [3, \infty)$ (say using the first derivative of $f$). Then deduce $f(x) \geq f(-1)$ for all $x \in [-1, 3]$ (say from part (a)). Finally, conclude $f(x) \geq f(-1)$ for all $x \in [-1, \infty)$.)

4. Using the method in Question 2, find the absolute maximum and minimum of the following functions on the following closed and bounded intervals. You should find the points where the maximum / minimum are achieved, and the values of the function at these maximum / minimum points.

   (a) $f(x) = xe^{-x}$, on the interval $-1 \leq x \leq 2$.

   (b) $g(x) = |x|(1 - x^2)^2$, on the interval $-2 \leq x \leq 2$.

   (c) $h(x) = \begin{cases} -x^2 - 2x + 4, & \text{if} \quad x \leq 1 \\ -x^2 + 6x - 4, & \text{if} \quad x > 1 \end{cases}$, on the interval $[-2, 2]$

   (d) $k(x) = |x^3 - 9x|$, on the interval $[-1, 4]$.

5. Find all critical points of the following functions on the indicated intervals. Determine whether these are relative maximums / minimums of the functions (they could be neither).

   (a) $f(x) = x^{1/3}(x - 4), \qquad (-1, \infty)$

   (b) $g(x) = x\sqrt{8 - x^2}, \qquad (-2\sqrt{2}, 2\sqrt{2})$

   (c) $h(x) = \begin{cases} x \ln |x|, & x \neq 0 \\ 0, & x = 0 \end{cases}, \qquad (-\infty, \infty)$

6. For each of the following function,

   (i) Determine where the function is increasing, and where it is decreasing;
   (ii) Find all relative maximums / minimums of the function on $(-\infty, \infty)$;
   (iii) Determine whether any of these is an absolute extremum of the function on $(-\infty, \infty)$. (For this you will need to understand the behaviour of the function at $\pm\infty$.)
   (iv) Determine where the function is convex, and where it is concave;
   (v) Sketch the graph of the function.

   (a) $f(x) = \dfrac{x}{x^2 + 1}$

   (b) $g(x) = \dfrac{|x - 1|}{|x| + 1}$

   (c) $h(x) = \dfrac{x|x - 1| + |x + 1|}{|x|^2 + 1}$

7. For each of the functions and intervals in Question 5, determine whether the given function have an absolute maximum / minimum on the indicated intervals. (You'll have to understand the behaviour of these functions as $x$ approaches the end-points of the intervals, and / or $\pm\infty$.) If yes, find the maximum / minimum values of the functions on the indicated intervals.

8. Determine whether the following functions have an absolute maximum / minimum on the indicated intervals. If yes, locate ALL points where the absolute maximum / minimum are achieved.

   (a) $f(x) = \dfrac{|x-1|(x-3)}{x-2}, \qquad (-\infty, 2)$

   (b) $g(x) = \dfrac{2x^2 - x^4}{x^4 - 2x^2 + 2}, \qquad [-1, \infty)$

   (c) $h(x) = 2e^{\cos x} + e^{2\cos x} + 3, \qquad [0, \infty)$

   (Hint: Do not proceed blindly! Come up with some tricks where you can.)

9. Determine whether the function
$$f(x) = 3 + 4\cos x + \cos 2x$$
   achieves an absolute minimum on $\mathbb{R}$.