

THE PRESSURE FUNCTION FOR PRODUCTS OF NON-NEGATIVE MATRICES

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Abstract. Let (Σ_A, σ) be a subshift of finite type and let $M(x)$ be a continuous function on Σ_A taking values in the set of non-negative matrices. We extend the classical scalar pressure function to this new setting and prove the existence of the Gibbs measure and the differentiability of the pressure function. We are especially interested on the case where $M(x)$ takes finite values M_1, \dots, M_m . The pressure function reduces to $P(q) := \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{J \in \Sigma_{A,n}} \|M_J\|^q$. The expression is important when we consider the multifractal formalism for certain iterated function systems with overlaps.

1. Introduction

Let σ be the shift map on $\Sigma = \{1, 2, \dots, m\}^{\mathbb{N}}$, $m \geq 2$. As usual Σ is endowed with the metric $d(x, y) = m^{-n}$ where $x = (x_k)$, $y = (y_k)$ and n is the smallest of the k such that $x_k \neq y_k$. Given an $m \times m$ matrix A with entries 0 or 1, we consider the subshift of finite type (Σ_A, σ) (see [B]). We shall always assume that A is primitive.

Suppose M is a continuous function on Σ_A taking values in the set of all non-negative $d \times d$ matrices. For $q \in \mathbb{R}$, we define the pressure function $P(q)$ of M by

$$P(q) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{J \in \Sigma_{A,n}} \sup_{x \in [J]} \|M(x)M(\sigma x) \dots M(\sigma^{n-1}x)\|^q, \quad (1.1)$$

where $\Sigma_{A,n}$ denotes the set of all admissible indices of length n over $\{1, \dots, m\}$; for $J = j_1 \dots j_n \in \Sigma_{A,n}$, $[J]$ denotes the cylinder set $\{x = (x_i) \in \Sigma_A : x_i = j_i, 1 \leq i \leq n\}$, $\|\cdot\|$ denotes the matrix norm defined by $\|B\| := \mathbf{1}^t B \mathbf{1}$, $\mathbf{1}^t = (1, 1, \dots, 1)$. By using a subadditive argument, it is easy to show that for $q > 0$, the limit in the above definition exists. With some additional conditions on the matrices (see Theorem 1.1), the limit exists for $q \in \mathbb{R}$.

The pressure function of the scalar case (i.e., $M(x) = e^{\phi(x)}$ where $\phi(x)$ is a real valued function called the *potential* of the subshift) has been studied in great detail

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in statistical mechanics and dynamical systems in conjunction with the Gibbs measure, the entropy and the variational principle (c.f., e.g., [B], [P], [R]); it has also been used to study the multifractal structure of the self-similar (or self-conformal) measures generated by iterated function systems (IFS) with no overlap (the *open set condition*) ([MU], [FL]). By identifying with the symbolic space, such self-similar measure μ is actually a Gibbs measure and the pressure function is directly related to the scaling spectrum of μ [FL, Theorem 3.3]. In all the above cases, the pressure functions under consideration are differentiable (actually real analytic). This property is essential to investigate the phase transition in thermodynamics and for the multifractal formalism in the dimension theory of fractals.

In the recent investigation of the self-similar measures generated by iterated function systems with *overlaps*, it is seen that in many interesting cases, such measure μ can be put into a vector form with a new non-overlapping IFS and with matrix weights ([LN1,2], [LNR], [Fe], [FeO]). In this way the validity of the multifractal formalism depends on the differentiability of the pressure function $P(q)$ in (1.1) (more precisely (1.4) in the following) [LN2]. In another direction, the expression of the matrix product in (1.1) also appears in the study of the scaling functions in wavelet theory (the matrices are allowed to be non-positive) in the form of L^q -joint spectral radius and the L^q -Lipschitz exponent ([DL1, 2], [LM]); the problem of differentiability of the $P(q)$ also appears there. So far there is no general theorem to guarantee this fact other than some special cases (e.g., [LN1], [FLN], [Fe], [FeO], [DL2]).

The main purpose of this paper is to consider the pressure functions and the Gibbs measures for the products of matrices. We first study the case that the matrices $M(x)$, $x \in \Sigma_A$ are positive, we prove the following fundamental theorems.

Theorem 1.1. *Suppose that M is a Hölder continuous function on Σ_A taking values in the set of positive $d \times d$ matrices. Then for any $q \in \mathbb{R}$, there is a unique σ -invariant, ergodic probability measure μ_q on Σ_A of which one can find constants $C_1 > 0$, $C_2 > 0$ such that*

$$C_1 \leq \frac{\mu_q([J])}{\exp(-nP(q)) \cdot \|M(x)M(\sigma x) \dots M(\sigma^{n-1}x)\|^q} \leq C_2 \quad (1.2)$$

for any $n > 0$, $J \in \Sigma_{A,n}$ and $x \in [J]$.

The above measure μ_q is called the *Gibbs measure* associated with M and q . We remark that the theorem generalizes the classical existence result of the Gibbs

measure for a real-valued $M(x)$ (see [B, §1.4]). The positivity of the matrices is used to yield the follow simple estimate (Lemma 2.1)

$$\|M(x) \cdots M(\sigma^{n+\ell-1}x)\| \approx \|M(x) \cdots M(\sigma^{n-1}x)\| \cdot \|M(\sigma^n x) \cdots M(\sigma^{n+\ell-1}x)\|. \quad (1.3)$$

By using this we can apply a technique of Brown, Michan and Peyriere [BMP] and Carleson [C] to construct a certain ergodic measure which is the Gibbs measure μ_q . The μ_q has the following *quasi-Bernoulli property* (Heurteaux [H]): there exists $C > 0$ such that for any $n, k \in \mathbb{N}$ with $I \in \Sigma_{A,n}$, $J \in \Sigma_{A,k}$ and $IJ \in \Sigma_{A,n+k}$

$$C^{-1}\mu_q([I])\mu_q([J]) \leq \mu_q([IJ]) \leq C\mu_q([I])\mu_q([J]) \quad (1.4)$$

This together with a result of Heurteux [H] imply

Theorem 1.2. *Under the condition of Theorem 1.1, $P(q)$ is differentiable for $q \neq 0$.*

As an application, we let

$$E(\alpha) := \left\{ x \in \Sigma_A : \lim_{n \rightarrow \infty} \frac{\log \|M(x)M(\sigma x) \cdots M(\sigma^{n-1}(x))\|}{n} = \alpha \right\}.$$

We prove the following dimension formula

Theorem 1.3. *Under the same assumption of Theorem 1.1, we have for any $\alpha = P'(q)$, $q \neq 0$,*

$$\dim_H E(\alpha) = \frac{1}{\log m}(-\alpha q + P(q))$$

where \dim_H denotes the Hausdorff dimension.

The above theorems depend very much on the positivity of the matrix-valued $M(x)$. In order to extend them to nonnegative matrix-valued functions, we have to impose more conditions on $M(x)$:

(H1) $M(x) = M_i$ if $x \in [i]$, $i = 1, \dots, m$;

(H2) M is irreducible in the following sense: there exists $r > 0$ such that for any $i, j \in \{1, 2, \dots, m\}$,

$$\sum_{k=1}^r \sum_{K \in \Sigma_{A,k;i,j}} M_K > \mathbf{0} \quad (1.5)$$

where $\Sigma_{A,k;i,j}$ denotes the set of all $K \in \Sigma_{A,k}$ such that $iKj \in \Sigma_{A,k+2}$.

We see that under the assumption (H1), the pressure function in (1.1) can be re-written as

$$P(q) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{J \in \Sigma_{A,n}} \|M_J\|^q. \quad (1.6)$$

where $M_J = M_{j_1} \cdots M_{j_n}$. If $\Sigma_A = \Sigma$ is the symbolic space with a full shift, then (H2) is equivalent to $\sum_{k=1}^r (M_1 + \cdots + M_m)^k > \mathbf{0}$.

In this new setting, we use (H2) to adjust (1.3) and the required lemmas, the Gibbs measure μ_q is shown to exist for $q > 0$. This time μ_q only satisfy $\mu_q([IJ]) \leq C\mu_q([I])\mu_q([J])$ instead of (1.4); nevertheless we can still prove the differentiability of $P(q)$, $q > 0$ as in Theorem 1.2. Theorem 1.3 can be adjusted likewise.

For the organization of the paper, we prove the above results for the positive matrix-valued functions in Section 2. In Section 3, we modify the proofs for the non-negative matrix-valued functions with (H1) and (H2). In Section 4, we give an illustration of reducing an IFS with overlap to a vector-valued IFS with no overlap, and the pressure function in (1.6) arises. We also give some remark on the theorems and raise a few unsettled problems.

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2. Positive Matrices

In this section we assume that M is a Hölder continuous function on Σ_A taking values in the set of all positive $d \times d$ matrices.

For any two families of positive numbers $\{a_i\}_{i \in \mathcal{I}}$, $\{b_i\}_{i \in \mathcal{I}}$, we write, for brevity, $a_i \approx b_i$ to mean the existence of a constant $C > 0$ such that $C^{-1}a_i \leq b_i \leq Ca_i$ for all $i \in \mathcal{I}$; $a_i \preceq b_i$ to mean the existence of a constant $C > 0$ such that $a_i \leq Cb_i$ for all $i \in \mathcal{I}$ and $a_i \succ b_i$ means $b_i \preceq a_i$.

We start with a simple lemma:

Lemma 2.1. *For any $x \in \Sigma_A$, $n, \ell \in \mathbb{N}$,*

$$\|M(x) \cdots M(\sigma^{n+\ell-1}x)\| \approx \|M(x) \cdots M(\sigma^{n-1}x)\| \cdot \|M(\sigma^n x) \cdots M(\sigma^{n+\ell-1}x)\|$$

(the involved constant in \approx is independent of n, ℓ and x).

Proof. It is clear that

$$\|M(x) \cdots M(\sigma^{n+\ell-1}x)\| \leq \|M(x) \cdots M(\sigma^{n-1}x)\| \cdot \|M(\sigma^n x) \cdots M(\sigma^{n+\ell-1}x)\|.$$

To prove the reverse inequality, we observe that M is positive and continuous, there is a constant $C > 0$ such that

$$\frac{\min_{i,j} M_{i,j}(x)}{\max_{i,j} M_{i,j}(x)} \geq C \quad \forall x \in \Sigma_A.$$

This implies that $M(x) \geq \frac{C}{d} EM(x)$ ($A \geq B$ means that $A_{i,j} \geq B_{i,j}$ for each index (i, j)) where $E = (E_{i,j})_{1 \leq i,j \leq d}$ is the matrix whose entries are all equal to 1. Let $\mathbf{1}$ be the d -dimensional column vector each coordinate of which is 1. Then

$$\begin{aligned} \|M(x) \cdots M(\sigma^{n+\ell-1}x)\| &\geq \|M(x) \cdots M(\sigma^{n-1}x) \frac{C}{d} EM(\sigma^n x) \cdots M(\sigma^{n+\ell-1}x)\| \\ &= \frac{C}{d} \|M(x) \cdots M(\sigma^{n-1}x) \mathbf{1}^\top \mathbf{1} M(\sigma^n x) \cdots M(\sigma^{n+\ell-1}x)\| \\ &= \frac{C}{d} \|M(x) \cdots M(\sigma^{n-1}x)\| \cdot \|M(\sigma^n x) \cdots M(\sigma^{n+\ell-1}x)\|. \end{aligned}$$

□

For any $I \in \Sigma_{A,n}$, we let

$$s_n(I, q) = \sup_{x \in [I]} \|M(x)M(\sigma x) \cdots M(\sigma^{n-1}x)\|^q, \quad s_n(q) = \sum_{I \in \Sigma_{A,n}} s_n(I, q). \quad (2.1)$$

Lemma 2.2. *For each $q \in \mathbb{R}$,*

$$s_n(I, q) \approx \|M(x)M(\sigma x) \cdots M(\sigma^{n-1}x)\|^q \quad \forall I \in \Sigma_{A,n}, x \in [I].$$

Proof. For any $n \in \mathbb{N}$, define

$$\eta_n = \sup \left\{ \frac{M_{i,j}(x)}{M_{i,j}(y)} : I \in \Sigma_{A,n}, x, y \in [I], 1 \leq i, j \leq d \right\}. \quad (2.2)$$

Since each $M_{i,j}$ is positive and Hölder continuous, we have $|\log \eta_n| \leq Cm^{-\alpha n}$ for some $0 < \alpha < 1$. It follows easily that $\eta := \prod_{n=1}^{\infty} \eta_n < \infty$ and hence for $x \in [I]$,

$$\|M(x) \cdots M(\sigma^{n-1}x)\|^q \leq s_n(I, q) \leq \eta^{|q|} \|M(x) \cdots M(\sigma^{n-1}x)\|^q.$$

□

We have assumed that A is primitive, there is an integer $p > 0$ such that $A^p > \mathbf{0}$. This implies that for any $I \in \Sigma_{A,n}, J \in \Sigma_{A,\ell}$, there exists $K \in \Sigma_{A,p}$ such that $IKJ \in \Sigma_{A,n+\ell+p}$.

Lemma 2.3. *Let p be such that $A^p > \mathbf{0}$. Then for each $q \in \mathbb{R}$,*

$$(i) \ s_\ell(q) \approx s_{\ell-p}(q) \text{ for all } \ell > p;$$

(ii) For $I \in \Sigma_{A,n}$, $\ell > p$, $\sum_J s_{n+\ell}(IJ, q) \approx \sum_J s_{n+\ell}(JI, q) \approx s_n(I, q)s_\ell(q)$ where the first (second) sum is taken over all $J \in \Sigma_{A,\ell}$ such that $IJ \in \Sigma_{A,n+\ell}$ ($JI \in \Sigma_{A,n+\ell}$ respectively);

(iii) $\sum_{K: IKJ \in \Sigma_{A,i}} s_i(IKJ, q) \approx s_n(I, q)s_\ell(J, q)s_{i-n-\ell}(q)$ for all $I \in \Sigma_{A,n}$, $J \in \Sigma_{A,\ell}$, $i > n + \ell + 2p$.

Proof. For any $I \in \Sigma_{A,\ell}$, write $I = KJ$ where $J \in \Sigma_{A,\ell-p}$. By Lemmas 2.1, 2.2, we have (note that p is fixed)

$$s_\ell(I, q) \approx s_{\ell-p}(J, q).$$

Since $A^p > \mathbf{0}$, for $J \in \Sigma_{A,\ell-p}$, we can find $K \in \Sigma_{A,p}$ such that $I = KJ \in \Sigma_{A,\ell}$. Hence when we take the sum of $I \in \Sigma_{A,\ell}$ on the left side of the expression, it is \approx to the right side summing over all $J \in \Sigma_{A,\ell-p}$. This implies (i).

To prove (ii), we fix $I \in \Sigma_{A,n}$ and take $J \in \Sigma_{A,\ell}$ such that $IJ \in \Sigma_{A,n+\ell}$. By Lemmas 2.2, 2.1, we have

$$s_{n+\ell}(IJ, q) \approx s_n(I, q)s_\ell(J, q).$$

Thus

$$\sum_J s_{n+\ell}(IJ, q) \preccurlyeq s_n(I, q)s_\ell(q).$$

For the reverse inequality we note for any $J' \in \Sigma_{A,\ell-p}$, there is $K \in \Sigma_{A,p}$ such that $IKJ' \in \Sigma_{A,n+\ell}$ and

$$s_{n+\ell}(IKJ') \approx s_n(I, q)s_p(K, q)s_{\ell-p}(J', q) \approx s_n(I, q)s_{\ell-p}(J', q).$$

Therefore summing over the above J' , we have

$$\sum_{IJ \in \Sigma_{A,n+\ell}} s_{n+\ell}(IJ, q) \succcurlyeq \sum_{J'} s_{n+\ell}(IKJ', q) \approx s_n(I, q)s_{\ell-p}(q) \approx s_n(I, q)s_\ell(q)$$

(we make used of $A^p > \mathbf{0}$ as in (i)). This proves one of the \approx in (ii). The remaining part follows from the same argument.

To prove (iii), we first observe that

$$\begin{aligned} \sum_{K: IKJ \in \Sigma_{A,i}} s_i(IKJ, q) &\approx \sum_{K: IKJ \in \Sigma_{A,i}} s_n(I, q)s_{i-n-\ell}(K, q)s_\ell(J, q) \\ &\preccurlyeq s_n(I, q)s_\ell(J, q)s_{i-n-\ell}(q) \end{aligned}$$

On the other hand, for any $K_1 \in \Sigma_{A,i-n-\ell-2p}$, there exists $K_2, K_3 \in \Sigma_{A,p}$ such that $IK_2K_1K_3J \in \Sigma_{A,i}$. Therefore

$$\begin{aligned} \sum_{K: IKJ \in \Sigma_{A,i}} s_i(IKJ, q) &\asymp s_n(I, q) s_\ell(J, q) \sum_{K_1 \in \Sigma_{A,i-n-\ell-2p}} s_{i-n-\ell-2p}(K_1, q) \\ &\approx s_n(I, q) s_\ell(J, q) s_{i-n-\ell-2p}(q) \\ &\approx s_n(I, q) s_\ell(J, q) s_{i-n-\ell}(q). \end{aligned}$$

□

Lemma 2.4. For $q \in \mathbb{R}$,

- (i) $s_{\ell+n}(q) \approx s_\ell(q) s_n(q)$.
- (ii) $s_n(q) \approx \exp(nP(q))$ where $P(q)$ is the pressure function defined in (1.1).

Proof. From Lemma 2.3 (ii), there exist $C, C' > 0$ such that

$$C' s_\ell(q) s_n(q) \leq s_{\ell+n}(q) \leq C s_\ell(q) s_n(q),$$

which proves (i). To prove (ii), we can write $C s_{\ell+n}(q) \leq (C s_\ell(q)) (C s_n(q))$. Hence the subadditivity property implies

$$P(q) = \lim_{n \rightarrow \infty} \frac{\log(C s_n(q))}{n} = \inf_n \frac{\log(C s_n(q))}{n},$$

so that $C^{-1} \exp(nP(q)) \leq s_n(q)$. The reverse inequality follows from a similar argument. □

For each integer $n > 0$, let \mathcal{B}_n be the σ -algebra generated by the cylinders $[I]$, $I \in \Sigma_{A,n}$. We define a sequence of probability measures $\{\nu_{n,q}\}$ on \mathcal{B}_n by

$$\nu_{n,q}([I]) = \frac{s_n(I, q)}{s_n(q)} \quad \forall I \in \Sigma_{A,n}. \quad (2.3)$$

Then there is a subsequence $\{\nu_{n_k,q}\}_{k \geq 1}$ converges in the weak-star topology to a probability measure ν_q . The following assertion shows that ν_q has the ‘‘Gibbs property’’.

Lemma 2.5. For fixed $q \in \mathbb{R}$, $\nu_q([I]) \approx s_n(I, q) \exp(-nP(q))$ for all $n > 0$, $I \in \Sigma_{A,n}$.

Proof. Let p be such that $A^p > \mathbf{0}$. For any $I \in \Sigma_{A,n}$ and $\ell > n + p$, we have

$$\begin{aligned} \nu_{\ell,q}([I]) &= \sum_{IJ \in \Sigma_{A,\ell}} \nu_{\ell,q}([IJ]) = \sum_{IJ \in \Sigma_{A,\ell}} \frac{s_\ell(IJ, q)}{s_\ell(q)} \\ &\approx s_n(I, q) \frac{s_{\ell-n}(q)}{s_\ell(q)} && \text{(by Lemma 2.3(ii))} \\ &\approx s_n(I, q) \exp(-nP(q)). && \text{(by Lemma 2.4).} \end{aligned}$$

Letting $\ell = n_k \uparrow \infty$, we obtain the desired result. \square

Proof of Theorem 1.1. Fix $q \in \mathbb{R}$. Let μ_q be a limit point of the subsequence of $\{\frac{1}{n}(\nu_q + \nu_q \circ \sigma^{-1} + \dots + \nu_q \circ \sigma^{-(n-1)})\}$ in the weak-star topology. Then μ_q is a σ -invariant measure on Σ_A . We have for each $I \in \Sigma_{A,n}$ and $\ell > p$,

$$\begin{aligned} \nu_q \circ \sigma^{-\ell}([I]) &= \sum_{JI \in \Sigma_{A,n+\ell}} \nu_q([JI]) \\ &\approx \sum_{JI \in \Sigma_{A,n+\ell}} s_{n+\ell}(JI, q) \exp(-(n+\ell)P(q)) && \text{(by Lemma 2.5)} \\ &\approx s_\ell(q) s_n(I, q) \exp(-(\ell+n)P(q)) && \text{(by Lemma 2.3(ii))} \\ &\approx s_n(I, q) \exp(-nP(q)) && \text{(by Lemma 2.4).} \end{aligned} \quad (2.4)$$

This proves that μ_q is a Gibbs measure. In what follows we prove that μ_q is ergodic. First we show that there is a constant $C > 0$ such that for each $I \in \Sigma_n$, $J \in \Sigma_\ell$,

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mu_q([I] \cap \sigma^{-i}([J])) \geq C \mu_q([I]) \mu_q([J]). \quad (2.5)$$

Since μ_q is supported on Σ_A , it suffices to prove (2.5) for $I \in \Sigma_{A,n}$ and $J \in \Sigma_{A,\ell}$. Note that when $i > n + 2p$,

$$\begin{aligned} &\mu_q([I] \cap \sigma^{-i}([J])) \\ &= \sum_{K: IKJ \in \Sigma_{A,i+\ell}} \mu_q([IKJ]) \\ &\asymp \sum_{K: IKJ \in \Sigma_{A,i+\ell}} s_{i+\ell}(IKJ, q) \exp(-(i+\ell)P(q)) \\ &\approx s_n(I, q) s_\ell(J, q) s_{i-n}(q) \exp(-(i+\ell)P(q)) && \text{(by Lemma 2.3(iii))} \\ &\approx s_n(I, q) s_\ell(J, q) \exp(-(n+\ell)P(q)) \\ &\approx \mu_q([I]) \mu_q([J]) \end{aligned}$$

from which (2.5) follows. Since the collection $\{[I] : I \in \Sigma_n, n \in \mathbb{N}\}$ is a semi-algebra that generates the Borel σ -algebra on Σ , a standard argument (e.g., see the proof

of [W, Theorem 1.17]) shows that for any Borel sets $A, B \subset \Sigma$,

$$\lim_{k \rightarrow \infty} \frac{1}{k} \sum_{i=0}^{k-1} \mu_q \left(A \cap \sigma^{-i}(B) \right) \geq C \mu_q(A) \mu_q(B).$$

This implies that for any Borel sets $A, B \subset \Sigma$ with $\mu_q(A) > 0$, $\mu_q(B) > 0$, there exists $n > 0$ with $\mu_q(A \cap \sigma^{-n}(B)) > 0$. By [W, Theorem 1.5], μ_q is ergodic.

For the uniqueness we recall that any two distinct ergodic measures must be singular to each other; but the Gibbs property (1.2) implies that any two μ_q must be absolutely continuous to each other. Hence μ_q must be unique. \square

Corollary 2.6. *Let μ_q be the Gibbs measure in Theorem 1.1, then there exist $C > 0$ such that for any $I \in \Sigma_{A,n}, J \in \Sigma_{A,\ell}$ with $IJ \in \Sigma_{A,n+\ell}$,*

$$C^{-1} \mu_q([IJ]) \leq \mu_q([I]) \mu_q([J]) \leq C \mu_q([I]) \mu_q([J]).$$

Proof. We have seen from the proof of Lemma 2.3 that for the above I, J , $s_{n+\ell}(IJ, q) \approx s_n(I, q) s_\ell(J, q)$ and from Lemma 2.4, $s_{\ell+n}(q) \approx s_n(q) s_\ell(q)$. By the definition of $\nu_{n,q}$, we have

$$\nu_{n,q}([IJ]) \approx \nu_{n,q}([I]) \nu_{n,q}([J])$$

which implies that the Gibbs measure μ_q has the same property. \square

The above property is called *quasi-Bernoulli property* by Heurteaux [H] (we remark that Heurteaux only introduced and studied it for measures in the full shift space Σ). To prove Theorem 1.2, we need a result in [H]. Let η be a probability measure on Σ . For $q \in \mathbb{R}$, let $\tau_\eta(q)$ be the L^q -spectrum of η , i.e.,

$$\tau_\eta(q) = \liminf_{n \rightarrow \infty} \frac{\log \sum_I \eta([I])^q}{\log m^{-n}},$$

where the summation is taken over all $I \in \Sigma_n$ with $\eta([I]) \neq 0$.

Proposition 2.7. ([H, Theorem 2.1]) *Let η be a probability measure on Σ . Assume that there exists a constant $C > 0$ such that*

$$\eta([IJ]) \leq C \eta([I]) \eta([J]) \quad \forall I \in \Sigma_n, J \in \Sigma_\ell. \quad (2.6)$$

Then $\tau'_\eta(1)$ exists if η is a Young measure (i.e., $\lim_{n \rightarrow \infty} \frac{\log \eta(I_n(x))}{\log m^{-n}} = \text{constant}$ for η almost all $x = (j_i) \in \Sigma$, here $I_n(x) = [j_i \dots j_n]$).

Proof of Theorem 1.2. For each q , let μ_q be the corresponding Gibbs measure in Theorem 1.1. We can view μ_q as a measure on Σ . For $t \in \mathbb{R}$, let $\tau_{\mu_q}(t)$ be the L^t -spectrum of μ_q . Since μ_q has the Gibbs property, it is easy to show by the definition of L^t -spectrum that

$$\tau_{\mu_q}(t) = \frac{tP(q) - P(qt)}{\log m} \quad (2.7)$$

Note that μ_q satisfies the condition (2.6). Since μ_q is ergodic on Σ , it is a Young measure by the Shannon-McMillan-Brieman theorem (i.e., $\lim_{n \rightarrow \infty} \frac{-\log \mu_q(I_n(x))}{n}$ equals the entropy of μ_q (with respect to σ) for μ_q -almost all $x = (j_i) \in \Sigma$ and $I_n(x) = [j_1 \dots j_n]$). Hence by Proposition 2.7, $\tau_{\mu_q}(t)$ is differentiable at $t = 1$. This implies that $P(q)$ is differential at any fixed $q \neq 0$, and

$$P'(q) = \frac{P(q) - \log m \cdot \tau'_{\mu_q}(1)}{q}$$

□

Proof of Theorem 1.3. Let $\alpha = \tau'(q)$ with $q \neq 0$. Let μ_q be the corresponding Gibbs measures in Theorem 1.1, then (2.7) implies that

$$\tau_{\mu_1}(q) = \frac{qP(1) - P(q)}{\log m}$$

and

$$E(\alpha) = \left\{ x \in \Sigma : \lim_{n \rightarrow \infty} \frac{\log \mu_1([x_1 \dots x_n])}{\log m^{-n}} = \frac{P(1) - \alpha}{\log m} \right\}$$

By [BMP, Theorem 1] or [LN2, Theorem 4.1], we have

$$\dim_H E(\alpha) \leq \left(\frac{P(1) - \alpha}{\log m} \right) q - \tau_{\mu_1}(q) = \frac{1}{\log m} (-\alpha q + P(q)) \quad \forall q \in \mathbb{R}. \quad (2.8)$$

For the reverse inequality, we see from the proof of Theorem 1.2 that $\tau'_{\mu_q}(1)$ exists and

$$\tau'_{\mu_q}(1) = \frac{-qP'(q) + P(q)}{\log m}.$$

By [N], we have for μ_q almost all $I = (i_1 \dots i_n \dots) \in \Sigma$,

$$\lim_{n \rightarrow \infty} \frac{\log \mu_q([i_1, \dots, i_n])}{\log m^{-n}} = \tau'_{\mu_q}(1) = \frac{-qP'(q) + P(q)}{\log m}.$$

This implies that

$$\lim_{n \rightarrow \infty} \frac{\log \|M(x)M(\sigma x) \dots M(\sigma^{n-1}x)\|}{n} = P'(q) = \alpha \quad \mu_q - a.a. x \in \Sigma.$$

Therefore we have

$$\dim_H E(\alpha) \geq \dim_H \mu_q = \frac{-q\alpha + P(q)}{\log m}.$$

□

3. Nonnegative matrices

In this section, we always assume that M is a function on Σ_A taking values in the set of all $d \times d$ non-negative matrices and satisfies (H1) and (H2) defined in Section 1. Let $q > 0$ be fixed. Then $s_n(I, q)$ and $s_n(q)$ in (2.1) are reduced to

$$s_n(I, q) = \|M_I\|^q \quad \forall I \in \Sigma_{A,n} \quad \text{and} \quad s_n(q) = \sum_{I \in \Sigma_{A,n}} s_n(I, q).$$

For convenience, we let

$$b = \min_{s,t \in \{1,2,\dots,m\}} \min_{1 \leq i,j \leq d} \left(\sum_{k=1}^r \sum_{K \in \Sigma_{A,k,s,t}} M_K \right)_{i,j}. \quad (3.1)$$

Then $b > 0$ by (H2).

We will reformulate the three theorems in the previous section. The proofs are almost the same and for simplicity, we only point out the differences. Here Lemmas 2.1, 2.2 do not hold anymore; on the other hand we can use (H2) to replace these lemmas to obtain an analog of Lemma 2.3:

Lemma 3.1. *For $q > 0$,*

- (i) $s_{\ell+1}(q) \approx s_\ell(q)$.
- (ii) For $I \in \Sigma_{A,n}$, $\sum_J s_{n+\ell}(IJ, q) \approx \sum_J s_{n+\ell}(JI, q) \approx s_n(I, q)s_\ell(q)$ where the first (second) sum is taken over all $J \in \Sigma_{A,\ell}$ such that $IJ \in \Sigma_{A,n+\ell}$ ($JI \in \Sigma_{A,n+\ell}$ respectively).
- (iii) $\sum_{k=1}^{2r} \sum_{K: IKJ \in \Sigma_{A,i+k}} s_{i+k}(IKJ, q) \approx s_n(I, q)s_\ell(J, q)s_{i-n-\ell}(q)$ for all $I \in \Sigma_{A,n}$, $J \in \Sigma_{A,\ell}$, $i > n + \ell$.

Proof. For any $I \in \Sigma_{A,\ell+1}$, write $I = iJ$ with $J \in \Sigma_{A,\ell}$. Using $\|M_I\| \leq \|M_i\| \|M_J\|$, we have for $q > 0$,

$$s_{\ell+1}(q) \leq m \left(\sup_{i \in \{1,2,\dots,m\}} \|M_i\|^q \right) s_\ell(q).$$

That is, $s_{\ell+1}(q) \preccurlyeq s_\ell(q)$. For the reverse inequality, since for any $J \in \Sigma_{A,\ell}$,

$$\sum_{k=1}^r \sum_{K \in \Sigma_{A,k}: KJ \in \Sigma_{A,\ell+k}} \|M_{KJ}\| = \left\| \left(\sum_{k=1}^r \sum_{K \in \Sigma_{A,k}: KJ \in \Sigma_{A,\ell+k}} M_K \right) M_J \right\| \geq b \|M_J\|,$$

it follows that

$$\sum_{k=1}^r \sum_{K \in \Sigma_{A,k}: KJ \in \Sigma_{A,\ell+k}} \|M_{KJ}\|^q \geq \left(\frac{b}{\sum_{k=1}^r m^k} \right)^q \|M_J\|^q.$$

This combines with $s_{\ell+1}(q) \asymp s_\ell(q)$ imply that

$$s_{\ell+1}(q) \gtrsim \sum_{k=1}^r s_{\ell+k}(q) \geq s_\ell(q)$$

and completes the proof of (i).

To prove (ii), it follows from $\|M_{IJ}\| \leq \|M_I\| \|M_J\|$ that $\sum_{J \in \Sigma_{A,\ell}: IJ \in \Sigma_{A,n+\ell}} s_{n+\ell}(IJ, q) \asymp s_n(I, q) s_\ell(q)$. For the reverse inequality, we use (H2) as above to conclude that for any $J \in \Sigma_{n,\ell}$,

$$\sum_{k=1}^r \sum_{K \in \Sigma_{A,k}: IKJ \in \Sigma_{A,n+\ell+k}} \|M_{IKJ}\| \gtrsim \|M_I\| \|M_J\|.$$

Hence $\sum_{k=1}^r \sum_{K \in \Sigma_{A,k}: IKJ \in \Sigma_{A,n+\ell+k}} \|M_{IKJ}\|^q \gtrsim \|M_I\|^q \|M_J\|^q$ and therefore, summing over the $J \in \Sigma_{n,\ell}$,

$$\sum_{k=1}^r \sum_{L \in \Sigma_{A,\ell+k}: IL \in \Sigma_{A,n+\ell+k}} s_{n+\ell+k}(IL, q) \gtrsim s_n(I, q) s_\ell(q).$$

Since

$$\sum_{J \in \Sigma_{A,\ell}: IJ \in \Sigma_{A,n+\ell}} s_{n+\ell}(IJ, q) \gtrsim \sum_{J \in \Sigma_{A,\ell+1}: IJ \in \Sigma_{A,n+\ell+1}} s_{n+\ell+1}(IJ, q),$$

we have

$$\sum_{J \in \Sigma_{A,\ell}: IJ \in \Sigma_{A,n+\ell}} s_{n+\ell}(IJ, q) \gtrsim \sum_{k=1}^r \sum_{J \in \Sigma_{A,\ell+k}: IJ \in \Sigma_{A,n+\ell+k}} s_{n+\ell+k}(IJ, q) \gtrsim s_n(I, q) s_\ell(q).$$

This completes the proof of an \approx in (ii); the other \approx follows from an identical argument.

To prove (iii), we have,

$$\begin{aligned} \sum_{k=1}^{2r} \sum_{K: IKJ \in \Sigma_{A,i+k}} s_{i+k}(IKJ, q) &\leq s_n(I, q) s_\ell(J, q) \sum_{k=1}^{2r} s_{i+k-n-\ell}(q) \\ &\approx s_n(q) s_\ell(J, q) s_{i-n-\ell}(q) \quad (\text{by (i)}). \end{aligned}$$

On the other hand, for any $W \in \Sigma_{A,i-n-\ell}(q)$, by (H2), there exist $1 \leq k_1 \leq r$, $K_1 \in \Sigma_{A,k_1}$ such that $IK_1W \in \Sigma_{A,i-\ell+k_1}$ and

$$\|IK_1W\| \geq \frac{b\|I\|\|W\|}{\sum_{k=1}^r m^k},$$

where b is defined by (3.1). By using (H2) again, there exist $1 \leq k_2 \leq r$, $K_2 \in \Sigma_{A,k_2}$ such that $IK_1WK_2J \in \Sigma_{A,i+k_1+k_2}$ and

$$\|IK_1WK_2J\| \geq \frac{b\|IK_1W\|\|J\|}{\sum_{k=1}^r m^k} \geq \frac{b^2\|I\|\|W\|\|J\|}{(\sum_{k=1}^r m^k)^2}.$$

Therefore we have

$$\sum_{k=1}^{2r} \sum_{K:IKJ \in \Sigma_{A,i+k}} s_{i+k}(IKJ, q) \gtrsim s_n(I, q)s_\ell(J, q)s_{i-n-\ell}(q)$$

and (iii) follows. \square

We now state the corresponding theorems as in Section 1.

Theorem 3.2. *Suppose M is a function on Σ_A taking values in the set of all $d \times d$ non-negative matrices and satisfies (H1) and (H2). Then for any $q > 0$, there is a unique Gibbs measure μ_q on Σ_A as in Theorem 1.1.*

The proof is almost identical with that of Theorem 1.1 using Lemma 3.1. The only adjustment is to replace

$$\mu_q([I] \cap \sigma^{-i}([J])) \approx \mu_q(I)\mu_q(J), \quad i > n + 2p.$$

by

$$\sum_{i=1}^{2r} \mu_q([I] \cap \sigma^{-i}([J])) \approx \mu_q(I)\mu_q(J) \quad \forall i > n.$$

We use the same proof as in Section 2 for the next two theorems.

Theorem 3.3. *Under the same conditions of Theorem 3.2, $P(q)$ is differentiable for any $q > 0$*

Theorem 3.4. *Under the same conditions of Theorem 3.2, we have for any $\alpha = P'(q)$, $q > 0$,*

$$\dim_H E(\alpha) = \frac{1}{\log m}(-\alpha q + P(q))$$

where $E(\alpha) = \{J = (j_i) \in \Sigma_A : \lim_{n \rightarrow \infty} \log \|M_{j_1} \cdots M_{j_n}\|/n = \alpha\}$.

To relate Theorem 3.4 to the classical random product of matrices, we let $\{Y_n\}$ be the i.i.d. random variables that take values M_1, \dots, M_m , invertible matrices and with uniform distribution, then $\lim_{n \rightarrow \infty} \frac{1}{n} \log \|Y_n \dots Y_1\| = \lambda$ a.s. and λ is called the upper Lyapunov exponent ([FK], [BL, Chapter 1]). In comparison with Theorem 3.4, we let $\Sigma_A = \Sigma$ be the space of full shift (i.e., all the entries of A are 1), then $P(0) = \log m$. The limit of the random variables corresponds to the case for $q = 0$, $\lambda = P'(0)$ and $\dim_H E(\lambda) = P(0)/\log m = 1$ (the existence of the derivative follows from some additional assumptions on the M_j ([BL, p.119])).

We remark that if $\Sigma_A = \Sigma$, then condition (H2) is reduced to a more simple form: $\sum_{k=1}^r H^k > \mathbf{0}$ where $H = M_1 + \dots + M_m$. The condition is essential for the theorems in Section 3. Indeed we have

Example 4.5. Let $M_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. Then $H = M_1 + M_2$ is reducible. Since $M_J = \begin{pmatrix} 2^n & 0 \\ 0 & 3^k \end{pmatrix}$ where $|J| = n$ and k is the number of 2's appeared in J . $\sum_{|J|=n} \|M_J\|^q = \sum_{k=0}^n \binom{n}{k} (2^n + 3^k)^q$. Note that

$$\sum_{k=0}^n \binom{n}{k} (2^n + 3^k)^q \geq \max \left\{ \sum_{k=0}^n \binom{n}{k} 2^{nq}, \sum_{k=0}^n \binom{n}{k} 3^{kq} \right\} = \max \{2^{n(q+1)}, (1 + 3^q)^n\}$$

and

$$\sum_{k=0}^n \binom{n}{k} (2^n + 3^k)^q \leq \sum_{k=0}^n \binom{n}{k} 2^q (2^{nq} + 3^{kq}) = 2^q (2^{n(q+1)} + (1 + 3^q)^n).$$

We have $P(q) = \max\{(q+1) \log 2, \log(1 + 3^q)\}$, which is not differentiable at $q = 1$.

We see that the Gibbs measure μ_q in Section 2 has the quasi-Bernoulli property. However for the case of non-negative matrices, only $\mu_q([IJ]) \leq C \mu_q([I]) \mu_q([J])$, $I \in \Sigma_n$, $J \in \Sigma_\ell$. The following example shows that the reverse inequality may not hold.

Example 4.6. Let $M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and M_2 an arbitrarily positive matrix, then $H = M_1 + M_2$ is an irreducible positive matrix. Let $J = 1 \dots 1$ (n -times), then $\|M_J\| = n + 2$ and hence

$$\|M_J\| \|M_J\| \geq \frac{n}{2} \|M_{JJ}\|.$$

Since $\mu_q(I) \approx \exp(-nP(q)) \cdot \|M_I\|$, $I \in \Sigma_n$, we see that there does not exist $C' > 0$ such that $C' \mu_q([J]) \mu([J]) \leq \mu([JJ])$.

4. Examples and remarks

Consider the classical Bernoulli convolution $X = (1 - \rho)^{-1} \sum_{n=1}^{\infty} \rho^n X_n$ where the X_n 's are i.i.d. random variables which take values 0, 1 and with probability 1/2 on each value. Let μ_ρ be the distribution measure of X . It is well known that for $0 < \rho < 1/2$, the measure is a Cantor type measures. It was proved recently that μ_ρ is absolutely continuous for almost all $1/2 < \rho < 1$ [S], however, it is still not clear which μ_ρ is absolutely continuous or singular. The question is subjected to intensive investigation, the reader can refer to the survey articles [L], [PSS] and the references there. The interest of the Bernoulli convolution in our setting is that the μ_ρ satisfies the *self-similar* identity

$$\mu_\rho = \frac{1}{2}\mu_\rho S_1^{-1} + \frac{1}{2}\mu_\rho S_2^{-1}$$

where $S_1 x = \rho x$, $S_2 x = \rho x + (1 - \rho)$; $\{S_1, S_2\}$ is the iterated function system (IFS). The support of μ_ρ is $[0, 1]$. For $0 < \rho < 1/2$, the $S_J(0, 1)$'s are disjoint (as in the basic intervals of the Cantor set); for $1/2 < \rho < 1$, the $S_J(0, 1)$ overlaps which is the source of difficulty.

For $\rho = (\sqrt{5} - 1)/2$, the reciprocal of the golden ratio, it was shown by Erdős that μ_ρ is singular. In order to consider the multifractal structure of μ_ρ , we can put the IFS $\{S_1, S_2\}$ to a new set of IFS $\{R_i\}_{i=1}^3$ with no overlap:

$$R_1(x) = \rho^2 x, \quad R_2(x) = \rho^3 x + \rho^2, \quad \text{and} \quad R_3(x) = \rho^2 x + \rho.$$

Then the measure μ_ρ satisfies

$$\mu_\rho([i_1 \dots i_n]) \approx \frac{1}{4^n} \|M_{i_1} \dots M_{i_n}\|$$

where $[i_1 \dots i_n] = R_{i_1} \dots R_{i_n}([0, 1])$ and

$$M_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad M_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

([LN1], [Fe], [FeO]). The $\{M_i\}_{i=1}^3$ satisfies the conditions (H2). Hence by Theorem 3.2, 3.3, $P(q)$ is differentiable for $q > 0$ and the multifractal formalism holds.

Actually more can be said about the L^q -spectrum $\tau(q)$ of μ_q : an explicit formula was given in [LN1] for $q > 0$ and was extended to $q < 0$ in [Fe]. By using the formula it was proved that $\tau(q)$ is differentiable (actually real analytic) on \mathbb{R} except one point in \mathbb{R}^- .

The above example of Bernoulli convolution gives rise to another interesting question. Note that the above example is a special case of the overlapping IFS that can

be reduced to new sets of IFS with no overlap and the calculation of the $\tau(q)$ can be converted into the product of matrices as in (1.4). Such IFS forms an important subclass of those that satisfy the *weak separation condition* ([LN2], [LNR]). (it will be interesting to classify this subclass of IFS. Under the weak separation condition it was proved that the multifractal formalism is valid provided that $\tau(q)$ is differentiable [LN2]. However we do not know its differentiability other than the case we just considered.

The behavior for $q < 0$ is also important for the multifractal analysis. It does not have a problem for M to be positive matrix-valued. For the non-negative matrix-valued M , $\|M_J\|$ can be $\mathbf{0}$, we have to modify the pressure function $P(q)$ in (1.6) slightly:

$$P(q) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{J \in \mathcal{N}_n} \|M_J\|^q \quad (4.1)$$

where \mathcal{N}_n consists of all the $J \in \Sigma_{A,n}$ such that $M_J \neq \mathbf{0}$. It is clear that if $M_J \neq \mathbf{0}$ for all $J \in \Sigma_{A,n}$, then the super-additivity of the sum in (4.1) implies that the limit exists. We include a simple proposition with $\Sigma_A = \Sigma$ to set up the consideration:

Proposition 4.4 Suppose M_1, \dots, M_m are non-negative matrices and $H = \sum_{i=1}^m M_i$ is irreducible, then the limit in (4.1) exists for each $q < 0$.

Proof. By the irreducibility, there exists integer r with $\sum_{k=1}^r H^k > 0$. Hence there is a constant $C > 0$ such that for any two finite indices I, J , there exists $K_0 \in \bigcup_{k=1}^r \Sigma_k$ satisfying

$$0 < \|M_{IK_0J}\| \leq C \|M_I\| \|M_J\|. \quad (4.2)$$

Denote by $s_n = \sum_{J \in \mathcal{N}_n} \|M_J\|^q$. Then (4.2) implies $s_n s_\ell \leq C^{-q} \sum_{k=1}^r s_{n+\ell+k}$. From (4.2) we also deduce that for any finite index I , there exists $i \in \Sigma_1$ such that $M_{Ii} \neq 0$; Since $\|M_{Ii}\| \leq C_1 \|M_I\|$ for some constant $C_1 > 0$, we have $s_n \leq C_1^{-q} s_{n+1}$ for any integer n, ℓ . It follows that $s_n s_\ell \leq C' s_{n+\ell+r}$ for some constant $C' > 0$ (depending on q), which implies that $a_n = \frac{1}{C'} s_{n-r}$ is super-multiplicative. This yields the existence of the limit. \square .

The differentiability of such $P(q)$ for $q < 0$ is still unknown. We know that in the above Bernoulli convolution of the golden ratio, it is possible for the $P(q)$ to be non-differentiable at a point of $q < 0$ [Fe]. On the other hand, it is known that by imposing some stronger conditions on the matrices, the pressure function $P(q)$ is analytic near $q = 0$ (see e.g., [BL, Theorem 4.3]).

Finally we remark that we do not know whether the theorems can be extended to *non-positive* matrices. An important theorem concerning this is in [BL, Theorem 4.3] for the analyticity of $P(q)$ near zero. Much closer to our development is the scaling functions: $f(x) = \sum_{i=0}^m c_i f(2x - i)$. It is known that such function can be put into matrix form as in the previous example [DL1]. Daubechies and Lagarias studied the multifractal formalism of the well known scaling function D_4 [DL2]. They showed the differentiability of the corresponding $\tau(q)$, but the consideration depends on the two 2×2 matrices involved to have a common eigenvector. There are some extensions in [LM].

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