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The Hausdorff dimension of recurrent sets in symbolic spaces

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Abstract
Let (Σ, σ) be the one-sided shift space on m symbols. For any $x = (x_i)_{i\geqslant 1} \in \Sigma$ and positive integer n, define

$$R_n(x) = \inf\{j \geqslant n : x_1x_2\cdots x_n = x_{j+1}x_{j+2}\cdots x_{j+n}\}.$$

We prove that for each pair of numbers $\alpha, \beta \in [0, \infty]$ with $\alpha \leqslant \beta$, the following recurrent set

$$E_{\alpha,\beta} = \left\{x \in \Sigma : \liminf_{n\to\infty} \frac{\log R_n(x)}{n} = \alpha, \limsup_{n\to\infty} \frac{\log R_n(x)}{n} = \beta\right\}$$

has Hausdorff dimension one.

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1. Introduction

Let (Σ, σ) be the one-sided shift space on m symbols $1, 2, \ldots, m$ $(m \geqslant 2)$. A commonly used metric on Σ is given by

$$d(x, y) = m^{-\inf\{k\geqslant 0: x_{k+1}\neq y_{k+1}\}}$$

for $x = (x_i)_{i=1}^{\infty}$ and $y = (y_i)_{i=1}^{\infty}$.

For any $x = (x_i)_{i\geqslant 1} \in \Sigma$ and positive integer n, define

$$R_n(x) = \inf\left\{j \geqslant n : x_1x_2\cdots x_n = x_{j+1}x_{j+2}\cdots x_{j+n}\right\}$$

that is, $R_n(x)$ is the first moment $j \geqslant n$ such that $\sigma^j(x)$ belongs to the n-cylinder $I_n(x) = \{y \in \Sigma : y_i = x_i \text{ for } 1 \leqslant i \leqslant n\}$. It was proved by Ornstein and Weiss (see [OW]) that for each σ-invariant ergodic Borel probability measure μ on Σ,

$$\mu\left\{x \in \Sigma : \lim_{n\to\infty} \frac{\log R_n(x)}{n} = h_\mu(\sigma)\right\} = 1$$

here $h_\mu(\sigma)$ denotes the measure-theoretic entropy of μ with respect to σ.

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In this paper, we would like to consider a more subtle question: for each pair of numbers $\alpha, \beta \in [0, \infty]$ with $\alpha \leqslant \beta$, what is the size of the following set:

$$E_{\alpha,\beta} = \left\{ x \in \Sigma : \liminf_{n\to\infty} \frac{\log R_n(x)}{n} = \alpha, \limsup_{n\to\infty} \frac{\log R_n(x)}{n} = \beta \right\}.$$

Since these sets are very complicated, we only consider their Hausdorff dimensions. The reader can refer to Falconer's book [Fal] for the definition of Hausdorff dimension. We should point out that under the metric d, the space Σ is of Hausdorff dimension one.

Our answer to the above question is the following theorem, which is somewhat surprising since the collection $\{E_{\alpha,\beta}\}$ is an uncountable partition of the space Σ.

Theorem 1. *For any $\alpha, \beta \in [0, \infty]$ with $\alpha \leqslant \beta$, we have*

$$\dim_H E_{\alpha,\beta} = 1$$

where $\dim_H$ *denotes the Hausdorff dimension.*

The main idea of the proof is constructing Cantor-like subsets of $E_{\alpha,\beta}$ so that their Hausdorff dimensions converge to one.

2. Proof of theorem 1

The proof of theorem 1 is based on the following lemma:

Lemma 1. *Let $\{\ell_n\}$ be a sequence of positive integers such that (a) $\exists n_0$, $\ell_{n+1} \geqslant \ell_n + 2n$ for any $n \geqslant n_0$ and (b) $\lim_{n\to\infty} \ell_n/n^2 = \infty$. Then the following set*

$$A^{\{\ell_n\}} := \{x \in \Sigma : \exists k_0, R_k(x) = \ell_k \text{ for } k \geqslant k_0\}$$

has Hausdorff dimension one.

Proof. Write for brevity $A = A^{\{\ell_n\}}$. Since $\dim_H \Sigma = 1$, it suffices to show that $\dim_H A \geqslant 1-\delta$ for any $\delta > 0$.

Fix $\delta > 0$. Choose an integer $p \geqslant \max\{6, n_0\}$ so that $\frac{p-2}{p} > 1 - \delta$, and define

$$F_p = \{x = (x_i)_{i\geqslant 1} \in \Sigma : x_j = m \text{ for } 1 \leqslant i \leqslant p, x_{pk+1} = x_{pk+p} = 1 \text{ for } k \geqslant 1\}.$$

Since the set $\{x = (x_i)_{i\geqslant 1} \in \Sigma : x_{pk+1} = x_{pk+p} = 1 \text{ for } k \geqslant 0\}$ can be viewed as a self-similar set generated by m^{p-2} many similitudes with ratio m^{-p}, its Hausdorff dimension is equal to

$$\frac{\log m^{p-2}}{\log m^p} = \frac{p-2}{p}$$

therefore $\dim_H F_p = \frac{p-2}{p} > 1 - \delta$.

In what follows, we will construct an injective map g from F_p to A such that g^{-1} is nearly Lipschitz on $g(F_p)$, i.e. $\forall\epsilon > 0$, $\exists M$ such that

$$d(g(x), g(y)) < m^{-k} \text{ implies } d(x, y) < m^{-(1-\epsilon)k}$$

for $k \geqslant M$. This means that $\dim_H g(F_p) \geqslant \dim_H F_p$ (see proposition 2.3 of [Fal]) and thus, $\dim_H A \geqslant \dim_H g(F_p) \geqslant \dim_H F_p \geqslant 1 - \delta$, as we desired.

For each $x = (x_i)_{i\geqslant 1} \in F_p$, we construct $x^* \in \Sigma$ by induction. Define $x^{(p-1)} = x$. Suppose we have defined $x^{(j)} = x_1^{(j)} x_2^{(j)} \cdots x_n^{(j)} \cdots$ for $p - 1 \leqslant j \leqslant k - 1$, we define $x^{(k)}$ as follows:

$$x^{(k)} = x_1^{(k-1)} x_2^{(k-1)} \cdots x_{l_k-1}^{(k-1)} \circ w_k \circ x_{l_k}^{(k-1)} x_{l_k+1}^{(k-1)} \cdots$$

where w_k is a word of length $k+3$ such that

$$w_k = 1x_1^{(k-1)}x_2^{(k-1)}\cdots x_k^{(k-1)}y_k1$$

here y_k is a letter not equal to $x_{k+1}^{(k-1)}$. That is, $x^{(k)}$ is obtained by inserting a word w_k into $x^{(k-1)}$ at the place ℓ_k.

The sequence $\{x^{(i)}\}_{i\geqslant p}$, defined above, satisfies the following properties:

(a) For any $k \geqslant p$, the prefix $x_1^{(k-1)}x_2^{(k-1)}\cdots x_{l_k-1}^{(k-1)} \circ w_k$ of $x^{(k)}$ is also the prefix of $x^{(k+1)}$. This is deduced from the condition $\ell_{k+1} \geqslant \ell_k + 2k$. Thus the sequence $\{x^{(i)}\}_{i\geqslant p}$ converges to a point x^*. Moreover, the first $\ell_{k+1} - 1 \geqslant \ell_k + 2k - 1$ symbols of x^* and $x^{(k)}$ coincide.

(b) For each $k \geqslant p$, $R_k(x^*) = l_k$.

To see property (b), note first that $x_1^{(k-1)}x_2^{(k-1)}\cdots x_{l_k-1}^{(k-1)} \circ w_k$ is the prefix of x^*, thus by the construction of w_k we have $d(\sigma^{\ell_k}(x^*), x^*) = m^{-k}$ for each $k \geqslant p$. Let

$$\underline{\theta} = \underbrace{mm\cdots m}_{p}.$$

The block $\underline{\theta}$ can only appear in the words w_k and the beginning of $x^{(k)}$ for each $k \geqslant p$. Combining this with the construction of $x^{(p)}$, we have $R_p(x^*) = \ell_p$. To finish the proof of (b), it is enough to show that for any $k \geqslant p$, $d(\sigma^{\ell_k+\mathrm{i}}(x^*), x^*) > m^{-k}$ for any $1 \leqslant i < \ell_{k+1} - \ell_k$. We prove this below by contradiction. Assume that $d(\sigma^{\ell_k+\mathrm{i}}(x^*), x^*) \leqslant m^{-k}$ for some $k \geqslant p$ and some $1 \leqslant i < \ell_{k+1} - \ell_k$. Note that $\underline{\theta}$ does not appear in the word $x_{\ell_k}^{(k-1)}x_{\ell_k+1}^{(k-1)}\cdots x_{\ell_{k+1}-1}^{(k-1)}$, we must have $i \leqslant k-1$ (using the fact that $y_k1x_{\ell_k}^{(k-1)}x_{\ell_k+1}^{(k-1)}\cdots x_{\ell_{k+1}-k-4}^{(k-1)}1$ is the prefix of $\sigma^{\ell_k+k}(x^*)$). Thus

$$x_{i+1}^{(k-1)}x_{i+2}^{(k-1)}\cdots x_k^{(k-1)}y_k1x_{\ell_k}^{(k-1)}x_{\ell_k+1}^{(k-1)}\cdots x_{\ell_k+\mathrm{i}-3}^{(k-1)} = x_1^{(k-1)}x_2^{(k-1)}\cdots x_k^{(k-1)}. \quad (1)$$

Since the left-hand side of (1) contains no copy of the blocks $\underline{\theta}$ and

$$\underbrace{mm\cdots m}_{p-1}$$

beginning after y_k, the word $x_1^{(k-1)}x_2^{(k-1)}\cdots x_k^{(k-1)}y_k1x_{\ell_k}^{(k-1)}x_{\ell_k+1}^{(k-1)}\cdots x_{\ell_k+\mathrm{i}-3}^{(k-1)}$ contains at most one more copy of $\underline{\theta}$ than $x_1^{(k-1)}x_2^{(k-1)}\cdots x_k^{(k-1)}$. In view of the fact that the number of times that $\underline{\theta}$ appears as a subword must be the same in both sides of (1), we must have $x_{k-p+2}^{(k-1)}x_{k-p+3}^{(k-1)}\cdots x_k^{(k-1)}y_k = \underline{\theta}$. Therefore,

$$x_{k-p+2}^{(k-1)}x_{k-p+3}^{(k-1)}\cdots x_k^{(k-1)} = \underbrace{mm\cdots m}_{p-1}.$$

Hence the block

$$\underbrace{mm\cdots m}_{p-1}$$

is the suffix of the word $x_{i+1}^{(k-1)}x_{i+2}^{(k-1)}\cdots x_k^{(k-1)}y_k1x_{\ell_k}^{(k-1)}x_{\ell_k+1}^{(k-1)}\cdots x_{\ell_k+\mathrm{i}-3}^{(k-1)}$, which is a contradiction to the fact that the block

$$\underbrace{mm\cdots m}_{p-1}$$

does not appear in the word $x_{\ell_k}^{(k-1)}x_{\ell_k+1}^{(k-1)}\cdots x_{\ell_k+\mathrm{i}-3}^{(k-1)}$. Thus we have proved that $R_k(x^*) = \ell_k$ for $k \geqslant p$.

Define $g : F_p \to A$ by $x \mapsto x^*$. It is clear from the construction that g is injective. In the following we show that g^{-1} is nearly Lipschitz on $g(F_p)$, i.e. $\forall \epsilon > 0$, $\exists M$ such that

$$d(x^*, y^*) < m^{-k} \qquad \text{implies} \quad d(x, y) < m^{-(1-\epsilon)k}$$

for $k \geqslant M$.

Since $\lim_{n\to\infty} \ell_n/n^2 = \infty$, there exists an integer $N > p$ such that $n^2/\ell_n < \epsilon/2$ for any $n \geqslant N$. Let $M = \ell_N$. If $d(x^*, y^*) < m^{-k}$ for some integer $k \geqslant M$, then we have $x_1^* x_2^* \cdots x_k^* = y_1^* y_2^* \cdots y_k^*$. Let q be the integer so that $\ell_q \leqslant k \leqslant \ell_{q+1}$. Since $k \geqslant \ell_N$, it follows immediately $q \geqslant N > p$. By the construction of x^* and y^*, we must have

$$x_1 x_2 \cdots x_{k'} = y_1 y_2 \cdots y_{k'}$$

where $k' = k - \sum_{j=p}^{q+1} (j+3)$. Note that

$$k' > k - 2q^2 \geqslant k - \epsilon \ell_q \geqslant k(1-\epsilon)$$

we have $d(x, y) \leqslant m^{-k'} \leqslant m^{-(1-\epsilon)k}$, as we desired. □

Proof of theorem 1. By lemma 1, to prove the theorem it suffices to show that for any given $\alpha, \beta \in [0, \infty]$ with $\alpha \leqslant \beta$ there exists a sequence of positive integers $\{\ell_n\}$ such that it satisfies the conditions (a) and (b) of lemma 1 and

$$\liminf_{n\to\infty} \frac{\log \ell_n}{n} = \alpha \qquad \limsup_{n\to\infty} \frac{\log \ell_n}{n} = \beta.$$

In the following we will give concrete constructions of $\{\ell_n\}$ for different cases.

Case 1. $\alpha = \beta = \infty$. In this case, define $\ell_n = [\mathrm{e}^{n^2}]$, here and afterwards $[x]$ denote the integral part of x.

Case 2. $0 < \alpha = \beta < \infty$. In this case, define $\ell_n = [\mathrm{e}^{n\alpha}]$.

Case 3. $\alpha = \beta = 0$. In this case, define $\ell_n = [\mathrm{e}^{\sqrt{n}}]$.

Case 4. $0 < \alpha < \beta < \infty$. In this case, set

$$u_n = \begin{cases} [\mathrm{e}^{n\alpha}] & \text{if } \sum_{j=1}^{2i-1} 2^{4^j} \leqslant n < \sum_{j=1}^{2i} 2^{4^j} \text{ for some integer } i > 0 \\ [\mathrm{e}^{n\beta}] & \text{otherwise} \end{cases}$$

and define $\ell_n = \sum_{i=1}^{n} u_i$.

Case 5. $0 < \alpha < \beta = \infty$. In this case, define

$$u_n = \begin{cases} [\mathrm{e}^{n\alpha}] & \text{if } \sum_{j=1}^{2i-1} 2^{4^j} \leqslant n < \sum_{j=1}^{2i} 2^{4^j} \text{ for some integer } i > 0 \\ [\mathrm{e}^{n^2}] & \text{otherwise} \end{cases}$$

and define $\ell_n = \sum_{i=1}^{n} u_i$.

Case 6. $0 = \alpha < \beta < \infty$. In this case, set

$$u_n = \begin{cases} [e^{\sqrt{n}}] & \text{if } \sum_{j=1}^{2i-1} 2^{4^j} \leqslant n < \sum_{j=1}^{2i} 2^{4^j} \text{ for some integer } i > 0. \\ [e^{n\beta}] & \text{otherwise} \end{cases}$$

and define $\ell_n = \sum_{i=1}^{n} u_i$.

Case 7. $0 = \alpha, \beta = \infty$. Define

$$u_n = \begin{cases} [e^{\sqrt{n}}] & \text{if } \sum_{j=1}^{2i-1} 2^{4^j} \leqslant n < \sum_{j=1}^{2i} 2^{4^j} \text{ for some integer } i > 0 \\ [e^{n^2}] & \text{otherwise} \end{cases}$$

and define $\ell_n = \sum_{i=1}^{n} u_i$. □

Remark 1. *Theorem 1 can be generalized to transitive subshifts of finite type.*

Let $A = (a_{i,j})$ be an $m \times m$ matrix with $a_{i,j} \in \{0, 1\}$. Define

$$\Sigma_A = \{(x_n) \in \Sigma : a_{x_n,x_{n+1}} = 1, \ \forall n \geqslant 1\}.$$

Note that $\sigma\Sigma_A \subset \Sigma_A$. The system (Σ_A, σ) is called a *subshift of finite type*. Suppose further that all the entries of A^M are strictly positive for some $M \geqslant 1$. Then the subshift is said to be *(topologically) transitive*. Using the similar argument, we can obtain $\dim_H E_{\alpha,\beta} = \dim_H \Sigma_A = \frac{\log \rho}{\log m}$, where ρ is the spectral radius of A.

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