MATH 2060 Mathematical Analysis II
Tutorial Class 10
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1. (a) Suppose $\sum_{n=1}^{\infty} x_n$ converge, show that $x_n \to 0$ and $\sum_{k=n}^{\infty} x_k \to 0$ as n goes to ∞.
   (b) State the Cauchy Criterion for convergence of series.
   (c) Prove the Comparsion Test. i.e. If $\{a_k\}$ and $\{b_k\}$ are two sequences of numbers such that $0 \le a_k \le b_k$ for all $k \in \mathbb{N}$. Then the convergence of $\displaystyle\sum_{n=1}^{\infty} b_n$ implies the convergence of $\displaystyle\sum_{n=1}^{\infty} a_n$.
   (d) Show that $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverge and $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$ converge.
   (e) Show that for any $\epsilon > 0$, the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^{2+\epsilon} - n + 1}$ converge.

2. (a) Suppose $x_n \ge 0$. Show that $\sum_{n=1}^{\infty} x_n$ converge if and only if its partial sum is bounded.
   (b) Suppose $x_n \ge 0$ and $\sum_{n=1}^{\infty} x_n$ converge. Show that the following series converge:
   (i) $\displaystyle\sum_{n=1}^{\infty} x_n^{1+\epsilon}$ (ii) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{x_n}}{n}$ (iii) $\displaystyle\sum_{n=1}^{\infty} \sqrt{x_n x_{n+1}}$.
   (c) Suppose $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are series of positive numbers such that
   $$\lim_{k\to\infty} \frac{a_k}{b_k} = l, \quad l > 0.$$
   Prove that the series $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.

3. (a) State the Ratio Test for the convergence of series.
   (b) Test the convergence of the series $\sum_{n=1}^{\infty} x_n$ with general term:
   (i) $x_n = (\frac{n}{2n+1})^n$ (ii) $x_n = \frac{3^n}{n^2}$ (iii) $x_n = \frac{n^n}{n!}$.

4. (a) State the Integral Test for convergence of series.
   (b) For $\alpha > 0$, consider the series
   $$\sum_{k=1}^{\infty} \frac{1}{(k+1)[\ln(k+1)]^{\alpha}},$$
   Find the values of α at which the series converge.
   (c) Give an exmaple of $x_n > 0$ such that $\lim_{n\to\infty} x_n = 0$ but $\displaystyle\sum_{n=2}^{\infty} \frac{x_n}{n \log n}$ diverge.