Math 1010C Term 1 2015
Supplementary exercises 3

1. Let $f\colon I \to \mathbb{R}$ be a function defined on an open interval I, and $c \in I$.

   (a) Show that if there exists a function $\phi\colon I \to \mathbb{R}$ such that

   $$f(x) = f(c) + \phi(x)(x-c) \quad \text{for all } x \in I,$$

   and such that ϕ is continuous at c, then f is differentiable at c, and $f'(c) = \phi(c)$. (Hint: Compute the quotient $\frac{f(x)-f(c)}{x-c}$, and let x tend to c.)

   (b) Show that the converse of (a) also holds, in the sense that if f is differentiable at c, then there exists a function $\phi\colon I \to \mathbb{R}$ such that

   $$f(x) = f(c) + \phi(x)(x-c) \quad \text{for all } x \in I, \tag{1}$$

   and such that ϕ is continuous at c. How does the value of $\phi(c)$ depend on f? (Hint: The identity (1) *defines* $\phi(x)$ for you already, for all $x \in I$ that is not equal to c. Just make $\phi(x)$ the subject of (1)! Now figure out what $\phi(x)$ should be when $x = c$, if ϕ were to be continuous at c.)

   Note that the above shows that if f is differentiable at c, then for $x \simeq c$, we have

   $$f(x) = f(c) + \phi(x)(x-c) \simeq f(c) + \phi(c)(x-c) = f(c) + f'(c)(x-c)$$

   The (linear) function $x \mapsto f(c) + f'(c)(x-c)$ is just the equation of the tangent line to the graph of f through $(c, f(c))$. Hence if f is differentiable at c, then when x is very close to c, $f(x)$ is very close to the tangent line through $(c, f(c))$. Every differentiable function is almost linear (locally)!

2. In order to prove the chain rule, sometimes the following heuristic argument is given: let f be differentiable at $g(c)$, and g be differentiable at c.

Then

$$\begin{aligned}
(f \circ g)'(c) &= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{x - c} \\
&= \lim_{x \to c} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \frac{g(x) - g(c)}{x - c} \\
&= \lim_{g(x) \to g(c)} \frac{f(g(x)) - f(g(c))}{g(x) - g(c)} \lim_{x \to c} \frac{g(x) - g(c)}{x - c} \\
&= f'(g(c))g'(c).
\end{aligned}$$

Why is this not quite a completely rigorous proof?

3. We are going to give a correct proof of the chain rule in this exercise. Let f be differentiable at $g(c)$, and g be differentiable at c.

   (a) Using Question 1b, show that there exists functions ϕ and ψ, such that

   $$g(x) = g(c) + \phi(x)(x - c) \quad \text{for all } x \text{ near } c,$$
   $$f(y) = f(g(c)) + \psi(y)(y - g(c)) \quad \text{for all } y \text{ near } g(c),$$

   and such that ϕ and ψ are continuous at c and $g(c)$ respectively. Note that $\phi(c) = g'(c)$, $\psi(g(c)) = f'(g(c))$.

   (b) Using part (a), show that

   $$f(g(x)) = f(g(c)) + \psi(g(x))\phi(x)(x - c) \quad \text{for all } x \text{ near } c.$$

   (c) Show that $\psi(g(x))\phi(x)$ is continuous at $x = c$, and is equal to $f'(g(c))g'(c)$ at $x = c$.

   (d) Using parts (b) and (c), together with Question 1a, conclude that $f \circ g$ is differentiable at $x = c$, with $(f \circ g)'(c) = f'(g(c))g'(c)$.

4. (Putnam 2011) Suppose f, g are (real-valued) functions defined on an open interval containing 0, such that g is continuous at 0, and $g(0) \neq 0$. For x sufficiently close to 0, define $u(x) = f(x)g(x)$, $v(x) = f(x)/g(x)$. Suppose u and v are both differentiable at 0. Show that f is also differentiable at 0.

   (Hint: If $f(0) \neq 0$, use $f(x) = \pm\sqrt{u(x)v(x)}$ for x near 0. Otherwise $f(0) = 0$, in which case one computes $\lim_{x \to 0} \frac{f(x)}{x}$ by using $f(x) = v(x)g(x)$ for x near 0.)