Category and dimension of compact subsets of $\mathbb{R}^n$

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IN fractal geometry, two classes of sets play important roles. One is the regular set (the set Hausdorff and packing dimension coincide), the other is the set whose Bouligand dimension exists. A natural question is how to measure "the size" of these sets mentioned above. In this note, by using category, we answer this question. The main result is Theorem 1.

Suppose that $E\subset\mathbb{R}^d$. We denote by $\dim_H E$, $\dim_p E$, $\overline{\dim}_B E$ and $\underline{\dim}_B E$, respectively, the Hausdorff dimension, packing dimension, upper Bouligand dimension and lower Bouligand dimension of E. If $\overline{\dim}_B E=\underline{\dim}_B E$, we say that the Bouligand dimension of E exists. For the details of definitions and properties of the above dimensions, see reference [1].

Given $\varepsilon>0$, let $V_\varepsilon(E)=\{x\in\mathbb{R}^d: d(x,E)\leqslant\varepsilon\}$, where $d(x,E)=\inf\{\|x-y\|:y\in E\}$, $\|\cdot\|$ denotes Euclidean metric. Let $N_\delta(E)$ denote the minimum number of balls of radius δ needed to cover E.

Since every set and its closure have the same upper, lower Bouligand dimension, we only consider the compact sets. Suppose that $\mathscr{K}^d$ is the collection of nonempty compact sets in $\mathbb{R}^d$ endowed with the Hausdorff metric ρ, then $(\mathscr{K}^d, \rho)$ is a complete metric space. A subset of $(\mathscr{K}^d, \rho)$ is said to be of first category if it can be represented as a countable union of nowhere dense sets. If a property holds except for a set of first category, we say that it holds almost all.

Lemma 1. *Let $E, F\in\mathscr{K}^d$. If $\rho(E,F)<\delta$, then*

$$4^{-d}N_\delta(F)\leqslant N_\delta(E)\leqslant 4^d N_\delta(F).$$

Proof. Since $\rho(E,F)<\delta$, $V_\delta(F)\supset E$. Let $B_i,\cdots,B_{N_\delta(F)}$ denote $N_\delta(F)$ balls with radius δ that cover F. For any $1\leqslant i\leqslant N_\delta(F)$, let $\widetilde{B}_i$ be the ball of radius 2δ with the same center as B_i. Then $\bigcup_{1\leqslant i\leqslant N_\delta(F)}\widetilde{B}_i\supset V_\delta(F)$. Notice that every $\widetilde{B}_i$ can be covered by 4^d balls with radius δ. Thus $V_\delta(F)$ can be covered by $4^dN_\delta(F)$ balls with radius δ. Therefore

$$N_\delta(E)\leqslant N_\delta(V_\delta(F))\leqslant 4^dN_\delta(F).$$

In a similar way , we can prove $N_\delta(F)\leqslant 4^dN_\delta(E)$.

Lemma 2. *For almost all $F\in\mathscr{K}^d$, $\underline{\dim}_B F=0$.*

Proof. Let $\mathbb{Q}$ denote the rational numbers in R^1 and let $\mathbb{Q}^d=\underbrace{\mathbb{Q}\times\cdots\times\mathbb{Q}}_{d}\subset R^d$. Let $J^d=\{A\subset\mathbb{Q}^d: \# A<\infty\}$. Then J^d is countable and dense in $\mathscr{K}^d$. In fact, since $\mathbb{Q}^d$ is dense in $\mathbb{R}^d$, we have for any $E\in\mathscr{K}^d$ and $\varepsilon>0$ the open balls with center in $\mathbb{Q}^d\cap V_\varepsilon(E)$ and radius ε that cover E. By the compactness of E, there exists $F\subset J^d$ such that $V_\varepsilon(F)\supset E$. Noticing that $F\subset V_\varepsilon(E)$, we have $\rho(E,F)<\varepsilon$.

For any $K\in J^d$, define the double sequence $\{\delta_j(K)\}_{j\geqslant 1}$ such that

$$0<\delta_i(K)<2^{-i};\ \frac{\log\# K}{-\log\delta_i(K)}<2^{-i}, i\geqslant 1.$$

For any $j \geqslant 1$, define $W_j = \{F \in \mathscr{K}^d$: there exists $K \in J^d$ such that $\rho(F, K) < \delta_j(K)\}$.

Since $W_j \supset J^d$, W_j is dense in $\mathscr{K}^d$. On the other hand, if $F \in W_j$, then there exists $K \in J^d$ such that $\rho(F, K) < \delta_j(K)$. Let $\varepsilon = \delta_j(K) - \rho(F, K)$. Then for any E in $\mathscr{K}^d$ with $\rho(F, E) < \varepsilon$, we have $\rho(E, K) < \delta_j(K)$. Therefore W_j is an open set of $(\mathscr{K}^d, \rho)$.

Let $W = \bigcap_{j \geqslant 1} W_j$. By Theorem 1.3 and 1.4 of ref. [2], we know that $\mathscr{K}^d \setminus W$ is of first category.

If $F \in W$, then for any $j \geqslant 1$, $F \in W_j$. By the definition of W_j, there exists $K_j \in J^d$ such that

$$\rho(F, K_j) < \delta_j(K_j).$$

By Lemma 1, we have

$$N_{\delta_j(K_j)}(F) \leqslant 4^d N_{\delta_j(K_j)}(K_j) \leqslant 4^d(\# K_j).$$

By the definition of $\delta_j(K_j)$, we have

$$\frac{\log N_{\delta_j(K_j)}(F)}{-\log \delta_j(K_j)} \leqslant \frac{\log(4^d \# K_j)}{-\log \delta_j(K_j)} \leqslant \frac{2d}{j} + 2^{-j}. \tag{1}$$

By Proposition 4.1 of ref. [1] and (1), we have

$$\underline{\dim}_B F \leqslant \liminf_{j \to \infty} \frac{\log N_{\delta_j(K_j)}(F)}{-\log \delta_j(K_j)} = 0.$$

Remark 1. For any $F \in \mathscr{K}^d$, $\dim_H F \leqslant \underline{\dim}_B F$. By lemma 2, we have $\dim_H F = 0$ for almost all $F \in \mathscr{K}^d$.

Lemma 3. *Let $x \in R^d$, $r > 0$. Then for almost all $K \in \mathscr{K}^d$,*

$$K \cap B_r(x) = \varnothing \text{ or } \overline{\dim}_B(K \cap B_{2r}(x)) = d,$$

where $B_r(x)$ denotes the open ball with center x and radius r.

Proof. Let

$$J_1^d = \{A : A \subset Q^d \setminus B_r(x), \# A < \infty\}, \text{ and } J_2^d = J^d \setminus J_1^d.$$

For any $F \in J_2^d$, we have $F \cap B_r(x) \neq \varnothing$. Choose any $z \in F \cap B_r(x)$. There exists $\delta > 0$ such that $B_\delta(z) \subset B_r(x)$. For any $i \geqslant 1$, let $F_i(F) = F_i = \overline{B}_{2^{-i}\delta}(z) \cup F$. It is easy to see that $\rho(F_i, F) \leqslant 2^{-i}\delta$.

Define $\mathscr{F} = \{F_i(F); F \in J_2^d, i \geqslant 1\}$. Then for any $E \in \mathscr{F}$, $(E \cap B_r(x))^0 \neq \varnothing$. Thus $\overline{\dim}_B(E \cap B_r(x)) = d$.

For any $E \in \mathscr{F}$, define the double sequence $\{\delta_i(E)\}_{E \in \mathscr{F}, i \geqslant 1}$ such that

$$0 < \delta_i(E) < 2^{-i}, \text{ and } \frac{\log N_{\delta_i(E)}(E \cap B_r(x))}{-\log \delta_i(E)} > d - 2^{-i}. \tag{2}$$

For any $j \geqslant 1$, define

$$V_j^{(i)} = \{G \in \mathscr{K}^d: \text{there exist } X \in J_1^d \text{ such that } \rho(G, X) < 2^{-j}\},$$

$$V_j^{(2)} = \{G \in \mathscr{K}^d: \text{there exist } E \in \mathscr{F} \text{ such that } \rho(G, E) < \delta_i(E)\}.$$

$$V_j = V_j^{(1)} \cup V_j^{(2)}.$$

It is obvious that $V_j \supset J_1^d \cup \mathscr{F}$ and V_j is dense in $\mathscr{K}^d$. Similar to Lemma 2, we can show that V_j is open in $(\mathscr{K}^d, \rho)$. Let $V = \bigcap_{j \geqslant 1} V_j$. Then $\mathscr{K}^d \setminus V$ is of first category.

Now for any $K \in V$, if $K \cap B_r(x) \neq \varnothing$, choose $y \in K \cap B_r(x)$. Let $r_1 = \| x - y \|$. Then $0 \leqslant r_1 < r$. Choose $l \in N$ large enough such that $r - r_1 > 2^{-l}$. Then for any $j > l$ and

$X \in J_1^d$, since $X \cap B_r(x) = \varnothing$, we have $\rho(X, \{y\}) \geq r - r_1 > 2^{-l}$. Therefore $\rho(X, K) > 2^{-j}$, thus $K \notin V_j^{(1)}$, so $K \in V_j^{(2)}$. By the definition of $V_j^{(2)}$. There exists $E \in \mathscr{F}$ such that

$$\rho(K, E) < \delta_j(E),$$

so

$$V_{\delta_j(E)}(K) \supset E \supset B_r(x) \cap E.$$

From the discussion above, we see that for any $u \in B_r(x) \cap E$, there exists $v \in K$ such that $\| u - v \| \leq \delta_j(E)$. Therefore

$$\| v - x \| \leq \| v - u \| + \| u - x \| \leq \delta_j(E) + r < 2r.$$

So $v \in K \cap B_{2r}(x)$, $V_{\delta_j(E)}(K \cap B_{2r}(x)) \supset E \cap B_r(x)$.

By lemma 1, we have

$$N_{\delta_j(E)}(K \cap B_{2r}(x)) \geq 4^{-d} N_{\delta_j(E)}(V_{\delta_j(E)}(K \cap B_{2r}(x))) \geq 4^{-d} N_{\delta_i(E)}(E \cap B_r(x)). \tag{3}$$

By (2) and (3),

$$\frac{\log N_{\delta_j(E)}(K \cap B_{2r}(x))}{-\log \delta_j(E)} \geq \frac{-2d}{j} + \frac{\log N_{\delta_j(E)}(E \cap B_r(x))}{-\log \delta_j(E)} \geq -\frac{2d}{j} + d - 2^{-j}. \tag{4}$$

By an analogous argument as in Lemma 2, we have

$$\overline{\dim}_B(K \cap B_{2r}(x)) \geq \lim_{j \to \infty}\left(\frac{-2d}{j} + d - 2^{-j}\right) = d.$$

Proposition 1. *For almost all $K \in \mathscr{K}^d$, we have*

$$\dim_p K = \overline{\dim}_B K = d.$$

Proof. Considering the open ball sequence $B_i(I)$, $i \in Q^+$, $I \in \mathbb{Q}^d$. By Lemma 3, for any $i \in Q^+$, $I \in \mathbb{Q}^d$, there exists $U_{i,I}$ such that $\mathscr{K} \setminus U_{i,I}$ is of first category and for any $F \in U_{i,I}$, either $F \cap B_i(I) = \varnothing$ or $\overline{\dim}_B(F \cap B_{2i}(I)) = d$.

Let

$$U = \bigcap_{i \in Q^+,\ I \in Q^d} U_{i,I}.$$

Then $\mathscr{K}^d \setminus U$ is of first category. For any $K \in U$ and any open set G in R^d such that $K \cap G \neq \varnothing$, there exist $i \in Q^+$, $I \in Q^d$ such that $K \cap B_i(I) \neq \varnothing$ and $B_{2i}(I) \subset G$. Thus

$$\overline{\dim}_B(K \cap G) \geq \overline{\dim}_B(K \cap B_{2i}(I)) = d.$$

Therefore, for any $K \in U$ and any open set G in R^d such that $K \cap G \neq \varnothing$, we have

$$\overline{\dim}_B K = \overline{\dim}_B(K \cap G) = d.$$

By Corollary 3.9 of ref [1], we have

$$\dim_P(K) = \overline{\dim}_B(K) = d.$$

Combining Lemma 2 with Proposition 1, we have the following theorem.

Theorem 1. *For almost all $K \in \mathscr{K}^d$,*

(i) $\dim_H K = \underline{\dim}_B K = 0$, $\dim_p K = \overline{\dim}_B K = d$;

(ii) *the Bouligand dimension of K does not exist*;

(iii) *K is not a regular set*;

(iv) *K is not an s-set.*

Remark 2. Let $h: R^+ \to R^+$ satisfy $\lim_{t \to 0} h(t) = 0$, $h(t) > 0$ if $t > 0$. Let $\mathscr{H}^h$ denote h-

Hausdorff measure[1]. Then for almost all $K\in\mathscr{K}^{d}$, $\mathscr{H}^{h}(K)=0$.

In fact, for any $E\in J^{d}$, we can construct a sequence $\{\delta_i(E)\}_{i\geqslant 1}$ such that $0<\delta_i(E)<2^{-i}$ and $h(2t)<(\# E)^{-1}2^{-i}$ if $0<t\leqslant\delta_i(E)$.

For any $j\geqslant 1$, let

$$V_j=\{F\in\mathscr{K}^{d}:\text{ there exists } E\in J^{d} \text{ such that } \rho(E,F)<\delta_j(E)\},\quad V=\bigcap_{j\geqslant 1}V_j.$$

Then $\mathscr{K}^{d}\setminus V$ is of first category.

For any $K\in V$, $j\in N$, there exists $E\in J^{d}$ such that

$$\rho(K,E)<\delta_j(E).$$

Therefore $\{B_{\delta_i(E)}(x)\}_{x\in E}$ is a $2\delta_i(E)$-cover of K. Thus

$$\mathscr{H}^{h}_{2\delta_i(E)}(K)\leqslant(\# E)h(2\delta_i(E))\leqslant 2^{-i}.$$

Letting $i\to\infty$, we have $\mathscr{H}^{h}(K)=0$.

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References

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2 Oxtoby, C., *Measure and Category*, New York: Springer-Verlag, 1980.

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