ON THE LIMIT RADEMACHER FUNCTIONS AND BERNOULLI CONVOLUTIONS

DE-JUN FENG *

Department of Applied Mathematics and Center for Advance Study, Tsinghua University, Beijing 100084, China, E-mail: dfeng@math.tsinghua.edu.cn.

(DEDICATED TO THE MEMORY OF PROFESSOR LIAO SHANTAO)

1. Introduction

For $0 < \rho < 1$, the limit Rademacher function f_ρ is defined as:

$$f_\rho(x) = (1-\rho)\sum_{n=0}^{\infty} \rho^n R(2^n x), \quad x \in [0,1]$$

where R denotes the Rademacher function: $R(x)$ is defined on the line $\mathbb{R}$ with period 1, takes values 0 and 1 on the intervals $[0,1/2)$ and $[1/2,1)$ respectively. The distribution of f_ρ induces a probability measure μ_ρ on $[0,1]$, which is called an *infinitely convolved Bernoulli measure* or simply a *Bernoulli convolution*. That is,

$$\mu_\rho(E) = \mathcal{L}\{x \in [0,1] : f_\rho(x) \in E\}, \ \forall E \subset [0,1] \text{ measurable}$$

where $\mathcal{L}$ denotes 1-dimensional Lebesgue measure. These measures have been studied for more than 60 years, revealing many connections with harmonic analysis, algebraic number theory, dynamical systems, and Hausdorff dimension estimation (for a good survey, see e.g., [15] or [1]). For $0 < \rho < 1/2$, the support of μ_ρ is a Cantor set of zero Lebesgue measure and μ_ρ is totally singular with respect to the Lebesgue measure. For $\rho = 1/2$, μ_ρ is just the 1-dimensional Lebesgue measure restriction on $[0,1]$. For $1/2 < \rho < 1$, μ_ρ is only partially understood still now. Solomyak [16] proved μ_ρ is absolute continuous for almost all $\rho \in (1/2,1)$. For integer $k \geq 2$, let λ_k is the positive root of the polynomial $1 - x - x^2 - \cdots - x^k$. Erdős [3] proved that the Bernoulli convolution μ_{λ_k} is totally singular to the Lebesgue measure for each $k \geq 2$.

In this note we give the explicit formulas for the Hausdorff dimension of the graphs of f_{λ_k}, the Hausdorff dimension of almost all level sets (respect to Lebesgue measure) of f_{λ_k}, the Hausdorff dimension, information dimension and the L^q-spectrum ($q \in \mathbb{R}$) of μ_{λ_k}, $k \geq 2$. We point out that for each $k \geq 2$, μ_{λ_k} is a locally infinitely-generated self-similar measure and the multifractal formalism of which holds. For the detailed proofs, see [5], [6]. The reader may refer to the books [4], [11], [13] for the relative definitions of dimensions.

* Supported by China Postdoc Foundation.

2. Main results

Set

$$M_0 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

For any $J = j_1 \cdots j_n \in \{0,1\}^n$, denote $M_J = M_{j_1} \circ \cdots \circ M_{j_n}$. For any 2×2 non-negative matrix B, denote by $||B|| = (1,1)B(1,1)'$.

We can formulate our main results as follows:

Theorem 2.1 *For each integer $k \geq 2$, let $\alpha_k = -\frac{\log \lambda_k}{\log 2}$. Then the Hausdorff dimension of the graph of the limit Rademacher function f_{λ_k} satisfies that*

$$\dim_H \ Graph(f_{\lambda_k}) = \frac{\log x_k}{\log \lambda_k},$$

where $0 < x_k \leq \lambda_{k-1}$ (defining $\lambda_1 = 1$) and x_k satisfies that

$$\frac{1 - 2x^{k-1} + x^k}{1 - 2x + x^k} \cdot \sum_{n=0}^{\infty} \Big(\sum_{|J|=n} ||M_J||^{\alpha_k} \Big) x^{kn+k+1} = 1.$$

Let $\alpha \in \mathbb{R}$, the α-level set of a function f is defined as $\{x : \ f(x) = \alpha\}$.

Theorem 2.2 *For $k \geq 2$, the Hausdorff dimension and box-counting dimension of t-level set of the limit Rademacher function f_{λ_k} are equal to*

$$\frac{(\lambda_k)^k (1 - 2(\lambda_k)^k)^2}{(2 - (k+1)(\lambda_k)^k) \log 2} \sum_{n=0}^{\infty} \Big((\lambda_k)^{kn} \sum_{|J|=n} \log ||M_J|| \Big)$$

for almost all $t \in [0,1]$ (respect to the Lebesgue measure).

Theorem 2.3 **(i)** *For any $q \in \mathbb{R}$, the L^q-spectrum $\tau_{\lambda_2}(q)$ of μ_{λ_2} is equal to*

$$\frac{q \log 2}{\log \lambda_2^{-1}} + \frac{\log \mathbf{x}(2,q)}{\log \lambda_2^{-1}},$$

where

$$\mathbf{x}(2,q) = \sup\{x \geq 0 : \sum_{n=0}^{\infty} \Big(\sum_{|J|=n} ||M_J||^q \Big) x^{2n+3} \leq 1\}.$$

There exists a unique $q_0 < -2$ such that $\sum_{n=0}^{\infty} (\sum_{|J|=n} ||M_J||^{q_0}) = 1$. When $q > q_0$, $\mathbf{x}(2,q)$ is the positive root of $\sum_{n=0}^{\infty} (\sum_{|J|=n} ||M_J||^q) x^{2n+3} = 1$, and it is *an infinitely differentiable function of q on $(q_0, +\infty)$. When $q \leq q_0$, $\mathbf{x}(2, q) = 1$. Moreover $\mathbf{x}(2, q)$ is not differentiable at $q = q_0$.*

(ii) *For any integer $k \geq 3$ and any real number q, the L^q -spectrum $\tau_{\lambda_k}(q)$ of the Bernoulli convolution μ_{λ_k} is equal to*

$$\frac{q \log 2}{\log \lambda_k^{-1}} + \frac{\log \mathbf{x}(k, q)}{\log \lambda_k^{-1}},$$

where $0 < \mathbf{x}(k, q) < \lambda_{k-1}$, $\mathbf{x}(k, q)$ satisfies that

$$\frac{1 - 2x^{k-1} + x^k}{1 - 2x + x^k} \cdot \sum_{n=0}^{\infty} (\sum_{|J|=n} ||M_J||^q) x^{kn+k+1} = 1.$$

Moreover $\mathbf{x}(k, q)$ is an infinitely differentiable function on the whole line.

Theorem 2.4 *For each integer $k \geq 2$, the Hausdorff dimension and the information dimension of the Bernoulli convolution μ_{λ_k} coincide, the common value is equal to*

$$-\frac{\log 2}{\log \lambda_k} + \left(\frac{2^k - 3}{2^k - 1} \right)^2 \cdot \frac{\sum\limits_{n=0}^{\infty} 2^{-kn-k-1} \sum\limits_{|J|=n} ||M_J|| \log ||M_J||}{\log \lambda_k}.$$

Theorem 2.5 *For each $k \geq 2$, the Bernoulli convolution μ_{λ_k} is a locally infinitely self-similar measure without overlap in the following sense: μ_{λ_k} is supported on the union of a sequence of disjoint intervals $\{I_i\}_i$; and for each i there exists a countable family of similitudes $\{g_{i,j}\}_j$ with contraction ratio $\{r_j\}_j$, and a probability weight $\{p_j\}_j$ with $\sum_j p_j = 1$, such that $g_{i,j}(I_i) \subset I_i$ for each j, $g_{i,j}(I_i) \cap g_{i,j'}(I_i) = \emptyset$ if $j \neq j'$, and*

$$\mu_{\lambda_k}^{(i)} = \sum_j p_j \mu_{\lambda_k}^{(i)} \circ g_{i,j}^{-1}$$

where $\mu_{\lambda_k}^{(i)}$ denote the restriction of μ_{λ_k} on the interval I_i, i.e., $\mu_{\lambda_k}^{(i)}(A) = \mu_{\lambda_k}(I_i \cap A)$ for each Borel set $A \subset \mathbb{R}$.

Theorem 2.6 **(i)** *Let q_0 defined as in Theorem 2.3(i). Then for each $q \in \mathbb{R} \backslash q_0$, the multifractal formalism of μ_{λ_2} holds for $\alpha = \alpha(q) := \tau'(q)$, where $\tau(q)$ is the L^q-spectrum of μ_{λ_2}.*

(ii) *For each integer $k \geq 3$ and $q \in \mathbb{R}$, the multifractal formalism of μ_{λ_k} holds for $\alpha = \alpha(q) := \tau'(q)$, where $\tau(q)$ is the L^q-spectrum of μ_{λ_k}.*

Remark. **(i)** Lau and Ngai [8] obtained the formula of the L^q-spectrum of μ_{λ_2} for $q > 0$, they proved that it is a differentiable function of q on $(0, +\infty)$.

Using this result, they showed [7] that the multifractal formula of μ_{λ_2} holds for $\alpha = \tau'(q)$, $q > 0$. Porzio [14] extended the differentiability range of L^q-spectrum to $(-\frac{1}{2}, +\infty)$, and showed that the multifractal formula of μ_{λ_2} holds for $\alpha = \tau'(q)$, $q \in (-\frac{1}{2}, +\infty)$.

(ii) Several people have obtained the explicit formula and numeral estimations of $\dim_H \mu_{\lambda_2}$. See [1], [2], [8], [9], [12], [17].

References

1. J.C. Alexander and J.A. Yorke, *Ergod. Theory & Dynam. Systems* **4**(1984), 1-23.
2. J.C. Alexander and D. Zagier, *J. London Math. Soc.* (2) **44** (1991), 121-134.
3. P. Erdős, *Amer. J. Math.* **61** (1939), 974-976.
4. K.J. Falconer, *Fractal Geometry, Mathematical Foundations and Applications* (Wiley, 1990).
5. D.-J. Feng, The limit Rademacher functions and Bernoulli convolutions associated Pisot numbers, Preprint.
6. D.-J. Feng, The similarity and multifractal analysis of Bernoulli convolutions, Preprint.
7. K.-S. Lau and S.-M. Ngai, Multifractal measure and a weak separation condition, *Adv. Math.*, to appear.
8. -, *Studia Math.*, **131,** no. 3 (1998), 225-251.
9. F. Ledrappier and A. Porzio, *J. Stat. Phys.* **76** (1994), 1307-1327.
10. -, *J. Statist. Phys.*, **82** (1996), 367-420.
11. P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*(Cambridge Univ. Press, 1995).
12. S.-M. Ngai, *Proc. Amer. Math. Soc.* **125** (1997), 2943-2951.
13. Y.B. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications* (The University of Chicago Press, 1997).
14. A. Porzio, *J. Stat. Phys.*, **91** (1998), 17-29.
15. Y. Peres, W. Schlag and B. Solomyak, Sixty years of Bernoulli convolutions. Preprint
16. B. Solomyak, *Ann. Math.*, **142** (1995), 611-625.
17. N. Sidorov and A. Vershik, Ergodic properties of Erdős measure, the entropy of the goldenshift and related problems. Preprint, 1997.