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Comparing Packing Measures to Hausdorff Measures on the Line

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Abstract. For each $0 < s < 1$, define

$$c(s) \;=\; \inf_E \frac{\mathcal{P}^s(E)}{\mathcal{H}^s(E)}\,,$$

where $\mathcal{P}^s$, $\mathcal{H}^s$ denote respectively the s–dimensional packing measure and Hausdorff measure, and the infimum is taken over all the sets $E \subset \mathbf{R}$ with $0 < \mathcal{H}^s(E) < \infty$. In this paper we give a nontrivial estimation of $c(s)$, namely, $2^s\,(1+v(s))^s \le c(s) \le 2^s \left(2^{\frac{1}{s}} - 1\right)^s$ for each $0 < s < 1$, where $v(s) = \min\left\{16^{-\frac{1}{1-s}},\, 8^{-\frac{1}{(1-s)^2}}\right\}$. As an application, we obtain a lower density theorem for Hausdorff measures.

1. Introduction

In this paper, we will compare packing measures to Hausdorff measures on the line. For given $E \subset \mathbf{R}^n$, a δ–packing of the set E is a countable family of disjoint closed balls of radii at most δ and with centers in E. For $s \ge 0$, the s–dimensional packing premeasure of E is defined as

$$P_0^s(E) \;=\; \inf_{\delta>0} \left\{ P_\delta^s(E) \right\},$$

where $P_\delta^s(E) = \sup\left\{ \sum_{B_i \in \mathcal{R}} |B_i|^s \;:\; \mathcal{R} \text{ is a } \delta\text{–packing of } E \right\}$ and $|B_i|$ denotes the diameter of B_i. The s–dimensional *packing measure* of E is defined as

$$\mathcal{P}^s(E) \;=\; \inf\left\{ \sum_{i=1}^{\infty} P_0^s(E_i) :\; E \subset \bigcup_{i=1}^{\infty} E_i \right\}.$$

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The *packing dimension* $\dim_{\mathcal{P}}$ is induced by packing measures by

$$\dim_{\mathcal{P}}(E) \;=\; \inf\{s \ge 0 :\ \mathcal{P}^s(E) = 0\} \;=\; \sup\{s \ge 0 :\ \mathcal{P}^s(E) = \infty\}\,.$$

The packing measure and packing dimension were introduced by TRICOT [13], TAYLOR and TRICOT [11, 12] and SULLIVAN [10]. As parameters to describe non–smooth sets, the packing measure and packing dimension play an important role in the study of fractal geometry in a manner dual to the Hausdorff measure and Hausdorff dimension (see [3, 8] for the definitions of Hausdorff measure and Hausdorff dimension). Let $\mathcal{H}^s(E)$ and $\dim_H(E)$ denote the s–dimensional Hausdorff measure and Hausdorff dimension respectively. It was proved in [9, 12] that $\mathcal{P}^s(E) \ge \mathcal{H}^s(E)$ for each $E \subset \mathbf{R}^n$ and $s \ge 0$. The condition $0 < \mathcal{P}^s(E) = \mathcal{H}^s(E) < \infty$ is a strong restriction: it implies that s must be an integer and $\mathcal{H}^s$–almost all of E can be covered with countably many s–dimensional C^1 submanifolds.

A natural question arises: if $s < n$ is not an integer, then what is the infimum of ratios $\mathcal{P}^s(E)/\mathcal{H}^s(E)$ where E runs over the subsets of $\mathbf{R}^n$ with $0 < \mathcal{H}^s(E) < \infty$? Denote by $c(s,n)$ this infimum. In this paper, we consider the case $n = 1$ where we write for simplicity $c(s) = c(s,1)$. The main result is the following.

Theorem 1.1. *For* $0 < s < 1$*, we have*

$$2^s(1 + v(s))^s \;\le\; c(s) \;\le\; 2^s\Big(2^{\frac{1}{s}} - 1\Big)^s\,,$$

where $v(s) = \min\Big\{16^{-\frac{1}{1-s}},\, 8^{-\frac{1}{(1-s)^2}}\Big\}$.

As an application, we obtain a lower density theorem for Hausdorff measures on the line as follows:

Theorem 1.2. *Given* $0 < s < 1$*, let* $F \subset \mathbf{R}$ *be a Borel set such that* $\mathcal{H}^s(F) < \infty$*. Then for* $\mathcal{H}^s$*–almost all* $x \in F$*,*

$$\liminf_{r \downarrow 0} \frac{\mathcal{H}^s(F \cap (x - r, x + r))}{(2r)^s} \;\le\; c(s)^{-1} \;\le\; 2^{-s}(1 + v(s))^{-s},$$

where $v(s)$ *is defined as in Theorem* 1.1.

It is well known that under the condition of Theorem 1.2,

$$\liminf_{r \downarrow 0} \frac{\mathcal{H}^s(F \cap (x - r, x + r))}{(2r)^s} \;\le\; 2^{-s} \quad \text{for } \mathcal{H}^s\text{–almost all } x \in F\,;$$

indeed this was first proved by BESCOVITCH [1] (it follows immediately from his estimates for one–sided upper and lower densities of Hausdorff measures). A similar estimate in a more general setting was obtained by MATTILA [7].

2. Proof of the main results

Fix $0 < s < 1$. First we give an elementary lemma.

Lemma 2.1. *Let $E \subset [a,b]$. Suppose $l \le \frac{b-a}{6}$ is a positive number such that $E \cap I \ne \emptyset$ for any subinterval I of $[a,b]$ with $|I| \ge l$. Then there exists a sequence of disjoint subintervals $\{I_i\}_{i=1}^m$ of $[a,b]$ with centers in E, such that*

$$\sum_{i=1}^m |I_i|^s \ \ge\ \frac{1}{3}(b-a)l^{s-1}.$$

Proof. Let l be a number satisfying the condition of the lemma, and M the unique integer such that $\frac{b-a}{l} \in (2M-1, 2M]$. The condition $\frac{b-a}{l} \ge 6$ implies $M-1 \ge \frac{b-a}{3l}$. For each positive integer $1 \le i \le M-1$, pick one point x_i from $E \cap (a+(2i-1)l,\ a+2il)$. The intervals $I_i = [x_i - l/2, x_i + l/2]$ are disjoint subintervals of $[a,b]$, satisfying

$$\sum_i |I_i|^s \ =\ (M-1)\cdot l^s \ \ge\ \frac{b-a}{3l}\, l^s \ =\ \frac{1}{3}(b-a)l^{s-1}. \qquad \square$$

Define $u = 8^{-\frac{1}{1-s}}$ and $v = \min\left\{16^{-\frac{1}{1-s}},\ 8^{-\frac{1}{(1-s)^2}}\right\}$. It is easily checked that

$$v^{s-1}\cdot \min\left\{u^s,\ \frac{1}{2}\right\} \ \ge\ 8. \tag{2.1}$$

Proposition 2.2. *Suppose $K \subset [a,b]$ is a Borel set with $0 < \mathcal{H}^s(K) < +\infty$. Then there exists a finite sequence of disjoint subintervals $\{I_i\}_{i=1}^m$ of $[a,b]$ with centers in K, such that*

(i) $\sum_{i=1}^m |I_i|^s > \frac{1}{6} uv\mathcal{H}^s(K)$.

(ii) $\sum_{i=1}^m |I_i|^s > 2^s(1+v)^s \cdot \sum_{i=1}^m \mathcal{H}^s(K \cap I_i)$.

Proof. Take $\epsilon > 0$ so that

$$1+\epsilon \ <\ \min\left\{\frac{16}{15},\ \left(1-v^2\right)^{-s/2}\right\}.$$

We shall first prove a version of the proposition under an extra assumption, that, in addition to the hypotheses of the proposition,

$$\mathcal{H}^s(K\cap I) \ <\ (1+\epsilon)\,|I|^s \tag{2.2}$$

for each interval $I \subset [a,b]$. We will prove that under this assumption there exists a finite sequence of disjoint subintervals $\{I_i\}_{i=1}^m$ of $[a,b]$ with centers in K, such that

(iii) $\sum_{i=1}^m |I_i|^s > \frac{1}{3} uv\mathcal{H}^s(K)$.

(iv) $\sum_{i=1}^m |I_i|^s > (1+\epsilon)2^s(1+v)^s \cdot \sum_{i=1}^m \mathcal{H}^s(K \cap I_i)$.

Define $l_1 = u \cdot (b-a)$. There are two possible cases:
(a) $K \cap I \neq \emptyset$ for any subinterval I of $[a,b]$ with $|I| \geq l_1$.
(b) there exists $(c,d) \subset [a,b]$ with $d-c > l_1$ so that $K \cap (c,d) = \emptyset$ and $c,d \in K$.

For the case (a), by Lemma 2.1 and (2.2), there exists a finite sequence of disjoint subintervals $\{I_i\}_{i=1}^m$ of $[a,b]$ with centers in K, such that

$$\sum_{i=1}^m |I_i|^s \;\geq\; \frac{1}{3}\,(b-a)\cdot[u(b-a)]^{s-1} \;=\; \frac{1}{3}(b-a)^s u^{s-1} \;=\; \frac{8}{3}\,(b-a)^s \;>\; \frac{5}{2}\,\mathcal{H}^s(K)$$

from which (iii) and (iv) follow immediately.

For the case (b), we may assume without loss of generality that $\mathcal{H}^s(K \cap [a,c]) \geq \frac{1}{2}\,\mathcal{H}^s(K)$. Thus by (2.2),

$$(c-a)^s \;>\; \frac{1}{1+\epsilon}\,\mathcal{H}^s\big(K \cap [a,c]\big) \;>\; \frac{1}{2(1+\epsilon)}\,\mathcal{H}^s(K)\,.$$

Let $h = \min\{d-c,\, c-a\}$. Noting that $d-c > l_1 = u\cdot(b-a)$, we have

$$\begin{aligned} h^s &= \min\big\{(d-c)^s, (c-a)^s\big\} \\ &> \min\left\{u^s(b-a)^s,\; \frac{1}{2(1+\epsilon)}\,\mathcal{H}^s(K)\right\} \\ &\geq \min\left\{u^s\cdot\frac{1}{(1+\epsilon)}\,\mathcal{H}^s(K),\; \frac{1}{2(1+\epsilon)}\,\mathcal{H}^s(K)\right\} \\ &\geq \min\left\{u^s, \frac{1}{2}\right\}\cdot\frac{1}{1+\epsilon}\,\mathcal{H}^s(K)\,. \end{aligned} \tag{2.3}$$

Define $l_2 = vh$. There are again two possible cases:
(b1) $K \cap I \neq \emptyset$ for every subinterval I of $[c-h,c]$ with $|I| \geq l_2$.
(b2) there exists $(e,f) \subset [c-h,c]$ with $f-e \geq l_2$ so that $K \cap (e,f) = \emptyset$ and $e,f \in K$.

For the case (b1), by Lemma 2.1, (2.2) and (2.3), there exists a finite sequence of disjoint subintervals $\{I_i\}_{i=1}^m$ of $[c-h,c]$ with centers in K, such that

$$\begin{aligned} \sum_{i=1}^m |I_i|^s \;>\; \frac{1}{3}\,h\cdot(vh)^{s-1} &= \frac{1}{3}\,h^s v^{s-1} \\ &\geq v^{s-1}\cdot\min\left\{u^s, \frac{1}{2}\right\}\frac{1}{3(1+\epsilon)}\,\mathcal{H}^s(K) \\ &\geq \frac{8}{3(1+\epsilon)}\,\mathcal{H}^s(K) \\ &> \frac{5}{2}\,\mathcal{H}^s(K)\,. \end{aligned} \tag{2.4}$$

from which (iii) and (iv) follow immediately. For the case (b2), let $I_1 = [e, 2c-e]$. Then the center of I_1 is c which is contained in K, and $K\cap I_1 = K \cap [e,c] = (K \cap [f,c]) \cup \{e\}$. By (2.3),

$$|I_1|^s \;\geq\; (f-e)^s \;\geq\; v^s h^s \;\geq\; \frac{1}{3}\,v^s u^s \mathcal{H}^s(K) \;>\; \frac{1}{3}\,uv\mathcal{H}^s(K)$$

from which (iii) follows. Note also that

$$c - f \;=\; c - e - (f - e) \;\le\; c - e - hv \;\le\; (c - e)(1 - v)\,,$$

so by (2.2), we have

$$\begin{aligned}\mathcal{H}^s(K \cap I_1) &= \mathcal{H}^s(K \cap [f, c]) \\ &\le (1+\epsilon)(c-f)^s \\ &\le (1+\epsilon)(c-e)^s(1-v)^s \\ &= 2^{-s}\,|I_1|^s(1+\epsilon)(1-v)^s \\ &< 2^{-s}\,|I_1|^s\,\frac{1}{(1+\epsilon)(1+v)^s}\end{aligned}$$

from which (iv) follows.

We have proved the stronger results (iii) and (iv) under the assumption (2.2), and we now get rid of this extra assumption. For each positive integer n, define

$$K_n \;=\; \left\{x \in K :\; \mathcal{H}^s(K \cap I) \le (1+\epsilon)\,|I|^s \text{ for all intervals } I \ni x \text{ with } |I| < \frac{1}{n}\right\}.$$

Then $\{K_n\}$ is a sequence of Borel sets with $K_n \subset K_{n+1}$, and $\lim_{n\to\infty} \mathcal{H}^s(K_n) = \mathcal{H}^s(K)$ (see Theorem 2.3 of [2] for a proof).

For a fixed positive integer n, choose an integer $M > n(b-a)$. For each $1 \le j \le M$, if

$$\mathcal{H}^s\left(K_n \cap \left[a + (j-1)\,\frac{b-a}{M},\; a + j\,\frac{b-a}{M}\right]\right) \;>\; 0\,,$$

then Proposition 2.2 (with the stronger conclusions (iii) and (iv)) remains true when K and $[a, b]$ are replaced by

$$K'_{n,j} \;:=\; K_n \cap \left[a + (j-1)\,\frac{b-a}{M},\; a + j\,\frac{b-a}{M}\right]$$

and

$$\left[a', b'\right] \;:=\; \left[a + (j-1)\,\frac{b-a}{M},\; a + j\frac{b-a}{M}\right]$$

respectively, since $K'_{n,j}$ satisfies (2.2). Denote by $\mathcal{A}_{n,j}$ a collection of disjoint subintervals I_i of $[a', b']$ with centers in $K'_{n,j}$ such that (iii) and (iv) hold, where K is replaced by $K'_{n,j}$. Let

$$\mathcal{A}_n = \bigcup_j \mathcal{A}_{n,j}$$

where j is taken so that $\mathcal{H}^s\big(K_n \cap \big[a + (j-1)\,\frac{b-a}{M}, a + j\,\frac{b-a}{M}\big]\big) > 0$. It is clear that the intervals in $\mathcal{A}_n$ satisfy (iii) and (iv) where K is replaced by K_n. Set $\alpha = \frac{1}{1+\epsilon}$. Take a large n so that

$$\mathcal{H}^s(K) - \mathcal{H}^s(K_n) \;\le\; \frac{1}{6}\,uv(1-\alpha)2^{-s}(1+v)^{-s}\mathcal{H}^s(K)\,.$$

It is clear the intervals in $\mathcal{A}_n$ satisfy (i); in what follows we show that they also satisfy (ii). To see this, we note that

$$\begin{aligned}\sum_{I\in\mathcal{A}_n}|I|^s &> (1-\alpha)\,\frac{1}{3}\,uv\mathcal{H}^s(K_n)+\alpha(1+\epsilon)2^s(1+v)^s\sum_{I\in\mathcal{A}_n}\mathcal{H}^s(K_n\cap I)\\ &\geq (1-\alpha)\,\frac{1}{6}\,uv\mathcal{H}^s(K)+2^s(1+v)^s\sum_{I\in\mathcal{A}_n}\mathcal{H}^s(K\cap I)\\ &\quad -2^s(1+v)^s(\mathcal{H}^s(K)-\mathcal{H}^s(K_n))\\ &\geq 2^s(1+v)^s\sum_{I\in\mathcal{A}_n}\mathcal{H}^s(K\cap I)\,,\end{aligned}$$

which concludes the proof. □

Remark 2.3. It is clear that Proposition 2.2 remains true if the interval $[a,b]$ therein is replaced by any set U which is the union of finitely many intervals.

Proposition 2.4. *Suppose $K\subset[a,b]$ is a Borel set with $0<\mathcal{H}^s(K)<+\infty$. Then there exists a finite or infinite sequence of disjoint subintervals $\{I_i\}_i$ of $[a,b]$ with centers in K, such that*

$$\sum_i |I_i|^s \;>\; 2^s(1+v)^s\mathcal{H}^s(K)\,.$$

Proof. Write for simplicity $r=\frac{1}{6}\,uv$ and $d=2^s(1+v)^s$. Assume the conclusion is not true, that is, for each sequence of disjoint intervals $\{I_i\}_i$ of $[a,b]$ with centers in K, $\sum_i|I_i|^s\leq d\,\mathcal{H}^s(K)$; in the following this will lead to a contradiction.

By Proposition 2.2, we can construct a collection $\mathcal{A}_1$ of finitely many disjoint subintervals of $[a,b]$ with centers in K, such that $\sum_{I\in\mathcal{A}_1}|I|^s>r\,\mathcal{H}^s(K)$ and $\sum_{I\in\mathcal{A}_1}|I|^s> d\,\mathcal{H}^s\big(K\cap\big(\bigcup_{I\in\mathcal{A}_1}I\big)\big)$. Define $V_1=\bigcup_{I\in\mathcal{A}_1}I$ and $U_1=[a,b]\backslash V_1$. Since $\sum_{I\in\mathcal{A}_1}|I|^s\leq d\,\mathcal{H}^s(K)$ by the assumption, we conclude that $\mathcal{H}^s(K\cap U_1)>0$.

By Proposition 2.2 and Remark 2.3, we can construct a collection $\mathcal{A}_2$ of finitely many disjoint subintervals of U_1 with centers in $K\cap U_1$, such that $\sum_{I\in\mathcal{A}_2}|I|^s> r\mathcal{H}^s(K\cap U_1)$ and $\sum_{I\in\mathcal{A}_2}|I|^s>d\,\mathcal{H}^s\big(K\cap\big(\bigcup_{I\in\mathcal{A}_2}I\big)\big)$. Define $V_2=\bigcup_{I\in\mathcal{A}_2}I$ and $U_2=[a,b]\backslash(V_1\cup V_2)$. Since $\sum_{I\in\mathcal{A}_1\cup\mathcal{A}_2}|I|^s\leq d\,\mathcal{H}^s(K)$ by the assumption, and $\sum_{I\in\mathcal{A}_1\cup\mathcal{A}_2}|I|^s>d\,\mathcal{H}^s(K\cap(V_1\cup V_2))$, we conclude $\mathcal{H}^s(K\cap U_2)>0$.

Continuing the above procedure, we obtain a sequence of collections $\mathcal{A}_n$ and sets V_n and U_n. For each n, $U_n=[a,b]\backslash(\bigcup_{i=1}^n V_i)$, and $\mathcal{A}_{n+1}$ is a collection of finitely many disjoint subintervals of U_n such that $\sum_{I\in\mathcal{A}_{n+1}}|I|^s>r\mathcal{H}^s\big(K\cap U_n\big)$ and $\sum_{I\in\mathcal{A}_{n+1}}|I|^s> d\mathcal{H}^s\big(K\cap\big(\bigcup_{\mathcal{A}_{n+1}}I\big)\big)$ and $V_{n+1}=\bigcup_{\mathcal{A}_{n+1}}I$.

Since $U_{n+1}\subset U_n$ for each n, it follows that the limit $\lim_{n\to\infty}\mathcal{H}^s\big(K\cap U_n\big)$ exists. If this limit is 0, then $\sum_{I\in\bigcup_{i=1}^\infty\mathcal{A}_i}|I|^s>d\,\mathcal{H}^s\big(K\cap\big(\bigcup_{i=1}^\infty V_i\big)\big)=d\mathcal{H}^s(K)$ which contradicts the assumption. If the limit is positive, then

$$\sum_{I\in\mathcal{A}_{n+1}}|I|^s \;>\; r\mathcal{H}^s(K\cap U_n)\;\geq\; r\lim_{n\to\infty}\mathcal{H}^s(K\cap U_n)\;>\;0\,,\quad\text{for all}\quad n$$

from which we have $\sum_{I\in\bigcup_{i=1}^{\infty}\mathcal{A}_i}|I|^s=\infty$, which also contradicts the assumption. □

Corollary 2.5. *For each set $E\subset\mathbf{R}$, we have*

$$P_0^s(E)\ \geq\ 2^s(1+v)^s\mathcal{H}^s(E)\,, \tag{2.5}$$

where P_0^s denotes the s–dimensional packing premeasure.

Proof. Denote by $\overline{E}$ the closure of E. Since $P_0^s(E)=P_0^s(\overline{E})\geq\mathcal{H}^s(\overline{E})$, we may assume $0<\mathcal{H}^s(\overline{E})<\infty$.

Let n be a positive integer. By Proposition 2.4, we know that for each integer l, either $\mathcal{H}^s\big(\overline{E}\cap\big[\frac{l}{n},\frac{l+1}{n}\big]\big)=0$ or there exists a sequence (finite or infinite) of disjoint subintervals $\{I_i\}_i$ of $\big[\frac{l}{n},\frac{l+1}{n}\big]$ with centers in $\overline{E}$ such that

$$\sum_i|I_i|^s\ >\ 2^s(1+v)^s\mathcal{H}^s\left(\overline{E}\cap\left[\frac{l}{n},\frac{l+1}{n}\right]\right).$$

Letting l run through $\mathbf{Z}$, we deduce that there exists a sequence (finite or infinite) of disjoint intervals $\{J_i\}_i$ of length less than $\frac{1}{n}$ and with centers in $\overline{E}$ such that

$$\sum_i|J_i|^s\ >\ 2^s(1+v)^s\mathcal{H}^s\big(\overline{E}\big)\,;$$

this implies that $P_{1/n}^s(E)=P_{1/n}^s\big(\overline{E}\big)>2^s(1+v)^s\mathcal{H}^s\big(\overline{E}\big)$. Letting n tends to infinity we get the desired result. □

From the above Corollary and the definition of packing measure, we have immediately the following

Corollary 2.6. *For each set $E\subset\mathbf{R}$,*

$$\mathcal{P}^s(E)\ \geq\ 2^s(1+v)^s\mathcal{H}^s(E)\,.$$

Proof of Theorem 1.1. By Corollary 2.6, $c(s)\geq 2^s(1+v)^s$. Now let E_s denote the unique self–similar set generated by the iterated function system $\big\{\frac{1-\beta}{2}x,\ \frac{1-\beta}{2}x+\frac{1+\beta}{2}\big\}$ where $\beta=1-2^{1-\frac{1}{s}}$. The set E_s is termed as the β–center Cantor set for which the packing dimension and Hausdorff dimension coincide with the common value s. It is well known that $\mathcal{H}^s(E_s)=1$ (see e. g. page 15 of [2] for a proof). On the other hand, Feng [5] showed recently that $\mathcal{P}^s(E_s)=2^s\big(2^{\frac{1}{s}}-1\big)^s$. Thus $c(s)\leq 2^s\big(2^{\frac{1}{s}}-1\big)^s$. □

To prove Theorem 1.2, we need the following lemma. For a proof, see Proposition 2.2 of [4] or Theorem 6.11 of [8].

Lemma 2.7. *Let $K\subset\mathbf{R}$ be a Borel set, μ a finite Borel measure on $\mathbf{R}$ and $0<t<\infty$. If $\liminf_{r\to0}\mu((x-r,x+r))/(2r)^s\geq t$ for all $x\in K$ then $\mathcal{P}^s(K)\leq\mu(K)/t$.*

Proof of Theorem 1.2. Assume the theorem is false, then there exists a real number $d>c(s)^{-1}$ and Borel set $E\subset\mathbf{R}$ with $0<\mathcal{H}^s(E)<\infty$ such that there is a Borel set $F\subset E$ with $\mathcal{H}^s(F)>0$,

$$\liminf_{r\downarrow0}\frac{\mathcal{H}^s(E\cap(x-r,x+r))}{(2r)^s}\ \geq\ d$$

for all $x \in F$. Let $\mu = \mathcal{H}^s|_E$, i. e, $\mu(B) = \mathcal{H}^s(E \cap B)$ for all $B \subset \mathbf{R}$. Then by Lemma 2.7, we have $\mathcal{P}^s(F) \leq \mu(F)/d = \mathcal{H}^s(F)/d < c(s)\mathcal{H}^s(F)$, which contradicts the definition of $c(s)$. □

We end this section by an unsolved question.

Question. Is it true that $c(s) = 2^s\left(2^{\frac{1}{s}} - 1\right)^s$ for all $0 < s < 1$?

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