MATH 1010A/K 2017-18

University Mathematics

Tutorial Notes V

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Question

(Q1) Show

$$\frac{\sin y - \sin x}{y - x} > \frac{\sin z - \sin y}{z - y}$$

for any $0 \le x < y < z \le \pi$.

(Q2) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with f' is strictly increasing.

Show $f'(x) < f(x+1) - f(x) < f'(x+1)$ for any $x \in \mathbb{R}$.

Hence, show

$$f'(1) + f'(2) + ... + f'(n-1) < f(n) - f(1) < f'(2) + f'(3) + ... + f'(n)$$

for any $n \in \mathbb{N}$.

(Q3) Let $x, y \in \mathbb{R}$ with $x < y$. Show that there exist some $\xi \in (x, y)$ such that

$$\frac{\sin x}{y} - \frac{\sin y}{x} = \left(\frac{1}{y} - \frac{1}{x}\right)\left(\sin \xi + \xi \cos \xi\right).$$

(Q4) Show

$$\frac{1}{\ln x} - \frac{e}{x \ln x} < \ln\left(\ln x\right) < \frac{x}{e} - 1$$

for any $x > e$.

(Q5) Let $x, y \in \mathbb{R}$ with $x < y$. Show that there exist some $\xi \in (x, y)$ such that

$$\frac{ye^{x^2} - xe^{y^2}}{y - x} = e^{x^2+y^2-\xi^2}\left(1 - 2\xi^2\right).$$

Further suppose $\frac{-1}{\sqrt{2}} < x < y < \frac{1}{\sqrt{2}}$. Show that

$$0 < \frac{ye^{x^2} - xe^{y^2}}{y - x} \le e^{x^2+y^2}.$$

Answer

(A1) Fixed x, y, z with $0 \le x < y < z \le \pi$. Let $f(w) = \sin w$.

Note f is continuous on $[x, y]$ and differentiable on (x, y) with $f'(w) = \cos w$,

by (Lagrange) Mean Value Theorem, there exist some $\xi \in (x, y)$, such that

$$\frac{\sin y - \sin x}{y - x} = \cos \xi.$$

Note f is continuous on $[y, z]$ and differentiable on (y, z) with $f'(w) = \cos w$,

by (Lagrange) Mean Value Theorem, there exist some $\eta \in (y, z)$, such that

$$\frac{\sin z - \sin y}{z - y} = \cos \eta.$$

Note that $0 \le x < \xi < y < \eta < z \le \pi$ and cos is decreasing on $(0, \pi)$,

hence $\cos \xi > \cos \eta$ and that is,

$$\frac{\sin y - \sin x}{y - x} > \frac{\sin z - \sin y}{z - y}.$$

(A2) Fixed $x \in \mathbb{R}$. Note f is continuous on $[x, x+1]$ and differentiable on $(x, x+1)$,

by (Lagrange) Mean Value Theorem, there exist some $\xi \in (x, x+1)$, such that

$$f(x+1) - f(x) = \frac{f(x+1) - f(x)}{x + 1 - x} = f'(\xi).$$

Since f' strictly increasing and $x < \xi < x + 1$, we have $f'(x) < f'(\xi) < f'(x+1)$, hence

$$f'(x) < f(x+1) - f(x) < f'(x+1).$$

It is true for any $x \in \mathbb{R}$. Hence, fixed $n \in \mathbb{N}$, we have

$$\begin{cases} f'(1) < f(2) - f(1) < f'(2) \\ f'(2) < f(3) - f(2) < f'(3) \\ \quad \vdots \\ f'(n-2) < f(n-1) - f(n-2) < f'(n-1) \\ f'(n-1) < f(n) - f(n-1) < f'(n) \end{cases},$$

sum all the inequality, we have

$$f'(1) + f'(2) + ... + f'(n-1) < f(n) - f(1) < f'(2) + f'(3) + ... + f'(n)$$

(A3) Fixed $x, y \in \mathbb{R}$ with $x < y$. Let $f(w) = w \sin w$.

Note f is continuous on $[x, y]$ and differentiable on (x, y) with $f'(w) = \sin w + w \cos w$,

by (Lagrange) Mean Value Theorem, there exist some $\xi \in (x, y)$, such that

$$\begin{aligned} \frac{y \sin y - x \sin x}{y - x} &= \sin \xi + \xi \cos \xi \\ \left(\frac{x - y}{xy}\right) \frac{y \sin y - x \sin x}{y - x} &= \left(\frac{1}{y} - \frac{1}{x}\right) \left(\sin \xi + \xi \cos \xi\right) \\ \frac{x \sin x - y \sin y}{xy} &= \left(\frac{1}{y} - \frac{1}{x}\right) \left(\sin \xi + \xi \cos \xi\right) \\ \frac{\sin x}{y} - \frac{\sin y}{x} &= \left(\frac{1}{y} - \frac{1}{x}\right) \left(\sin \xi + \xi \cos \xi\right). \end{aligned}$$

(A4) Fixed $x \in \mathbb{R}$ with $x > e$. Let $f(w) = \ln\left(\ln w\right)$.

Note f is continuous on $[e, x]$ and differentiable on (e, x) with $f'(w) = \dfrac{1}{w \ln w}$,

by (Lagrange) Mean Value Theorem, there exist some $\xi \in (e, x)$, such that

$$\frac{\ln\left(\ln x\right)}{x - e} = \frac{\ln\left(\ln x\right) - \ln\left(\ln e\right)}{x - e} = \frac{1}{\xi \ln \xi}.$$

Note $\dfrac{1}{w \ln w}$ strictly decreasing (since w and $\ln w$ strictly decreasing and POSITIVE),

and $e < \xi < x$, we have $\dfrac{1}{e} = \dfrac{1}{e \ln e} > \dfrac{1}{\xi \ln \xi} > \dfrac{1}{x \ln x}$, hence

$$\begin{aligned} \frac{1}{x \ln x} &< \frac{\ln\left(\ln x\right)}{x - e} < \frac{1}{e} \\ \frac{x - e}{x \ln x} &< \ln\left(\ln x\right) < \frac{x - e}{e} \\ \frac{1}{\ln x} - \frac{e}{x \ln x} &< \ln\left(\ln x\right) < \frac{x}{e} - 1. \end{aligned}$$

(A5) Fixed $x, y \in \mathbb{R}$ with $x < y$. Let $f(w) = we^{-w^2}$.

Note f is continuous on $[x, y]$ and differentiable on (x, y) with $f'(w) = \left(1 - 2w^2\right) e^{-w^2}$,

by (Lagrange) Mean Value Theorem, there exist some $\xi \in (x, y)$, such that

$$\begin{aligned} \frac{ye^{-y^2} - xe^{-x^2}}{y - x} &= \left(1 - 2\xi^2\right) e^{-\xi^2} \\ e^{x^2+y^2} \cdot \frac{ye^{-y^2} - xe^{-x^2}}{y - x} &= \left(1 - 2\xi^2\right) e^{-\xi^2} \cdot e^{x^2+y^2} \\ \frac{ye^{x^2} - xe^{y^2}}{y - x} &= e^{x^2+y^2-\xi^2} \left(1 - 2\xi^2\right). \end{aligned}$$

Now if $\dfrac{-1}{\sqrt{2}} < x < y < \dfrac{1}{\sqrt{2}}$, then $\dfrac{-1}{\sqrt{2}} < \xi < \dfrac{1}{\sqrt{2}}$,

that means $0 < 1 - 2\xi^2 \leq 1$ and $0 < e^{-\xi^2} \leq 1$(Why?) and hence

$$0 < \left(1 - 2\xi^2\right) e^{-\xi^2} \leq 1,$$

and we have

$$0 < \frac{ye^{x^2} - xe^{y^2}}{y - x} \leq e^{x^2+y^2}.$$