ON THE STRUCTURES OF GENERATING ITERATED FUNCTION SYSTEMS OF CANTOR SETS

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Abstract. A generating IFS of a Cantor set F is an IFS whose attractor is F. For a given Cantor set such as the middle-3rd Cantor set we consider the set of its generating IFSs. We examine the existence of a minimal generating IFS, i.e. every other generating IFS of F is an iterating of that IFS. We also study the structures of the semi-group of homogeneous generating IFSs of a Cantor set F in $\mathbb{R}$ under the open set condition (OSC). If $\dim_H F < 1$ we prove that all generating IFSs of the set must have logarithmically commensurable contraction factors. From this Logarithmic Commensurability Theorem we derive a structure theorem for the semi-group of generating IFSs of F under the OSC. We also examine the impact of geometry on the structures of the semi-groups. Several examples will be given to illustrate the difficulty of the problem we study.

1. Introduction

In this paper, a family of contractive affine maps $\Phi = \{\phi_j\}_{j=1}^N$ in $\mathbb{R}^d$ is called an *iterated function system* (IFS). According to Hutchinson [12], there is a unique non-empty compact $F = F_\Phi \subset \mathbb{R}^d$, which is called the *attractor of* Φ, such that $F = \bigcup_{j=1}^N \phi_j(F)$. Furthermore, F_Φ is called a *self-similar set* if Φ consists of similitudes.

It is well known that the standard middle-third Cantor set C is the attractor of the iterated function system (IFS) $\{\phi_0, \phi_1\}$ where

$$\phi_0(x) = \frac{1}{3}x, \qquad \phi_1(x) = \frac{1}{3}x + \frac{2}{3}. \tag{1.1}$$

A natural question is: Is it possible to express C as the attractor of another IFS?

Surprisingly, the general question whether the attractor of an IFS can be expressed as the attractor of another IFS, which seems a rather fundamental question in fractal geometry, has

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hardly been studied, even for some of the best known Cantor sets such as the middle-third Cantor set.

A closer look at this question reveals that it is not as straightforward as it may appear. It is easy to see that for any given IFS $\{\phi_j\}_{j=1}^N$ one can always iterate it to obtain another IFS with identical attractor. For example, the middle-third Cantor set C satisfies

$$\begin{aligned} C &= \phi_0(C) \cup \phi_1(C) \\ &= \phi_0 \circ \phi_0(C) \cup \phi_0 \circ \phi_1(C) \cup \phi_1(C) \\ &= \phi_0 \circ \phi_0(C) \cup \phi_0 \circ \phi_1(C) \cup \phi_1 \circ \phi_0(C) \cup \phi_1 \circ \phi_1(C). \end{aligned}$$

Hence C is also the attractor of the IFS $\{\phi_0 \circ \phi_0, \phi_0 \circ \phi_1, \phi_1\}$ and the IFS $\{\phi_0 \circ \phi_0, \phi_0 \circ \phi_1, \phi_1 \circ \phi_0, \phi_1 \circ \phi_1\}$, as well as infinitely many other iterations of the original IFS $\{\phi_0, \phi_1\}$. The complexity doesn't just stop here. Since C is centrally symmetric, $C = -C + 1$, we also have

$$C = \Big(-\frac{1}{3}C + \frac{1}{3}\Big) \cup \Big(-\frac{1}{3}C + 1\Big).$$

Thus C is also the attractor of the IFS $\{-\frac{1}{3}x + \frac{1}{3}, -\frac{1}{3}x + 1\}$, or even $\{-\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}x + \frac{2}{3}\}$.

Definition 1.1. Let $\Phi = \{\phi_i\}_{i=1}^N$ and $\Psi = \{\psi_j\}_{j=1}^M$ be two IFSs. We say that Ψ is *derived* from Φ if for each $1 \le j \le M$, $\psi_j = \phi_{i_1} \circ \cdots \circ \phi_{i_k}$ for some $1 \le i_1, \dots, i_k \le N$. We say that Ψ is an *iteration* of Φ if Ψ is derived from Φ and $F_\Phi = F_\Psi$, where F_Φ and F_Ψ denote the attractors of Φ and Ψ, respectively.

We point out that the multiplicities in an IFS are not counted in our study. An IFS Φ, after an iteration, may contain redundant maps. For example, let $\Phi = \{\phi_0, \phi_1, \phi_2\}$ where $\phi_i = \frac{1}{2}(x+i)$. Then in $\{\phi_i \circ \phi_j : \ 0 \le i, j \le 2\}$ both $\frac{1}{4}(x+2)$ and $\frac{1}{4}(x+4)$ appear twice. After removing redundancies we have $\Psi = \{\frac{1}{4}(x+j) : 0 \le j \le 6\}$ as an iteration of Φ.

Definition 1.2. Let F be a compact set in $\mathbb{R}^d$. A *generating IFS of* F is an IFS Φ whose attractor is F. A *generating IFS family of* F is a set $\mathcal{I}$ of generating IFSs of F. A generating IFS family $\mathcal{I}$ of F is said to have a *minimal element* $\Phi_0 \in \mathcal{I}$ if every $\Psi \in \mathcal{I}$ is an iteration of Φ_0.

The objective of this paper is to study the existence of a minimal IFS in a generating IFS family of a self-similar set $F \subset \mathbb{R}$. We have already pointed out the complexity of this problem even for the middle-third Cantor set. Naturally, one cannot expect the existence of a minimal IFS in a generating IFS family $\mathcal{I}$ of a set F to be the general rule — not without first imposing restrictions on $\mathcal{I}$ and F. But what are these restrictions? A basic restriction is the *open set condition* (OSC) [12]. Recall that an IFS $\Phi = \{\phi_i\}_{i=1}^N$ in $\mathbb{R}^d$ is said to satisfy the OSC if there exists a nonempty open set $V \subset \mathbb{R}^d$ such that $\phi_i(V)$, $i = 1, \ldots, N$, are disjoint subsets of V. Without the OSC either the existence of a minimal IFS is hopeless, or the problem appears rather intractable. But even with the OSC a compact set may have generating IFSs that superficially seem to bear little relation to one another. One such example is the unit interval $F = [0,1]$. For each integer $N \geq 2$ the IFS $\Phi_N = \{\frac{1}{N}(x+j) : \ 0 \leq j < N\}$ is a generating IFS for F satisfying the OSC, and for $N_2 > N_1$ that is not a power of N_1, Φ_{N_2} is not an iteration of Φ_{N_1}. It is evident that other restrictions will be needed. We study this issue in this paper.

While the questions we study in the paper appear to be rather fundamental questions of fractal geometry in themselves, our study is also motivated by several questions in related areas. One of the well known questions in tiling is whether there exists a 2-reptile that is also a 3-reptile in the plane ([5]). A compact set T with $T = \overline{T^o}$ is called a *k-reptile* if there exists a measure disjoint partition $T = \bigcup_{j=1}^k T_j$ of T such that each T_j is similar to T and all T_j are congruent. Suppose that $T_j = \phi_j(T)$ for some similarity ϕ_j. Then T is the attractor of the IFS $\{\phi_j\}_{j=1}^k$. So this question, or more generally whether an m-reptile can also be an n-reptile, is a special case of the questions we study here.

Another motivation comes from the application of fractal geometry to image compression, see Barnsley [2] or Lu [15]. The basic premise of fractal image compression is that a digital image can be partitioned into pieces in which each piece is the attractor of an affine IFS. So finding a generating IFS of a given set plays the central role in this application. Naturally, better compressions are achieved by choosing a minimal generating IFS for each piece if possible, see also Deliu, Geronimo and Shonkwiler [6].

Although not directly related, there are two other questions that have also motivated our study. One is a question raised by Mattila: Is there a non-trivial self-similar subset F of the middle-third Cantor set C in the sense that F has a generating IFS that is not derived from the generating IFS $\{\phi_0, \phi_1\}$ of C given in (1.1)? We shall give a positive answer in §6. The other question concerns the symmetry of a self-similar set such as the Sierpinski Gasket, see e.g. [4], [10] and [23]. We have already seen from the middle-third Cantor set that symmetry complicates the study of existence of minimal IFSs. How the two questions relate is perhaps a problem worth further exploiting.

For any IFS Φ we shall use F_Φ to denote its attractor. We call an IFS $\Phi = \{\rho_j x + a_j\}_{j=1}^N$ *homogeneous* if all contraction factors ρ_j are identical. In this case we use ρ_Φ to denote the homogeneous contraction factor. We call Φ *positive* if all $\rho_j > 0$. A fundamental result concerning the structures of generating IFSs of a self-similar set is the Logarithmic Commensurability Theorem stated below. It is the foundation of many of our results in this paper.

Theorem 1.1 (**The Logarithmic Commensurability Theorem**). *Let F be the attractor of a homogeneous IFS $\Phi = \{\phi_i\}_{i=1}^N$ in $\mathbb{R}$ satisfying the OSC.*

(i) *Suppose that $\dim_H F = s < 1$. Let $\psi(x) = \lambda x + d$, $\lambda \neq 0$, such that $\psi(F) \subseteq F$. Then $\log|\lambda|/\log|\rho_\Phi| \in \mathbb{Q}$.*
(ii) *Suppose that $\dim_H F = 1$ and F is not a finite union of intervals. Let $\psi(x) = \lambda x + d$, $\lambda > 0$, such that $\psi(F) \subseteq F$ and $\min(F) \in \psi(F)$. Then $\log\lambda/\log|\rho_\Phi| \in \mathbb{Q}$.*

An immediate corollary of the above theorem is:

Corollary 1.2. *Let F be the attractor of a homogeneous IFS $\Phi = \{\phi_i\}_{i=1}^N$ satisfying the OSC. Suppose that $\Psi = \{\psi_j(x) = \lambda_j x + b_j\}_{j=1}^M$ is another generating IFS of F.*

(i) *If $\dim_H F = s < 1$, then $\log|\lambda_j|/\log|\rho_\Phi| \in \mathbb{Q}$ for all $1 \leq j \leq M$.*
(ii) *If $\dim_H F = 1$ and F is not a finite union of intervals, and if Ψ is homogeneous, then $\log|\rho_\Psi|/\log|\rho_\Phi| \in \mathbb{Q}$.*

Note that the set of all homogeneous generating IFSs of a self-similar set F forms a semigroup. Let $\Phi = \{\phi_i\}_{i=1}^N$ and $\Psi = \{\psi_j\}_{j=1}^M$ be two generating IFSs of F. We may define $\Phi \circ \Psi$ by $\Phi \circ \Psi = \{\phi_i \circ \psi_j : 1 \leq i \leq N, 1 \leq j \leq M\}$. Then clearly $\Phi \circ \Psi$ is also a generating IFS of F.

Definition 1.3. *Let F be any compact set in $\mathbb{R}$. We shall use $\mathcal{I}_F$ to denote the set of all homogeneous generating IFSs of F satisfying the OSC, augmented by the "identity" $Id = \{id(x) := x\}$. We shall use $\mathcal{I}_F^+$ to denote the set of all positive homogeneous generating IFSs of F satisfying the OSC, augmented by the identity Id.*

Clearly both $\mathcal{I}_F$ and $\mathcal{I}_F^+$, equipped with the composition as product, are semi-groups. If F is not the attractor of a homogeneous IFS with OSC then $\mathcal{I}_F$ is trivial. The Logarithmic Commensurabilty Theorem leads to the following structure theorem for $\mathcal{I}_F$ and $\mathcal{I}_F^+$:

Theorem 1.3. *Let F be a compact set in $\mathbb{R}$ that is not a finite union of intervals. Then $\mathcal{I}_F$ is Abelian. Let $\Phi = \{\phi_i\}_{i=1}^N \in \mathcal{I}_F$, $N > 1$.*

(i) *Both $\mathcal{I}_F$ and $\mathcal{I}_F^+$ are finitely generated semi-groups.*
(ii) *Suppose that N is not a power of another integer. If $\rho_\Phi > 0$ then Φ is the minimal element for $\mathcal{I}_F^+$, namely $\mathcal{I}_F^+ =<\Phi>:= \{\Phi^m : m \geq 0\}$. If $\rho_\Phi < 0$ then either $\mathcal{I}_F^+$ has a minimal element, or $\mathcal{I}_F^+ =<\Phi^2, \Psi>$ for some Ψ with $\rho_\Psi = \rho_\Phi^q$ where $q \in \mathbb{N}$ is odd and $\Psi^2 = \Phi^{2q}$.*
(iii) *Suppose that N is not a power of another integer. Then either $\mathcal{I}_F =<\Phi>$ or $\mathcal{I}_F =<\Phi, \Psi>$ for some Ψ with $\rho_\Psi = -\rho_\Phi^q$ where $q \in \mathbb{N}$ and $\Psi^2 = \Phi^{2q}$.*

Due to the technical nature of the proof of the Logarithmic Commensurability Theorem we shall postpone it until §5. Theorem 1.3 establishes the structures of $\mathcal{I}_F$ and $\mathcal{I}_F^+$ purely on algebraic grounds. However, the structures of the semi-groups are also dictated by the geometric structures of F. We shall exploit the impact of geometry on the structures of the semi-groups in §2. In §3 we further study the structures of the semi-groups under the convex open set condition. In §4 we study the existence of minimal IFSs for IFS families with non-homogeneous contraction factors. Geometry plays a considerably bigger role in this setting. In §5 we prove the Logarithmic Commensurability Theorem, along with other related results. Finally in §6 we present various counterexamples, including an example to Mattila's question.

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2. Structures of the Semi-groups

In this section we prove Theorem 1.3, and examine the impact of geometry to the structures of the semi-groups $\mathcal{I}_F$ and $\mathcal{I}_F^+$. Although the proof of Theorem 1.1 and Corollary 1.2

will be given later in §5, their proofs do not depend on the results in this section. Hence we shall assume their validity in this section and use them to prove our results.

Proposition 2.1. *Let Φ and Ψ be two homogeneous IFSs in $\mathbb{R}$ satisfying the OSC. If $\rho_\Phi = \rho_\Psi$ and $F_\Phi = F_\Psi$, then $\Phi = \Psi$.*

Proof. Let $\Phi = \{\phi_i(x) := \rho x + a_i\}_{i=1}^N$ and $\Psi = \{\psi_j(x) := \rho x + b_j\}_{j=1}^M$. Denote $F = F_\Phi = F_\Psi$. To see $N = M$ observe that by the OSC of Φ and Ψ we have $\dim_H F = \frac{\log N}{-\log|\rho|} = \frac{\log M}{-\log|\rho|}$. It follows that $N = M$.

Let ν_F be the normalized s-dimensional Hausdorff measure restricted to F, where $s = \dim_H F$, i.e. $\nu_F = \frac{1}{\mathcal{H}^s(F)}\mathcal{H}^s$. We assert that ν_F is the self-similar measure defined by Φ with equal weights, i.e.

$$\nu_F = \frac{1}{N}\sum_{j=1}^N \nu_F \circ \phi_j^{-1}.$$

For any $E \subset F$ let $E_i = \phi_i(F) \cap E$. Note that $\phi_j^{-1}(E_i) \cap F \subseteq \phi_j^{-1}(\phi_i(F) \cap \phi_j(F))$. The OSC now implies $\nu_F(\phi_j^{-1}(E_i)) = 0$, as well as $\nu_F(E_i \cap E_j) = 0$ for any $i \neq j$. Therefore

$$\nu_F \circ \phi_j^{-1}(E) = \nu_F(\phi_j^{-1}(E_j)) = \rho^{-s}\nu_F(E_j) = N\nu_F(E_j).$$

It follows that

$$\frac{1}{N}\sum_{j=1}^N \nu_F \circ \phi_j^{-1}(E) = \sum_{j=1}^N \nu_F(E_j) = \nu_F(E).$$

This proves the assertion. Similarly, ν_F is also the self-similar measure defined by the IFS Ψ with equal weights.

Now taking the Fourier transform of ν_F and applying the self-similarity yield

$$\widehat{\nu}_F(\xi) = A(\rho\xi)\widehat{\nu}_F(\rho\xi) = B(\rho\xi)\widehat{\nu}_F(\rho\xi)$$

where $A(\xi) := \frac{1}{N}\sum_{j=1}^N e^{-2\pi i a_j \xi}$ and $B(\xi) := \frac{1}{M}\sum_{j=1}^M e^{-2\pi i b_j \xi}$. Observe that A, B and $\widehat{\nu}_F$ are real analytic, not identically zero on $\mathbb{R}$. Let $V \subset \mathbb{R}$ be a non-empty open set so that $\widehat{\nu}_F(\xi) \neq 0$ for any $\xi \in V$. Then $A(\xi) = B(\xi)$ for $\xi \in V$, which implies $A = B$ on $\mathbb{R}$. Hence we have $N = M$ and $\{a_j\} = \{b_j\}$, proving the proposition. $\square$

Proposition 2.2. *Let Φ and Ψ be two homogeneous IFSs satisfying the OSC. Then $F_\Phi = F_\Psi$ if and only if $\Phi \circ \Psi = \Psi \circ \Phi$.*

Proof. Suppose that $F_\Phi = F_\Psi$ then both $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are generating IFSs of F with identical contraction factors, and both satisfy the OSC. Hence $\Phi \circ \Psi = \Psi \circ \Phi$ by Proposition 2.1.

Conversely, if $\Phi \circ \Psi = \Psi \circ \Phi$ then $\Phi \circ \Psi^m = \Psi^m \circ \Phi$ for any $m \in \mathbb{N}$. Therefore $\Phi \circ \Psi^m(F_\Psi) = \Psi^m \circ \Phi(F_\Psi)$. But $\Psi^m(E) \longrightarrow F_\Psi$ as $m \longrightarrow \infty$ in the Hausdorff metric for any compact set E. Taking limit we obtain $\Phi(F_\Psi) = F_\Psi$. Therefore $F_\Phi = F_\Psi$. $\square$

Proof of Theorem 1.3. By Proposition 2.2, $\mathcal{I}_F$ is Abelian. Let k be the largest integer such that $N = L^k$ for some $L \in \mathbb{N}$. Suppose that $\Psi = \{\psi_j\}_{j=1}^M \in \mathcal{I}_F$ and $\Psi \neq Id$. By Corollary 1.2, $\log|\rho_\Psi|/\log|\rho_\Phi| \in \mathbb{Q}$. Then the dimension formula $N^{-s} = |\rho_\Phi|$ and $M^{-s} = |\rho_\Psi|$ where $s = \dim_H F$ implies that $\log M/\log N \in \mathbb{Q}$. It follows that $M = L^m$ and $\rho_\Psi = \pm|\rho_\Phi|^{\frac{m}{k}}$ for some $m \in \mathbb{N}$.

We first prove (ii). By assumption $N = L$. Let $\Psi = \{\psi_j\}_{j=1}^M \in \mathcal{I}_F^+$. Then $M = N^m$ for some m, which implies that $\rho_\Psi = |\rho_\Phi|^m$. If $\rho_\Phi > 0$ then $\Psi = \Phi^m$ via Proposition 2.1 because they have the same contraction factor. Thus $\mathcal{I}_F^+ =< \Phi >$. Suppose that $\rho_\Phi < 0$. We have two cases: Either every $\Psi \in \mathcal{I}_F^+$ has $\rho_\Psi = |\rho_\Phi|^{2m'}$ for some m', or there exists a $\Psi \in \mathcal{I}_F^+$ with $\rho_\Psi = |\rho_\Phi|^m$ for some odd m. In the first case every $\Psi \in \mathcal{I}_F^+$ has $\Psi = (\Phi^2)^m$ again by Proposition 2.1. Hence $\mathcal{I}_F^+ =< \Phi^2 >$. In the second case, let q be the smallest odd integer such that $\rho_{\Psi_0} = |\rho_\Phi|^q$ for some $\Psi_0 \in \mathcal{I}_F^+$. For any $\Psi \in \mathcal{I}_F^+$ we have $\rho_\Psi = |\rho_\Phi|^m$. If $m = 2m'$ then $\Psi = (\Phi^2)^{m'}$. If m is odd then $m \geq q$ and $m - q = 2m'$. Thus $\rho_\Psi = \rho_{\Phi^{2m'} \circ \Psi_0}$, and hence $\Psi = \Phi^{2m'} \circ \Psi_0$. It follows that $\mathcal{I}_F^+ =< \Phi^2, \Psi_0 >$ with $\Psi_0^2 = (\Phi^2)^q$. This proves (ii).

We next prove (iii), which is rather similar to (ii). Again, any $\Psi \in \mathcal{I}_F$ must have $\rho_\Psi = \pm|\rho_\Phi|^m$ for some m. If $\mathcal{I}_F =< \Phi >$ we are done. Otherwise there exists a $\Psi_0 \in \mathcal{I}_F$ such that $\Psi_0 \notin < \Phi >$ and it has the largest contraction factor in absolute value. Since $\rho_{\Psi_0} = \pm|\rho_\Phi|^q$ for some q, and $\Psi_0 \neq \Phi^q$, we must have $\rho_{\Psi_0} = -\rho_\Phi^q$. We show that $\mathcal{I}_F =< \Phi, \Psi_0 >$. For any $\Psi \in \mathcal{I}_F$ either $\Psi = \Phi^m$ for some m or $\rho_\Psi = -\rho_\Phi^m$. In the latter case $m \geq q$. So $\rho_\Psi = \rho_{\Phi^{m-q} \circ \Psi_0}$, implying that $\Psi = \Phi^{m-q} \circ \Psi_0$ by Proposition 2.1. Also it is clear $\Psi_0^2 = \Phi^{2q}$ because they have the same contraction factor. We have proved (iii).

Finally we prove (i). We have already seen that $\rho_\Psi = \pm|\rho_\Phi|^{\frac{m}{k}}$ for some $m \in \mathbb{N}$ for any $\Psi \in \mathcal{I}_F$. Set $\rho = |\rho_\Phi|^{\frac{1}{k}}$. Then $\rho_\Psi = \pm\rho^m$.

Define $\mathcal{P}^+ = \{m : \ \rho^m = \rho_\Psi$ for some $\Psi \in \mathcal{I}_F\}$ and $\mathcal{P}^- = \{m : \ \rho^m = -\rho_\Psi$ for some $\Psi \in \mathcal{I}_F\}$. We will show that $\mathcal{I}_F^+$ is finitely generated. Set $a = \gcd(\mathcal{P}^+)$. Let $\Psi_1, \dots, \Psi_n \in \mathcal{I}_F$ with $\rho_{\Psi_j} = \rho^{m_j}$ such that $\gcd(m_1, m_2, \dots, m_n) = a$. By a standard result in elementary number theory every sufficiently large integer $ma \geq N_0$ can be expressed as $ma = \sum_{j=1}^n p_j m_j$ with $p_j \geq 0$. Thus every $\Psi \in \mathcal{I}_F^+$ with $\rho_\Psi = \rho^{ma}$, $ma \geq N_0$, can be expressed as $\Psi = \Pi_{j=1}^n \Psi_j^{p_j}$ since the two IFSs have the same contraction factor. Let $\{\Psi_{n+1}, \dots, \Psi_K\} \subseteq \mathcal{I}_F^+$ consist of all elements $\Psi \in \mathcal{I}_F^+$ with $\rho_\Psi \geq \rho^{N_0}$ that are not already in $\{\Psi_1, \dots, \Psi_n\}$. Then $\mathcal{I}_F^+ =< \Psi_1, \Psi_2, \dots, \Psi_K >$, and it is finitely generated.

The proof that $\mathcal{I}_F$ is finitely generated is virtually identical, and we omit it. □

3. The Convex Open Set Condition

In this section, we study the attractors of homogeneous IFSs satisfying the convex open set condition.

Definition 3.1. Let $\Phi = \{\phi_j\}_{j=1}^N$ be an IFS in $\mathbb{R}$. We say Φ satisfies the *separation condition* (SC) if $\phi_i(F_\Phi) \cap \phi_j(F_\Phi) = \emptyset$ for all $i \neq j$. We say Φ satisfies the *convex open set condition* (COSC) if Φ satisfies the OSC with a convex open set.

The following is another main theorem in this paper:

Theorem 3.1. *Let $F \subset \mathbb{R}$ be a compact set that is not a finite union of intervals such that F is the attractor of a homogeneous IFS satisfying the COSC. We have:*

- (i) *The semi-group $\mathcal{I}_F^+$ has a minimal element Φ_0, namely $\mathcal{I}_F^+ =< \Phi_0 >$.*
- (ii) *Suppose that F is not symmetric. Then $\mathcal{I}_F$ has a minimal element Φ_0, $\mathcal{I}_F =< \Phi_0 >$.*
- (iii) *Suppose that F is symmetric. Then there exist Φ_+ and Φ_- in $\mathcal{I}_F$ with $\rho_{\Phi_+} = -\rho_{\Phi_-} > 0$ such that every $\Psi \in \mathcal{I}_F$ can be expressed as $\Psi = \Phi_+^m$ if $\rho_\Psi > 0$ and $\Psi = \Phi_+^m \circ \Phi_-$ if $\rho_\Psi < 0$ for some $m \in \mathbb{N}$.*

We shall first prove several results leading up to our main theorem.

Lemma 3.2. *Let $\Phi = \{\phi_j\}$ be an IFS in $\mathbb{R}$. Then Φ satisfies the COSC if and only if for all $i \neq j$ we have $\phi_i(x) \leq \phi_j(y)$ for all $x, y \in F_\Phi$ or $\phi_i(x) \geq \phi_j(y)$ for all $x, y \in F_\Phi$.*

Proof. Suppose that Φ satisfies the COSC. Then the convex open set for the OSC must be an interval $U = (a, b)$. Since $\phi_i(U) \cap \phi_j(U) = \emptyset$ for all $i \neq j$, and noting that $F_\Phi \subseteq \overline{U}$, we immediately know that $\phi_i(F)$ must lie entirely on one side of $\phi_j(F)$.

Conversely, suppose that $\phi_i(F)$ lies entirely on one side of $\phi_j(F)$, $i \neq j$. Let U be the interior of the convex hull of F, which is an interval. Then $\phi_i(U) \cap \phi_j(U) = \emptyset$, and clearly $\phi_i(F) \subset F$ implies that $\phi_i(U) \subset U$. Hence Φ satisfies the COSC. $\square$

Lemma 3.3. *Under the assumption of Theorem 3.1, let Φ be any generating homogeneous IFS of F with the OSC. Then Φ also satisfies the COSC.*

Proof. Let Ψ be a generating homogeneous IFS of F with the COSC. By Corollary 1.2 and Proposition 2.1, there exist integers m, n such that $\rho_\Phi^m = \rho_\Psi^n$. It follows from Lemma 2.1 that $\Phi^m = \Psi^n$. Assume that Φ does not satisfy the COSC. Then there exist $\phi_i, \phi_j \in \Phi$ so that $\phi_i(x) < \phi_j(y)$ and $\phi_i(z) > \phi_j(w)$ for some $x, y, z, w \in F$. The same or the opposite inequalities will hold if we replace ϕ_i, ϕ_j by $\phi_1^{m-1} \circ \phi_i$ and $\phi_1^{m-1} \circ \phi_j$, respectively. But this is impossible because both $\phi_1^{m-1} \circ \phi_i$ and $\phi_1^{m-1} \circ \phi_j$ are in Φ^m, and hence in Ψ^n, which satisfies the COSC. $\square$

The following lemma is also needed in the proof of the theorem.

Lemma 3.4. *Let Φ and Ψ be two homogeneous IFSs such that $\rho_\Phi = -\rho_\Psi$ and $F_\Phi = F_\Psi$. Assume that Φ satisfies the COSC and Ψ satisfies the OSC. Then F_Φ must be symmetric.*

Proof. Let $\Phi = \{\phi_i(x) := \rho x + a_i\}_{i=1}^N$ and $\Psi = \{\psi_j(x) := -\rho x + b_j\}_{j=1}^M$. Ψ also satisfies the COSC by Lemma 3.3. Without loss of generality we assume that $\rho > 0$ and $a_1 < a_2 < \cdots < a_N$, $b_1 < b_2 < \cdots < b_M$. Denote $\mathcal{A} = \{a_i\}$ and $\mathcal{B} = \{b_j\}$. By Proposition 2.1, the OSC for Φ and Ψ as well as $\rho_{\Phi^2} = \rho_{\Psi^2}$ imply $\Phi^2 = \Psi^2$. Observe that

$$\Phi^2 = \{\rho^2 x + a_i + \rho a_j\}_{i,j=1}^N, \qquad \Psi^2 = \{\rho^2 x + b_i - \rho b_j\}_{i,j=1}^M.$$

It follows from the COSC for Φ^2 that the lexicographical order for $\{a_i + \rho a_j\}_{i,j=1}^N$ also yields a strictly increasing order for the set. Similarly, the lexicographical order for $\{b_i - \rho b_{M+1-j}\}_{i,j=1}^M$ also yields a strictly increasing order for the set. Therefore $M = N$ and $a_i + \rho a_j = b_i - \rho b_{N+1-j}$ for all i, j. Fix $j = 1$ yields $a_i = b_i + c$ for some constant c. Fix $i = 1$ yields $a_j = -b_{N+1-j} + c'$ for some constant c'. Thus $a_j = a_{N+1-j} + c''$ for some constant c''. Hence $\mathcal{A}$ is symmetric, which implies that F_Φ is symmetric. $\square$

Proof of Theorem 3.1. We first give the following claim.

Claim. Let Φ, Ψ be any two elements in $\mathcal{I}_F$ with $|\rho_\Phi| > |\rho_\Psi|$. Then there exists a $\Gamma \in \mathcal{I}_F$ such that $\Psi = \Phi \circ \Gamma$, where $\Phi \circ \Gamma := \{\phi \circ \gamma : \phi \in \Phi, \gamma \in \Gamma\}$.

Proof of Claim. Let $\Phi = \{\phi_i(x)\}_{i=1}^N$ and $\Psi = \{\psi_j(x)\}_{j=1}^M$. Since both Φ and Ψ satisfy the COSC, we may without loss of generality assume that $\phi_1(F) \le \cdots \le \phi_M(F)$ and $\psi_1(F) \le \cdots \le \psi_N(F)$, where $X \le Y$ for two sets X and Y means $x \le y$ for all $x \in X$ and $y \in Y$. Set $a = \min F$, $b = \max F$ and $F_0 = [a, b]$. Clearly each $\phi_i(F_0)$ (resp. $\psi_j(F_0)$) is a sub-interval of F_0, with end points $\phi_i(a)$ and $\phi_i(b)$ (resp. $\psi_i(a)$ and $\psi_i(b)$). The COSC for Φ and Ψ now imply that $\phi_1(F_0) \le \cdots \le \phi_M(F_0)$ and $\psi_1(F_0) \le \cdots \le \psi_N(F_0)$.

It follows from Corollary 1.2 that $\frac{\log|\rho_\Phi|}{\log|\rho_\Psi|} = \frac{n}{m}$ for some positive integers m and n with $\gcd(m, n) = 1$. Thus $N^m = M^n$, or $N = M^{\frac{n}{m}}$, by $\dim_H F = \frac{\log N}{-\log|\rho_\Phi|} = \frac{\log M}{-\log|\rho_\Psi|}$. This forces $K = M^{\frac{1}{m}}$ to be an integer, for otherwise the co-primeness of n, m makes $N = M^{\frac{n}{m}}$ an irrational number. Therefore $M = K^m$ and $N = K^n$. In particular, $\frac{M}{N} = L \in \mathbb{Z}$.

Now $\Phi^q = \Psi^r$ by Proposition 2.1, where $q = 2m$ and $r = 2n$. For each $\mathbf{i} = i_1 i_2 \cdots i_q \in \{1, \ldots, N\}^q$ denote $\phi_\mathbf{i} := \phi_{i_1} \circ \cdots \circ \phi_{i_q}$, and similarly define $\psi_\mathbf{j}$ for $\mathbf{j} \in \{1, \ldots, M\}^r$. Then $\Phi^q = \{\phi_\mathbf{i} : \mathbf{i} \in \{1, \ldots, N\}^q\}$ and $\Psi^r = \{\psi_\mathbf{i} : \mathbf{i} \in \{1, \ldots, M\}^r\}$. It is clear that both Φ^q and Ψ^r satisfy the COSC. We rank the maps in Φ^q in the increasing order of $\phi_\mathbf{i}(F_0)$, and rank the maps in Ψ^r in the increasing order of $\psi_\mathbf{j}(F_0)$ respectively. Then the first N^{q-1} maps in $(\phi_\mathbf{i} : \mathbf{i} \in \{1, \ldots, N\}^q)$ are $\mathcal{J}_1 = \{\phi_{1\mathbf{i}'} : \mathbf{i}' \in \{1, \ldots, N\}^{q-1}\}$, while the first N^{q-1} maps in $(\psi_\mathbf{j} : \mathbf{j} \in \{1, \ldots, M\}^r)$ are $\mathcal{J}_2 = \{\psi_{j_1\mathbf{j}'} : 1 \le j_1 \le L,\ \mathbf{j}' \in \{1, \ldots, M\}^{r-1}\}$. Therefore $\mathcal{J}_1 = \mathcal{J}_2$. Note that

$$\bigcup_{\varphi \in \mathcal{J}_1} \varphi(F) = \phi_1(F), \qquad \bigcup_{\varphi \in \mathcal{J}_2} \varphi(F) = \bigcup_{j=1}^{L} \psi_j(F).$$

It follows that $F = \bigcup_{j=1}^L \phi_1^{-1} \circ \psi_j(F)$, so $\Gamma_1 := \{\phi_1^{-1} \circ \psi_j\}_{j=1}^L$ is a generating IFS for F. It clearly satisfies the COSC.

We can continue the same argument by counting the next N^{q-1} elements in the two sequences. This yields $F = \bigcup_{j=L+1}^{2L} \phi_2^{-1} \circ \psi_j(F)$, so $\Gamma_2 := \{\phi_2^{-1} \circ \psi_j\}_{j=L+1}^{2L}$ is a generating IFS for F. Continue to the end yields $\Gamma_1, \ldots, \Gamma_N$ in $\mathcal{I}_F$, with the property that

$$\{\psi_j : (k-1)L + 1 \le j \le kL\} = \{\phi_k \circ \varphi : \varphi \in \Gamma_k\}. \tag{3.1}$$

But by Proposition 2.1, all Γ_k are equal because they have the same contraction factor. It follows from (3.1) that $\Psi = \Phi \circ \Gamma$, with $\Gamma := \Gamma_k$. This proves the Claim. $\square$

To prove part (i) of the theorem, let $\Phi_0 \in \mathcal{I}_F^+$ have the largest contraction factor. Such a Φ_0 exists because for any $\Phi = \{\phi_i\}_{i=1}^N \in \mathcal{I}_F^+$ we must have $\rho_\Phi = N^{-\dim_H(F)}$. Now by Proposition 2.1, for any $\Phi \neq \Phi_0$ in $\mathcal{I}_F^+$ we have $\rho_\Phi < \rho_{\Phi_0}$. By the Claim $\Phi = \Phi_0 \circ \Gamma_1$ for some $\Gamma_1 \in \mathcal{I}_F^+$. If $\Gamma_1 = \Phi_0$ then $\Phi = \Phi_0^2$, and we finish the proof. If not then $\rho_{\Gamma_1} < \rho_{\Phi_0}$, yielding $\Gamma_1 = \Phi_0 \circ \Gamma_2$ for some $\Gamma_2 \in \mathcal{I}_F^+$. Apply the Claim recursively, and the process will eventually terminate since $|\rho_\Phi| > 0$. Hence $\Phi = \Phi_0^k$ for some k. The proof of part (i) is now complete.

To prove part (ii) of the theorem, if $\mathcal{I}_F = \mathcal{I}_F^+$ then there is nothing we need to prove. Assume that $\mathcal{I}_F \neq \mathcal{I}_F^+$. Let $\mathcal{I}_F^- \subset \mathcal{I}_F$ consisting of all homogeneous IFSs with negative contraction factors, and $\Phi_- \in \mathcal{I}_F^-$ have the largest contraction factor in absolute value. Let $\Phi_+ \in \mathcal{I}_F^+$ have the largest contraction factor in $\mathcal{I}_F^+$. If $|\rho_{\Phi_-}| = \rho_{\Phi_+}$ then F is symmetric by Lemma 3.4, a contradiction. So $|\rho_{\Phi_-}| \neq \rho_{\Phi_+}$. Note that $\Phi_-^2 = \Phi_+^m$ for some m by part (i). Thus $m = 1$ or $m > 2$. If $m > 2$ then $\rho_{\Phi_+} > |\rho_{\Phi_-}|$. Following the Claim we have $\Phi_- = \Phi_+ \circ \Gamma$ for some $\Gamma \in \mathcal{I}_F$. But $\rho_\Gamma < 0$ and $|\rho_\Gamma| > |\rho_{\Phi_-}|$. This is a contradiction. Therefore $m = 1$ and $\Phi_-^2 = \Phi_+$. Part (ii) of the theorem follows from part (i) and the Claim.

Finally we prove (iii). If F is symmetric, then for any IFS $\Psi \in \mathcal{I}_F$ there is another $\Psi' \in \mathcal{I}_F$ such that $\rho_\Psi = -\rho_{\Psi'}$ because $F = -F + c$ for some c. Let Φ_+ and Φ_- be the elements in $\mathcal{I}_F$ whose contraction factors have the largest absolute values, $\rho_{\Phi_+} = -\rho_{\Phi_-} > 0$. Proposition 2.1 and the same argument to prove part (i) now easily apply to prove that for any $\Psi \in \mathcal{I}_F$, $\Psi = \Phi_+^m$ if $\rho_\Psi > 0$ and $\Psi = \Phi_+^m \circ \Phi_-$ if $\rho_\Psi < 0$ for some $m \in \mathbb{N}$. This finishes the proof of Theorem 3.1. $\square$

The COSC in Theorem 3.1 cannot be replaced by the condition SC. We give a counterexample in §6.

4. Non-homogeneous IFS

When we do not require that the contraction factors be homogeneous, the main result in §2 no longer holds. In §6 we give a counterexample showing that the COSC no longer guarantees the existence of a minimal element. For the existence to hold we need stronger assumptions. The following is our main result:

Theorem 4.1. *Let $F \subset \mathbb{R}$ be a compact set such that F is the attractor of an IFS $\Phi = \{\rho_i x + c_i\}_{i=1}^N$ with the OSC. Assume that $\dim_H F = s < 1$ and $\mathcal{H}^s(F) = (b-a)^s$, where $a = \min F$ and $b = \max F$.*

(i) *Let $\mathcal{G}_F^+$ denote the set of all positive generating IFSs of F with the OSC. Then $\mathcal{G}_F^+$ contains a minimal element.*

(ii) *Let $\mathcal{G}_F$ denote the set of all generating IFSs of F with the OSC. Suppose that F is not symmetric. Then $\mathcal{G}_F$ contains a minimal element.*

(iii) *Suppose that F is symmetric. Then there exists a $\Phi_0 = \{\phi_i\}_{i=1}^M$ in $\mathcal{G}_F$ such that for any $\Psi \in \mathcal{G}_F$ and each $\psi \in \Psi$, there exist $i_1, \ldots, i_\ell \in \{1, \ldots, M\}$ so that*

$$\psi = \phi_{i_1} \circ \cdots \circ \phi_{i_\ell} \quad \text{or} \quad \overline{\psi} = \phi_{i_1} \circ \cdots \circ \phi_{i_\ell},$$

where $\overline{\psi}$ denote the unique affine map on $\mathbb{R}$ satisfying $\overline{\psi}(a) = \psi(b)$ and $\overline{\psi}(b) = \psi(a)$.

Remark 4.1. The condition $\mathcal{H}^s(F) = (b-a)^s$ is very strong. In general if F is the attractor of an IFS in $\mathbb{R}$ with the OSC, we always have $0 < \mathcal{H}^s(F) \le (b-a)^s$. Let $\mu = \mathcal{H}^s / \mathcal{H}^s(F)$. Then the condition $\mathcal{H}^s(F) = (b-a)^s$ is equivalent to

$$\mu([u,v]) \le \left(\frac{v-u}{b-a}\right)^s \qquad \text{for any interval } [u,v] \subset \mathbb{R}. \tag{4.1}$$

(See e.g. (1.3) and (1.5) in [1].) Let $\Phi = \{\phi_i\}_{i=1}^N \in \mathcal{G}_F^+$. Assume that $\phi_1(a) < \phi_2(a) < \ldots < \phi_N(a)$. Marion [18] showed that $\mathcal{H}^s(F) = (b-a)^s$ if and only if

$$\frac{\sum_{i=m}^n |\rho_i|^s}{|\phi_n(b) - \phi_m(a)|^s} \le (b-a)^{-s}, \qquad \forall\, 1 \le m < n \le N. \tag{4.2}$$

See [1, Theorem 4.2] for a shorter proof. We remark that (4.2) is an easily checkable condition. For any given $\rho_1, \ldots, \rho_N > 0$ with $\sum_{i=1}^N \rho_i < 1$, one can choose $c_1, \ldots, c_N$ such that (4.2) holds for the IFS $\{\rho_i x + c_i\}_{i=1}^N$. For instance, one may take $c_1 = 0$ and $c_j = \sum_{k=1}^{j-1} (\rho_k + \ell_k)$ for $2 \le j \le N$, where

$$\begin{cases} \ell_j = (\rho_j^s + \cdots + \rho_N)^{1/s} - (\rho_{j+1} + \cdots + \rho_N^s)^{1/s} - \rho_j \text{ for } 1 \le j < N-1, \\ \ell_{N-1} = (\rho_{N-1}^s + \rho_N^s)^{1/s} - \rho_{N-1} - \rho_N. \end{cases}$$

Furthermore (4.2) remains valid if we perturb the above c_j's slightly (see [1, Corollary 4.5.]). If we drop the condition $\mathcal{H}^s(F) = (b-a)^s$, then Theorem 4.1 is no longer true. We present a counterexample in §6.

Proof of Theorem 4.1. We first prove the following claim:

Claim. Let ψ_1, ψ_2 be any two contractive affine maps with $\psi_1(F) \subset F$ and $\psi_2(F) \subset F$. Then one of the following cases must happen:

(A) $\psi_1(F) \cap \psi_2(F) = \emptyset$;
(B) $\psi_1(F) \supseteq \psi_2(F)$;
(C) $\psi_2(F) \supseteq \psi_1(F)$.

Proof of Claim. Let ν_F denote the s-dimensional Hausdorff measure restricted to F. It follows from (4.1) that for all intervals $[u, v]$,

$$\nu_F([u, v]) \leq (v - u)^s. \tag{4.3}$$

Denote $[a_1, b_1] := \psi_1([a, b])$ and $[a_2, b_2] := \psi_2([a, b])$. There are at most 5 different possible scenarios for these two intervals:

(1) $[a_1, b_1] \cap [a_2, b_2] = \emptyset$;
(2) $[a_1, b_1] \supseteq [a_2, b_2]$;
(3) $[a_2, b_2] \supseteq [a_1, b_1]$;
(4) $a_1 < a_2 \leq b_1 < b_2$;
(5) $a_2 < a_1 \leq b_2 < b_1$.

We prove the claim by examining $\psi_1(F)$ and $\psi_2(F)$ in each of the above scenarios.

It is clear that with scenario (1) we have $\psi_1(F) \cap \psi_2(F) = \emptyset$. We show that $\psi_1(F) \supseteq \psi_2(F)$ with scenario (2) by contradiction. Assume it is not true. Then there exists an $x_0 \in F$ such that $\mathrm{dist}(\psi_2(x_0), \psi_1(F)) > 0$. This means there exists a small cylinder $E = \phi_{i_1} \circ \cdots \circ \phi_{i_n}(F)$ of the IFS Φ containing x_0 such that $\psi_2(E) \cap \psi_1(F) = \emptyset$. Note that by the scaling property of the measure ν_F we have $\nu_F(\psi_2(E)) > 0$. Hence

$$\nu_F([a_1, b_1]) \geq \nu_F(\psi_1(F) \cup \psi_2(F)) \geq \nu_F(\psi_1(F)) + \nu_F(\psi_2(E)) > \nu_F(\psi_1(F)).$$

But because $\nu_F(F) = (b - a)^s$ we also have $\nu_F(\psi_1(F)) = (b_1 - a_1)^s$ by the scaling property of ν_F and the fact that $\psi_1(F) \subseteq F$. Therefore $\nu_F([a_1, b_1]) > (b_1 - a_1)^s$, a contradiction to (4.3). Similarly $\psi_2(F) \supseteq \psi_1(F)$ with scenario (3).

Now we prove that scenarios (4) and (5) never occur. Assume this is false. Without loss of generality we assume that scenario (4) has occurred. Then

$$\begin{aligned}\nu_F([a_1,b_2]) &= \nu_F([a_1,b_1])+\nu_F([a_2,b_2])-\nu_F([a_2,b_1])\\ &= (b_1-a_1)^s+(b_2-a_2)^s-\nu_F([a_2,b_1])\\ &\geq (b_1-a_1)^s+(b_2-a_2)^s-(b_1-a_2)^s\\ &> (b_2-a_1)^s,\end{aligned}$$

a contradiction. Note that for the last inequality we have employed an easily checked fact

$$(x+y)^s+(y+z)^s-y^s>(x+y+z)^s$$

with $x=a_2-a_1>0$, $y=b_1-a_2\geq 0$ and $z=b_2-b_1>0$. So have completed the proof of the claim. □

Going back to the proof, suppose that $\Phi_0=\{\phi_i\}_{i=1}^M$ is an element in $\mathcal{G}_F$ (resp. $\mathcal{G}_F^+$) with the smallest integer M. By the claim Φ_0 satisfies the SC. To prove the theorem, it suffices to prove that if $\psi(x)=\rho x+b$ is an affine map (reps. $\rho>0$) satisfying $\psi(F)\subset F$, then

$$\psi(F)=\phi_{i_1}\circ\cdots\circ\phi_{i_\ell}(F)$$

for some indices $i_1,\dots,i_\ell\in\{1,\dots,M\}$.

First we assert that $\psi(F)\subseteq\phi_i(F)$ for some $i\in\{1,\dots,M\}$. To see this, denote

$$\Lambda=\{j:\ 1\leq j\leq M \text{ and } \psi(F)\cap\phi_j(F)\neq\emptyset\}.$$

We only need to show that Λ is a singleton. Assume it is not true. That is, $\#\Lambda\geq 2$. Then by the claim we have $\psi(F)\supseteq\bigcup_{j\in\Lambda}\phi_j(F)$, and thus $\psi(F)=\bigcup_{j\in\Lambda}\phi_j(F)$. It follows that $\{\psi\}\cup\{\phi_{j'}\}_{1\leq j'\leq M, j'\notin\Lambda}$ constitutes an IFS for F, which contradicts the minimality of M.

Now let ℓ be the largest integer such that

$$\psi(F)\subseteq\phi_{i_1}\circ\cdots\circ\phi_{i_\ell}(F)$$

for some indices $i_1,\dots,i_\ell$. We show that $\psi(F)=\phi_{i_1}\circ\cdots\circ\phi_{i_\ell}(F)$ as required. Denote $\hat{\psi}:=\phi_{i_\ell}^{-1}\circ\cdots\circ\phi_{i_1}^{-1}\circ\psi$. Then $\hat{\psi}(F)\subseteq F$. Assume that $\psi(F)\neq\phi_{i_1}\circ\cdots\circ\phi_{i_\ell}(F)$, that is, $\hat{\psi}(F)\neq F$. Then again $\hat{\psi}(F)\subseteq\phi_{i_{\ell+1}}(F)$ for some index $i_{\ell+1}$. Therefore $\psi(F)\subseteq\phi_{i_1}\circ\cdots\circ\phi_{i_{\ell+1}}(F)$, contradicting the maximality of ℓ.

Observe that by the scaling property of ν_F again, $\psi(F) = \phi_{i_1} \circ \cdots \circ \phi_{i_\ell}(F)$ implies that the two maps on both side of the equality must have the same contraction factor in absolute values. Therefore $\hat{\psi} = x+c$ or $-x+c$ for some c. If $\hat{\psi} = x+c$ then $\hat{\psi}(F) = F$ yields $c = 0$, so $\psi = \phi_{i_1} \circ \cdots \circ \phi_{i_\ell}$. In the case of $\mathcal{G}_F^+$ this is the only possibility. If $\hat{\psi} = -x+c$ then $\hat{\psi}(F) = F$ implies F is symmetric and $\hat{\psi} = \overline{\psi}$. The proof of the theorem is now complete. □

5. Logarithmic Commensurability of Contraction Factors

In this section we prove Theorem 1.1 and Corollary 1.2. The most difficult part of the proof by far is for part (i) of Theorem 1.1, which is rather tedious and technical, requiring delicate estimates and analysis. We first prove a stronger form of part (ii) of Theorem 1.1. A compact set F is said to satisfy the *no interval condition* if $F \not\supseteq [\min(F), \min(F)+\varepsilon]$ for any $\varepsilon > 0$.

Lemma 5.1. *Let $\Phi = \{\phi_i(x) := \rho_i x + c_i\}_{i=1}^N$ be an IFS in $\mathbb{R}$ with attractor F satisfying the no interval condition. Assume that $x_0 := \min(F) \in \phi_1(F)$ but $x_0 \notin \phi_j(F)$ for all $j > 1$, and $\rho_1 > 0$. Let $\psi(x) = \lambda x + b$ such that $x_0 \in \psi(F) \subset F$ and $\lambda > 0$. Then $\log \lambda / \log \rho_1 \in \mathbb{Q}$.*

Proof. Since $\rho_1, \lambda > 0$ it is clear that x_0 is a fixed point of ϕ_1 and ψ, i.e. $x_0 = \phi_1(x_0) = \psi(x_0)$. By making a translation $F' = F - x_0$ it is easy to see that we may without loss of generality assume that $x_0 = \min(F) = 0$, which forces $\phi_1(x) = \rho_1 x$ and $\psi(x) = \lambda x$.

From the definition and $\phi_1(x) = \rho_1 x$, we have

$$\rho_1^{-m} F = \rho_1^{-m} \bigcup_{\phi \in \Phi^m} \phi(F) = F \cup \Big(\bigcup_{\phi \in \Phi^m \setminus \{\phi_1^m\}} \rho_1^{-m} \phi(F) \Big), \tag{5.1}$$

Since $\psi^n(F) = \lambda^n F \subset F$, by (5.1) we have

$$\rho_1^{-m} \lambda^n F \subseteq \rho_1^{-m} F = F \cup \Big(\bigcup_{\phi \in \Phi^m \setminus \{\phi_1^m\}} \rho_1^{-m} \phi(F) \Big). \tag{5.2}$$

Observe that $0 \in \phi_1(F)$ but $\operatorname{dist}(0, \phi_j(F)) \geq \delta$ for some $\delta > 0$ and all $j > 1$. This means $0 \in \phi_1^m(F)$ but $\operatorname{dist}(0, \phi(F)) \geq \rho_1^m \delta$ for all other $\phi \in \Phi^m$. Hence $\operatorname{dist}(0, \rho_1^{-m}\phi(F)) \geq \delta$ for all $\phi \in \Phi^m \setminus \{\phi_1^m\}$. Now, $[0, \delta] \not\subseteq F$ by the no interval condition. So there exists an interval $I_0 \subseteq (0, \delta) \setminus F$. Clearly I_0 has no intersection with the set on the righthand side of (5.2). Assume that $\log \lambda / \log \rho_1 \notin \mathbb{Q}$. Then $\{-m \log \rho_1 + n \log \lambda\}$ is dense in $\mathbb{R}$, and hence $\rho_1^{-m} \lambda^n$ is dense in $\mathbb{R}^+$. In particular we may choose m, n such that $\rho_1^{-m} \lambda^n \max(F) \in I_0$. For such m, n (5.2) is clearly violated, yielding a contradiction. □

We remark that Lemma 5.1 does not require the IFS to satisfy the OSC. Clearly the no interval condition is satisfied if $\dim_H F < 1$. If a homogeneous IFS Φ satisfying the OSC and $\dim_H F_\Phi = 1$, then the no interval condition is equivalent to F_Φ is not a finite union of intervals:

Proposition 5.2. *Let Φ be a homogeneous IFS with the OSC. Suppose that F_Φ does not satisfy the no interval condition. Then $\rho_\Phi = \frac{1}{p}$ for some integer p and F_Φ is a finite union of intervals.*

Proof. This is proved in Lagarias and Wang [13], using a result of Odlyzko [19]. In fact, the structure of Φ is known. □

We now prove part (i) of Theorem 1.1. This is done by breaking it down into several lemmas.

Lemma 5.3. *Under the assumptions of Theorem 1.1, there exists a positive number t (depending on F) such that*

$$\mathcal{H}^s(F \cap [a,b]) \le t\,(b-a)^s, \qquad \forall\,[a,b] \subset \mathbb{R}. \tag{5.3}$$

Proof. It is included implicitly in the proof of Theorem 8.6 in Falconer [9]. □

As a result of the above lemma, we introduce

$$d_{\max} = \sup\{\mathcal{H}^s(F \cap [a,b])/(b-a)^s : \ \ [a,b] \subset \mathbb{R}\},$$

and clearly $0 < d_{\max} < \infty$. The following lemma plays a central role in the proof of Theorem 1.1.

Lemma 5.4. *There exist an interval $[a,b]$ and an integer $k > 0$ such that*

(i) $[a,b] \cap F \neq \emptyset$.
(ii) $\big[x - |\rho_\Phi|^k \operatorname{diam} F,\ x + |\rho_\Phi|^k \operatorname{diam} F\big] \cap F = \emptyset$ *for* $x = a,\ b$.
(iii) *Denote* $\mathcal{M} = \big\{\mathbf{i} \in \{1, \cdots, N\}^k :\ \phi_{\mathbf{i}}(F) \subset [a,b]\big\}$ *and* $M = \#\mathcal{M}$, *then*

$$(M + 1/2)|\rho_\Phi|^{ks}\mathcal{H}^s(F) > d_{\max}(b-a)^s. \tag{5.4}$$

Proof. Denote $\rho = |\rho_\Phi|$. Since $0 < s < 1$, using L'Hospital's rule we have

$$\lim_{x \to 0} \frac{(1+ux)^s - 1}{x^s} = 0, \qquad \forall\, u > 0.$$

Therefore there exist $\ell \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\frac{1}{2}\,\rho^{\ell s}\mathcal{H}^s(F) - \varepsilon > d_{\max}\left((1 + 8\rho^{\ell-1}\mathrm{diam}F)^s - 1\right). \tag{5.5}$$

By the definition of $d_{\max}$ there exists an interval $[c,d]$ such that $[c,d] \cap F \neq \emptyset$ and

$$\mathcal{H}^s(F \cap [c,d]) \geq (d_{\max} - \varepsilon)\,(d-c)^s.$$

Let r be the integer so that $\rho^{r+1} < d - c \leq \rho^r$. Then we have

$$\begin{aligned} \mathcal{H}^s(F \cap [c,d]) + \frac{1}{2}\rho^{(r+\ell)s}\mathcal{H}^s(F) &> (d_{\max} - \varepsilon)(d-c)^s + \frac{1}{2}\rho^{\ell s}\mathcal{H}^s(F)(d-c)^s \\ &\geq \left(d_{\max} - \varepsilon + \frac{1}{2}\rho^{\ell s}\mathcal{H}^s(F)\right)(d-c)^s \\ &\geq d_{\max}\left(1 + 8\rho^{\ell-1}\mathrm{diam}F\right)^s (d-c)^s \qquad \text{(by (5.5))} \\ &\geq d_{\max}\left(d - c + 8\rho^{\ell-1}(d-c)\mathrm{diam}F\right)^s \\ &\geq d_{\max}\left(d - c + 8\rho^{\ell+r}\mathrm{diam}F\right)^s. \end{aligned}$$

That is

$$\mathcal{H}^s(F \cap [c,d]) + \frac{1}{2}\rho^{(r+\ell)s}\mathcal{H}^s(F) > d_{\max}\left(d - c + 8\rho^{\ell+r}\mathrm{diam}F\right)^s. \tag{5.6}$$

Define $k = \ell + r$ and $[a,b] = [c - 2\rho^k\mathrm{diam}\,F,\ d + 2\rho^k\mathrm{diam}\,F]$. We show that $[a,b]$ and k satisfy (i), (ii) and (iii). Part (i) is obvious since $[a,b] \supset [c,d]$. Assume that (ii) is not true. Then

$$F \cap \Big([c - 3\rho^k\mathrm{diam}\,F,\ c - \rho^k\mathrm{diam}\,F] \cup [d + \rho^k\mathrm{diam}\,F,\ d + 3\rho^k\mathrm{diam}\,F]\Big) \neq \emptyset.$$

Therefore there exists at least one $\mathbf{i} \in \{1, \ldots, N\}^k$ such that

$$\phi_{\mathbf{i}}(F) \subset [c - 4\rho^k\mathrm{diam}\,F,\ c] \cup [d, d + 4\rho^k\mathrm{diam}\,F].$$

Then it would follow from (5.6) that

$$\begin{aligned} \mathcal{H}^s(F \cap [c - 4\rho^k\mathrm{diam}\,F,\ d + 4\rho^k\mathrm{diam}\,F]) &\geq \mathcal{H}^s(F \cap [c,d]) + \rho^{ks}\mathcal{H}^s(F) \\ &> d_{\max}(d - c + 8\rho^k\mathrm{diam}\,F)^s, \end{aligned}$$

which leads to a contradiction. This finishes the proof of part (ii). According to (ii), we have $\bigcup_{\mathbf{i}\in\mathcal{M}} \phi_{\mathbf{i}}(F) \supseteq F \cap [c,d]$. Thus $M\rho^{ks}\mathcal{H}^s(F) \geq \mathcal{H}^s(F \cap [c,d])$. Hence by (5.6),

$$\begin{aligned} (M + 1/2)\rho^{ks}\mathcal{H}^s(F) &\geq \mathcal{H}^s(F \cap [c,d]) + \frac{1}{2}\rho^{ks}\mathcal{H}^s(F) \\ &> d_{\max}(d - c + 8\rho^k\mathrm{diam}F)^s \\ &> d_{\max}(b-a)^s, \end{aligned}$$

proving part (iii). $\square$

Proof of Theorem 1.1. Part (ii) is a corollary of Lemma 5.1 and Proposition 5.2. To see it, we observe that under the assumption of (ii), F satisfies the no interval condition by Proposition 5.2. Then we apply Lemma 5.1 to the positive homogeneous IFS Φ^2 to yield $\log\lambda/\log\rho_\Phi^2 \in \mathbb{Q}$.

In the following we prove part (i) of the theorem. Note that $\mathcal{I}_F$ and $\mathcal{I}_F^+$ are Abelian as a result of Proposition 2.2. Let $[a,b]$, k, $\mathcal{M}$ and M be given as in Lemma 5.4. Assume that Theorem 1.1 is false, that is, $\log|\lambda|/\log|\rho_\Phi| \notin \mathbb{Q}$. We derive a contradiction.

Let $\varepsilon > 0$ be a small number such that $(1-\varepsilon)^s(M+1) \geq (M+1/2)$. Since $\log|\lambda|/\log|\rho_\Phi| \notin \mathbb{Q}$, there exist $m,n \in \mathbb{N}$ such that

$$1-\varepsilon < |\rho_\Phi|^m/|\lambda|^n < 1.$$

Define $J = \psi^n([a,b])$. We show that

$$\mathcal{H}^s(J \cap F) > d_{\max}|\mathrm{diam} J|^s, \tag{5.7}$$

which contradicts the maximality of $d_{\max}$.

To show (5.7), let

$$\tilde{J} := \psi^n\left[a + |\rho_\Phi|^k \mathrm{diam}\, F,\ b - |\rho_\Phi|^k \mathrm{diam}\, F\right].$$

By Lemma 5.4 (ii),

$$\begin{aligned}
\tilde{J} \cap F &\supseteq \tilde{J} \cap \psi^n(F) \\
&= \psi^n\left(\left[a + |\rho_\Phi|^k \mathrm{diam} F,\ b - |\rho_\Phi|^k \mathrm{diam}\, F\right] \cap F\right) \\
&= \psi^n([a,b] \cap F) \\
&= \psi^n\left(\bigcup_{\mathbf{i}\in\mathcal{M}} \phi_{\mathbf{i}}(F)\right).
\end{aligned}$$

Hence

$$\mathcal{H}^s(\tilde{J} \cap F) \geq \mathcal{H}^s\left(\psi^n\Big(\bigcup_{\mathbf{i}\in\mathcal{M}} \phi_{\mathbf{i}}(F)\Big)\right) = M\ |\lambda|^{ns}\ |\rho_\Phi|^{ks}\ \mathcal{H}^s(F). \tag{5.8}$$

Define

$$\mathcal{R} := \left\{\mathbf{i} \in \{1,\cdots,N\}^{m+k} :\ \phi_{\mathbf{i}}(F) \cap \tilde{J} \neq \emptyset\right\}$$

and $R = \#\mathcal{R}$. Then $\phi_{\mathbf{i}}(F) \subset J$ for any $\mathbf{i} \in \mathcal{R}$, and $\bigcup_{\mathbf{i}\in\mathcal{R}} \phi_{\mathbf{i}}(F) \supset \tilde{J} \cap F$. Thus

$$\mathcal{H}^s(J \cap F) \geq \mathcal{H}^s(\bigcup_{\mathbf{i}\in\mathcal{R}} \phi_{\mathbf{i}}(F)) = R\ |\rho_\Phi|^{(m+k)s}\ \mathcal{H}^s(F) \geq \mathcal{H}^s(\tilde{J} \cap F).$$

Combining the second inequality with (5.8) and with $|\rho_\Phi|^m/|\lambda|^n < 1$ we obtain $R > M$, and thus $R \geq M + 1$. Hence we have

$$\begin{aligned}
\mathcal{H}^s(J \cap F) &\geq (M+1)\ |\rho_\Phi|^{(m+k)s}\ \mathcal{H}^s(F) \\
&> (M+1)\ (1-\varepsilon)^s\ |\lambda|^{ns}\ |\rho_\Phi|^{ks}\ \mathcal{H}^s(F) \\
&> (M+1/2)\ |\lambda|^{ns}\ |\rho_\Phi|^{ks}\ \mathcal{H}^s(F) \\
&> d_{\max}\ |\lambda|^{ns}\ (b-a)^s \qquad (\text{ by } (5.4)) \\
&= d_{\max}\ |J|^s.
\end{aligned}$$

This is a contradiction, proving part (i) of the theorem. $\square$

6. Counterexamples and Open Questions

In this section we present various counterexamples, including an example to Mattila's question. We also propose some open questions.

Let us first give an example to show that the condition COSC in Theorem 3.1 cannot be replaced with the SC.

Example 6.1. Let F be the attractor of the IFS $\Phi = \{\frac{1}{16}(x+a) : \ a \in \mathcal{A}\}$ where $\mathcal{A} = \{0, 1, 64, 65\}$. It is not difficult to check that Φ satisfies the SC but does not satisfy the COSC. We prove that $\mathcal{I}_F^+$ does not contain a minimal element by contradiction. Assume this is not true. Let $\Phi_0 = \{\rho x + c_i\}_{i=1}^N$ be the minimal element of $\mathcal{I}_F^+$. By the dimension formula and Corollary 1.2, $\log \rho / \log 16^{-1} = \log N / \log 4 \in \mathbb{Q}$. Therefore $N = 2$ and $\rho = \frac{1}{4}$ or $N = 4$ and $\rho = \frac{1}{16}$. But it is easy to check that if $N = 2$ then the IFS Φ_0 must satisfy the COSC, but Φ does not, a contradiction to Theorem 3.1. Hence we must have $N = 4$ and hence $\Phi_0 = \Phi$ by Lemma 2.1. Now let $\Psi = \{\frac{1}{64}(x+b) : \ b \in \mathcal{B}\}$ where $\mathcal{B} = \{0, 1, 16, 17, 256, 257, 272, 273\}$. One can check directly $\mathcal{B} + 64\mathcal{B} = \mathcal{A} + 16\mathcal{A} + 16^2\mathcal{A}$. Thus $\Psi^2 = \Phi^3$, which implies $\Psi \in \mathcal{I}_F$. However Ψ is not derived from Φ, which leads to a contradiction. Hence $\mathcal{I}_F^+$ does not contain a minimal element.

Now we give a counterexample showing that the COSC no longer guarantees the existence of a minimal element if we do not require that the contraction factors be homogeneous.

Example 6.2. Let F be the attractor of the IFS $\Phi = \{\frac{1}{10}(x+a) : \ a \in \mathcal{A}\}$ where $\mathcal{A} = \{0,1,5,6\}$. As before let $\mathcal{G}_F^+$ denotes the set of all positive generating IFSs of F satisfying the OSC. We claim that Φ satisfies the COSC and $\mathcal{G}_F^+$ does not contain a minimal element. Indeed one can check directly that F is symmetric with $\min F = 0$ and $\max F = 2/3$. Since $F < F+1 < F+5 < F+6$, F satisfies the COSC by Lemma 3.2. To see that $\mathcal{G}_F^+$ does not contain a minimal element, note that any ϕ in a generating IFS of F must map F either to the left or to the right part of F, because the hole in the middle (having length $\mathrm{diam}(F)/2$) would be too large for a subset of F to be similar to F. Thus ϕ must have contraction factor $\leq 1/4$. Assume that $\mathcal{G}_F^+$ contains a minimal element Φ_0. Then $\Phi_0 = \Phi$, because each map in Φ (with contraction factors $> 1/16$) cannot be a composition of two maps in Φ_0. Consider

$$\Psi := \left\{ \frac{x}{100}, \frac{x+1}{100}, \frac{x+1/2}{10}, \frac{x+15}{100}, \frac{x+16}{100}, \frac{x+5}{10}, \frac{x+6}{10} \right\}.$$

We see that Ψ is a generating IFS of F, since F satisfies the following relation:

$$\begin{aligned} F &= \frac{F+\{0,1,5,6\}}{10} = \frac{F+\{0,1,5,6,10,11,15,16\}}{100} \cup \frac{F+\{5,6\}}{10} \\ &= \frac{F+\{0,1,15,16\}}{100} \cup \frac{F+1/2}{10} \cup \frac{F+\{5,6\}}{10}. \end{aligned}$$

Furthermore Ψ satisfies the OSC (one can take $(0,2/3)$ as the open set). Thus $\Psi \in \mathcal{G}_F^+$. This is a contradiction because the map $\frac{x+1/2}{10}$ is not the composition of elements in Φ_0. Hence $\mathcal{G}_F^+$ does not contain a minimal element.

Remark 6.1. Example 6.2 also shows that the condition $\mathcal{H}^s(F) = (\mathrm{diam} F)^s$ in Theorem 1.3 cannot be dropped. An example essentially identical to Example 6.2 has recently been obtained in [8] independently.

Example 6.3. In this example we consider the questions raised by Mattila (see *Period. Math. Hungar.* **37** (1998), 227–237) : What are the self-similar subsets of the middle-third Cantor set C? Is there a non-trivial self-similar subset F of C, in the sense that F has a generating IFS that is not derived from the generating IFS $\{\phi_0, \phi_1\}$ of C given in (1.1)?

We give a positive answer to the second question here by constructing a concrete example. In fact for the first question, we have obtained a complete classification of self-similar subsets of C with positive homogeneous contraction factors and with minimum 0. This will be presented in a separate note [11]. For now, let $\Phi = \{\frac{1}{9}x, \frac{1}{9}(x+2)\}$. Choose a sequence $(\epsilon_k)_{k=1}^{\infty}$ with $\epsilon_k \in \{0,2\}$ so that $w = \sum_{k=1}^{\infty} \epsilon_k 3^{-2k+1}$ is an irrational number. Then by looking at the ternary expansion of the elements in $F_\Phi + w$ it is easy to see that $F_\Phi + w \subset C$. Observe that $F_\Phi + w$ a self-similar subset of C since it is the attractor of the IFS $\Psi = \{\frac{1}{9}(x+8w), \frac{1}{9}(x+2+8w)\}$. However any generating IFS of F_Ψ can not be derived from the original IFS $\{\phi_0, \phi_1\}$, since $w = \min F_\Psi$ can not be the fixed point of any map $\phi_{i_1 i_2 \dots i_n}$ composed from ϕ_0, ϕ_1 due to the irrationality of w.

Open Question 1. We pose the following question concerning the symmetry of a self-similar set: Let Φ and Ψ be two homogeneous IFSs satisfying the OSC, with $\rho_\Phi = -\rho_\Psi$ and $F_\Phi = F_\Psi$. Does it follow that F is symmetric?

This is answered in affirmative under the strong assumption of COSC. But is it true in general? If so, then the results in part (ii) and (iii) of Theorem 1.3 will be much cleaner.

It should be pointed out that this is *not* true for self-similar measures. We'll leave to the readers to construct a counterexample.

Open Question 2. We do not have a good way to generalize our results to higher dimensions.

The challenge here is to generalize the Logarithmic Commensurability Theorem to higher dimensions for affine IFSs. There is a possibility to do it for similitude IFSs.

Remark 6.2. Recently Elekes, Keleti and Máthé [8] have proved the Logarithmic Commensurability Theorem for similitude IFSs in $\mathbb{R}^d$ with the SC. Shmerkin [22] told the authors that an argument using the results on sum of Cantor sets in [20] can prove the Logarithmic Commensurability Theorem for any IFS in $\mathbb{R}$ of the form $\{\rho^{k_i}x + c_i\}_{i=1}^N$, $k_i \in \mathbb{N}$, under the assumption that the Hausdorff dimension of the attractor is less than 1 and coincides with its self-similar dimension.

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