Suggested Solution to Homework 1

Yu Mei†

P32, 2. If (x_n) is Cauchy and has a convergent subsequence, say, $x_{n_k} \to x$, show that (x_n) is convergent with the limit x.

Proof. Let $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence in metric space (X, d) which has a convergent subsequence $(x_{n_k})_{k=1}^{\infty}$ with the limit x. Then, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. for all $n, m, k > N$,

$$d(x_n, x_m) < \frac{\epsilon}{2} \quad \text{and} \quad d(x_{n_k}, x) < \frac{\epsilon}{2}.$$

Therefore, note that $n_k \geq k$, we have

$$d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

which implies $x_n \to x$. □

P32, 8. If d_1 and d_2 are metrics on the same set X and there are positive numbers a and b such that for all $x, y \in X$,

$$ad_1(x, y) \leq d_2(x, y) \leq bd_1(x, y),$$

show that the Cauchy sequences in (X, d_1) and (X, d_2) are the same.

Proof. Let (x_n) be a Cauchy sequence in (X, d_1). Then, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. for all $n, m \geq N$,

$$d_1(x_n, x_m) < \frac{\epsilon}{b}.$$

It follows that

$$d_2(x_n, x_m) \leq bd_1(x_n . x_m) < \epsilon,$$

which yields (x_n) is a Cauchy sequence in (X, d_2).

Similarly, let (y_n) be a Cauchy sequence in (X, d_2). Then, $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t. for all $n, m \geq N$,

$$d_2(y_n, y_m) < a\epsilon.$$

It follows that

$$d_1(y_n, y_m) \leq \frac{d_2(y_n \cdot y_m)}{a} < \epsilon,$$

which yields (y_n) is a Cauchy sequence in (X, d_1). □

P40, 7. Let X be the set of all positive integers and $d(m, n) = |m^{-1} - n^{-1}|$. Show that (X, d) is not complete.

Proof. It is easy to check d is a metric on X. Now we show that (X, d) is not complete.

Set $x_n = n$. Then (x_n) is a Cauchy sequence in (X, d). Indeed, $\forall \epsilon > 0$, $\exists N = [\frac{2}{\epsilon}] + 1$, s.t. for all $n, m > N$,

$$d(x_n, x_m) = d(n, m) = |m^{-1} - n^{-1}| \leq 2N^{-1} < \epsilon.$$

Now, we claim that (x_n) cannot converge in (X, d). Otherwise, assume $x_n \to k$ for some $k \in \mathbb{Z}^+$. Then, for all $n > 2k^{-1}$

$$d(n, k) = |k^{-1} - n^{-1}| \geq 2k^{-1}.$$

A contradiction! □

† Email address: ymei@math.cuhk.edu.hk. (Any questions are welcome!)