MATH 1010A/K 2017-18

University Mathematics

Notes of $\lim_{x\to\infty} e^{-x}x^k$, $\lim_{x\to 0} x\sin\frac{1}{x}$

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Caution This notes need to use some basic knowlegde of calculus.

Theorem For all $x \geq 0$, $e^x \geq 1 + x + \frac{x^2}{2!} ... + \frac{x^k}{k!}$ for any $k \in \mathbb{N} \cup \{0\}$.

Proof We use induction on k.

Let $P(k)$ be the statement that "For all $x \geq 0$, $e^x \geq 1 + x + \frac{x^2}{2!} ... + \frac{x^k}{k!}$."

Note $P(0)$ is true since $e^x \geq 1$ for any $x \geq 0$.

Assume $P(i)$ is true for some $i \in \mathbb{N} \cup \{0\}$. That is

$$\begin{aligned} e^t &\geq 1 + t + \frac{t^2}{2!} ... + \frac{t^i}{i!} && \forall\, t \geq 0 \\ \int_0^x e^t dt &\geq \int_0^x \left(1 + t + \frac{t^2}{2!} ... + \frac{t^i}{i!}\right) dt && \forall\, x \geq 0 \\ e^x - 1 &\geq x + \frac{x}{2} + \frac{x}{3!} + ... + \frac{x^{i+1}}{(i+1)!} && \forall\, x \geq 0 \\ e^x &\geq 1 + x + \frac{x}{2} + \frac{x}{3!} + ... + \frac{x^{i+1}}{(i+1)!} && \forall\, x \geq 0. \end{aligned}$$

Hence, $P(i+1)$ is true.

By the first principal of Mathematical Induction, $P(k)$ is true for any $k \in \mathbb{N} \cup \{0\}$.

Corollary $\lim_{x\to\infty} \frac{x^k}{e^x} = 0$ for any $k \in \mathbb{N} \cup \{0\}$.

Proof By last theorem, we have $e^x \geq 1 + x + \frac{x^2}{2!} ... + \frac{x^{k+1}}{(k+1)!} \geq \frac{x^{k+1}}{(k+1)!} > 0$ for any $x > 0$.

That is, $0 \leq \frac{x^k}{e^x} \leq \frac{x^k\,(k+1)!}{x^{k+1}} = \frac{(k+1)!}{x}$. Note $\lim_{x\to\infty} 0 = 0 = \lim_{x\to\infty} \frac{(k+1)!}{x}$.

By Sandwich Theorem, $\lim_{x\to\infty} \frac{x^k}{e^x}$ exists and $\lim_{x\to\infty} \frac{x^k}{e^x} = 0$.

Corollary $\lim_{x\to\infty} \frac{(\ln x)^k}{x} = 0$ for any $k \in \mathbb{N} \cup \{0\}$.

Proof $\lim_{x\to\infty} \frac{(\ln x)^k}{x} \overset{y=\ln x}{=} \lim_{y\to\infty} \frac{y^k}{e^y} = 0$.

Theorem Let $f : \mathbb{R} \to \mathbb{R}$ be a function, and $c \in \mathbb{R} \cup \{\pm\infty\}$.

If $\lim_{x\to c} |f(x)| = 0$, then $\lim_{x\to c} f(x)$ exists and equals to 0.

Proof Note $-|w| \leq w \leq |w|$ for any $w \in \mathbb{R}$.

Hence, $-|f(x)| \leq f(x) \leq |f(x)|$ for any $x \in \mathbb{R}$.

Note $\lim_{x\to c} -|f(x)| = 0 = \lim_{x\to c} |f(x)|$.

By Sandwich Theorem, $\lim_{x\to c} f(x)$ exists and equals to 0.

Theorem $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

Proof We prove it from one-sided limit.

For $x > 0$, we have $-x \leq x \sin \frac{1}{x} \leq x$.

Note $\lim_{x \to 0^+} -x = 0 = \lim_{x \to 0^+} x$, by Sandwich Theorem, we have $\lim_{x \to 0^+} x \sin \frac{1}{x}$ exists and equals to 0.

For $x < 0$, we have $x \leq x \sin \frac{1}{x} \leq -x$.

Note $\lim_{x \to 0^-} x = 0 = \lim_{x \to 0^-} -x$, by Sandwich Theorem, we have $\lim_{x \to 0^-} x \sin \frac{1}{x}$ exists and equals to 0.

Hence, $\lim_{x \to 0^-} x \sin \frac{1}{x} = 0 = \lim_{x \to 0^+} x \sin \frac{1}{x}$.

Therefore, $\lim_{x \to 0} x \sin \frac{1}{x}$ exists and equals to 0.