Suggested Solution to Homework 2

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P71, 12. A *seminorm* on a vector space X is a mapping $p : X \to \mathbf{R}$ satisfying (N1), (N3), (N4) in Sec. 2.2. Show that

$$p(0) = 0,$$
$$|p(x) - p(y)| \le p(x, y).$$

(Hence if $p(x) = 0$ implies $x = 0$, then p is a norm.)

Proof. The property (N3) yields that, for any $\alpha \in \mathbf{R}$,

$$p(0) = p(\alpha 0) = |\alpha| p(0)$$

So, $p(0) = 0$.

It follows from the property (N4) that, for any $x, y \in X$,

$$p(y) = p(y - x + x) \le p(y - x) + p(x).$$

Similarly,

$$p(x) \le p(x - y) + p(y)$$

Hence, $|p(x) - p(y)| \le p(x - y)$, where (N3) has been used. □

P71, 13. Show that in Prob. 12, the elements $x \in X$ such that $p(x) = 0$ form a subspace N of X and a norm on X/N(c.f. Prob. 14, Sec. 2.1) is defined by $\|\hat{x}\|_0 = p(x)$, where $x \in \hat{x}$ and $\hat{x} \in X/N$.

Proof.

(1) For any $x, y \in N$(i.e.$p(x) = p(y) = 0$), it follows from (N1), (N4) and (N3) that

$$0 \le p(\alpha x + \beta y) \le p(\alpha x) + p(\beta y) = |\alpha| p(x) + |\beta| p(y) = 0, \quad \alpha, \beta \in \mathbf{R}.$$

So, $\alpha x + \beta y \in N$ which implies that N is a subspace of X.

(2) First, for any $x_1, x_2 \in \hat{x}$, there exist $n_1, n_2 \in X$ such that $x_1 = x + n_1, x_2 = x + n_2$. Then,

$$|p(x_1) - p(x_2)| \le |p(x_1 - x_2)| = |p(n_1 - n_2)| = 0,$$

since N is a subspace. So, $p(x_1) = p(x_2)$, i.e. $\|\hat{x}\|_0 = p(x)$ is well-defined, which is independent of the choice of represent element x. Now, we verify that $\|\cdot\|_0$ satisfies (N1)-(N4):

(N1) Since $p(x) \ge 0$, $\|\hat{x}\|_0 \ge 0$.
(N2) If $\|\hat{x}\|_0 = 0$, then $p(x) = 0$, so that $x \in N$. Hence $\hat{x} = N$, i.e. $\hat{x} = \hat{0} \in X/N$.
(N3) Since $\alpha\hat{x} = \alpha x + N$, it holds that, for some $n \in N$,

$$\|\alpha\hat{x}\|_0 = p(\alpha x + n) = p(\alpha(x + n/\alpha)) = |\alpha| p(x + n/\alpha) = |\alpha| \|\hat{x}\|_0, \quad for \ \ \alpha \ne 0.$$

It is clear that, for $\alpha = 0$,

$$\|0\hat{x}\|_0 = 0 = 0\|\hat{x}\|_0.$$

(N4) For any $\hat{x} = x + N, \hat{y} = y + N$, $\hat{x} + \hat{y} = x + y + N$. Then,

$$\|\hat{x} + \hat{y}\|_0 = p(x + y) \le p(x) + p(y) = \|\hat{x}\|_0 + \|\hat{y}\|_0.$$

□

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P101, 5. Show that the operator $T : \ell^\infty \to \ell^\infty$ defined by $y = (\eta_j) = Tx, \eta_j = \xi_j/j, x = (\xi_j)$, is linear and bounded.

Proof. For any $x_1 = (\xi_j^1), x_2 = (\xi_j^2)$,

$$T(\alpha x_1 + \beta x_2) = ((\alpha\xi_j^1 + \beta\xi_j^2)/j) = (\alpha\xi_j^1/j) + (\beta\xi_j^2/j) = \alpha Tx + \beta Ty, \quad for \quad \alpha, \beta \in \mathbf{R}.$$

So, T is linear.

On the other hand, since $\xi_j/j \le \xi_j$ for any $j \in \mathbb{N}^+$,

$$\|Tx\|_{\ell^\infty} = \sup_{j\ge1} |\xi_j/j| \le \sup_{j\ge1} |\xi_j| = \|x\|_{\ell^\infty}.$$

So, T is bounded. □

P101, 9. Let $T : C[0,1] \to C[0,1]$ be defined by

$$y(t) = \int_0^t x(\tau)d\tau.$$

Find $\mathscr{R}(T)$ and $T^{-1} : \mathscr{R}(T) \to C[0,1]$. Is T^{-1} linear and bounded?

Proof. By the Fundamental Theorem of Calculus, one has

$$\mathscr{R}(T) = \{y(t)|y(t) \in C^1[0,1], y(0) = 0\} \subset C[0,1].$$

and $T^{-1} : \mathscr{R}(T) \to C[0,1]$ is

$$T^{-1}y(t) = y'(t).$$

Since the differentiation is linear, so is T^{-1}. But T^{-1} is unbounded. Indeed, for $y_n(t) = t^n, t \in [0,1], n \in \mathbb{N}^+$, it is clear that

$$y_n(t) \in \mathscr{R}(T) \subset C[0,1], \quad \text{and} \quad \|y_n(t)\|_{C_0} = 1, \quad \text{for any} \quad n \in \mathbb{N}^+,$$

where $\|f(t)\|_{C_0} := \sup_{t\in[0,1]} |f(t)|$ for any $f(t) \in C[0,1]$. However,

$$\|T^{-1}(y_n)\|_{C_0} = \|y_n'(t)\|_{C_0} = \|nt^{n-1}\|_{C_0} = n \to +\infty, \quad as \quad n \to +\infty.$$

Hence, T^{-1} is not bounded. □