Math 1010A Term 1 2017
Supplementary exercise about derivatives of some elementary functions

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The following exercises aim to illustrate how some properties of exp, sin, cos follow from differentiation, and how these properties in turn gives formula for the derivatives of these functions.

1. Suppose f is a differentiable function on $\mathbb{R}$, with
$$f'(x) = f(x) \quad \text{for all } x \in \mathbb{R}, \quad \text{and} \quad f(0) = 1. \tag{1}$$
   (a) Compute $f'(0)$ using (1). Hence show that
$$\lim_{x \to 0} \frac{f(x) - 1}{x} = 1. \tag{2}$$
   (b) Fix any $y \in \mathbb{R}$. Let
$$F(x) = f(y) - f(x)f(y - x) \quad \text{for any } x \in \mathbb{R}.$$
   Show that $F'(x) = 0$ for all $x \in \mathbb{R}$. Also find the value of $F(0)$. Hence show that $F(x) = 0$ for all $x \in \mathbb{R}$. This shows that $f(y) = f(x)f(y - x)$ for any $x, y \in \mathbb{R}$, which is equivalent to saying that
$$f(x + y) = f(x)f(y) \quad \text{for any } x, y \in \mathbb{R}. \tag{3}$$
   (c) In particular, if all that we knew about e^x is that it is differentiable on $\mathbb{R}$ and satisfies (1), then we can *deduce* that
$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \quad \text{and} \quad e^{x+y} = e^x e^y \quad \text{for any } x, y \in \mathbb{R}.$$
   Clearly we can also deduce that e^x is continuous on $\mathbb{R}$ in that case.

2. Conversely, suppose f is a *continuous* function on $\mathbb{R}$ that satisfies (2) and (3) in the previous question.
   (a) Show that $\lim_{x \to 0} f(x) = 1$ by using (2). Hence conclude that $f(0) = 1$.
   (b) From first principles, show that f is differentiable at every point of $\mathbb{R}$, with $f'(x) = f(x)$ for all $x \in \mathbb{R}$.
   (c) In particular, if all we knew about e^x is that it is continuous on $\mathbb{R}$ and satisfies (2) and (3), then we can *deduce* that e^x is differentiable on $\mathbb{R}$,
$$\frac{d}{dx} e^x = e^x \quad \text{for all } x \in \mathbb{R}, \quad \text{and} \quad e^0 = 1.$$

3. Suppose g and h are differentiable real-valued functions on $\mathbb{R}$, with
$$\begin{cases} g'(x) = h(x) \\ h'(x) = -g(x) \end{cases} \quad \text{for all } x \in \mathbb{R}, \quad \text{and} \quad \begin{cases} g(0) = 0 \\ h(0) = 1. \end{cases} \tag{4}$$

(a) Compute $g'(0)$ and $h'(0)$ using (4). Hence show that

$$\lim_{x\to 0}\frac{g(x)}{x} = 1 \quad \text{and} \quad \lim_{x\to 0}\frac{h(x)-1}{x} = 0. \tag{5}$$

(b) Fix any $y \in \mathbb{R}$. Let

$$\begin{cases} G(x) = g(x+y) - g(x)h(y) - h(x)g(y) \\ H(x) = h(x+y) - h(x)h(y) + g(x)g(y) \end{cases} \quad \text{for any } x \in \mathbb{R}.$$

(i) Show that

$$\begin{cases} G'(x) = H(x) \\ H'(x) = -G(x) \end{cases} \quad \text{for any } x \in \mathbb{R}, \quad \text{and} \quad \begin{cases} G(0) = 0 \\ H(0) = 0. \end{cases}$$

(ii) Let

$$Z(x) = [G(x)]^2 + [H(x)]^2 \quad \text{for any } x \in \mathbb{R}.$$

Show that $Z'(x) = 0$ for all $x \in \mathbb{R}$. Also find the value of $Z(0)$. Hence show that $Z(x) = 0$ for all $x \in \mathbb{R}$. This shows that $G(x) = H(x) = 0$ for all $x \in \mathbb{R}$, and hence

$$\begin{cases} g(x+y) = g(x)h(y) + h(x)g(y) \\ h(x+y) = h(x)h(y) - g(x)g(y) \end{cases} \quad \text{for any } x, y \in \mathbb{R}. \tag{6}$$

(c) In particular, if all we knew about $\sin x$ and $\cos x$ are that they are differentiable real-valued functions on $\mathbb{R}$ and satisfies (4), then we can *deduce* that

$$\lim_{x\to 0}\frac{\sin x}{x} = 1 \quad \text{and} \quad \lim_{x\to 0}\frac{\cos x - 1}{x} = 0,$$

and that the compound angle formula holds:

$$\begin{cases} \sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y) \\ \cos(x+y) = \cos(x)\cos(y) - \sin(x)\sin(y) \end{cases} \quad \text{for any } x, y \in \mathbb{R}.$$

4. Conversely, suppose g and h are *continuous* functions on $\mathbb{R}$ that satisfies (5) and (6) in the previous question.

(a) Show that $\lim_{x\to 0} g(x) = 0$ and $\lim_{x\to 0} h(x) = 1$ by using (5).
Hence conclude that $g(0) = 0$, $h(0) = 1$.

(b) From first principles, show that g and h are differentiable at every point of $\mathbb{R}$, with $g'(x) = h(x)$ and $h'(x) = -g(x)$ for all $x \in \mathbb{R}$.

(c) In particular, if all we knew about $\sin x$ and $\cos x$ are that they are continuous on $\mathbb{R}$ and satisfies (5) and (6), then we can *deduce* that $\sin x$ and $\cos x$ are differentiable on $\mathbb{R}$,

$$\begin{cases} \frac{d}{dx}\sin x = \cos x \\ \frac{d}{dx}\cos x = -\sin x \end{cases} \quad \text{for all } x \in \mathbb{R}, \quad \text{and} \quad \begin{cases} \sin 0 = 0 \\ \cos 0 = 1. \end{cases}$$