Suggested Solution to Homework 4

Yu Mei†

P101, 5. Show that the operator $T : \ell^\infty \to \ell^\infty$ defined by $y = (\eta_j) = Tx,\ \eta_j = \dfrac{\xi_j}{j}, x = (\xi_j)$, is linear and bounded.

Proof. Let $x = (\xi_j), y = (\eta_j)$ be in ℓ^∞. For any $\alpha, \beta \in \mathbb{R}$, $\alpha x + \beta y = (\alpha\xi_j + \beta\eta_j)$. Then, by the definition of T,

$$T(\alpha x + \beta y) = \frac{\alpha\xi_j + \beta\eta_j}{j} = \alpha\frac{\xi_j}{j} + \beta\frac{\eta_j}{j} = \alpha Tx + \beta Ty.$$

So, T is linear. Furthermore,

$$\|Tx\| = \sup_j |\frac{\xi_j}{j}| \le \sup_j |\xi_i| = \|x\|.$$

So, T is bounded. □

P101, 6.(Range) Show that the range $\mathscr{R}(T)$ of a bounded linear operator $T : X \to Y$ need not be closed in Y.

Proof. Let T be the linear bounded operator in Prob.5. We claim that $\mathscr{R}(T)$ is not closed. It sufficies to show that there exists a sequece $(y_n) \subset \mathscr{R}(T)$ which converges to some y, but $y \notin \mathscr{R}(T)$. Indeed, set $x_n = (1, 2, 3, \cdots, n, 0, \cdots 0, \cdots)$. Then $x_n \in \ell^\infty$ and $y_n = Tx_n = (1, 1, \cdots, 1, 0, \cdots)$ with first n terms being 1, others zero. It is clear that $y_n \in \mathscr{R}(T)$ and $y_n \to y = (1, \cdots, 1, \cdots)$. However, $y \notin \mathscr{R}(T)$, since $x = T^{-1}y = (1, 2, 3, \cdots, n \cdots)$ is not in ℓ^∞. □

P101, 7.(Inverse operator) Let T be a bounded linear opertator from a normed space X onto a normed space Y. If there is a positive b such that

$$\|Tx\| \ge b\|x\|, \qquad \text{for all} \quad x \in X,$$

show that then $T^{-1} : Y \to X$ exists and is bounded.

Proof. Assume that $Tx_1 = Tx_2$. Then

$$0 = \|Tx_1 - Tx_2\| \ge b\|x_1 - x_2\|$$

which yields that $x_1 = x_2$. So, T is injective. Since it is also onto, T is bijective. Thus, T^{-1} exists. Moreover,

$$\|T^{-1}y\| \le \frac{1}{b}\|T(T^{-1}y)\| = \|y\|.$$

Therefore, T is bounded. □

† Email address: ymei@math.cuhk.edu.hk. (Any questions are welcome!)