# GROWTH RATE FOR BETA-EXPANSIONS

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Abstract. Let $\beta > 1$ and let $m > \beta$ be an integer. Each $x \in I_\beta := [0, \frac{m-1}{\beta-1}]$ can be represented in the form

$$x = \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k},$$

where $\varepsilon_k \in \{0, 1, \dots, m-1\}$ for all $k$ (a $\beta$-expansion of $x$). It is known that a.e. $x \in I_\beta$ has a continuum of distinct $\beta$-expansions. In this paper we prove that if $\beta$ is a Pisot number, then for a.e. $x$ this continuum has one and the same growth rate. We also link this rate to the Lebesgue-generic local dimension for the Bernoulli convolution parametrized by $\beta$.

When $\beta < \frac{1+\sqrt{5}}{2}$, we show that the set of $\beta$-expansions grows exponentially for every internal $x$.

## 1. Introduction

Let $\beta > 1$ and let $m > \beta$ be an integer. Put $I_\beta = [0, (m-1)/(\beta-1)]$. As is well known, each $x \in I_\beta$ can be represented as a $\beta$-*expansion*

$$x = \sum_{n=1}^{\infty} \varepsilon_n \beta^{-n}, \quad \varepsilon_n \in \{0, 1, \dots, m-1\}.$$

Since we do not impose any extra restrictions on the "digits" $\varepsilon_n$, one might expect a typical $x$ to have multiple $\beta$-expansions. Indeed, it was shown that a.e. $x \in I_\beta$ has $2^{\aleph_0}$ such expansions – see [17, 2, 18].

The main purpose of this paper is to study the rate of growth of the set of $\beta$-expansions for a generic $x$ when $\beta$ is a Pisot number (see below). We also show

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that if $\beta$ is smaller than the golden ratio, then every $x$, except the endpoints, has a continuum of $\beta$-expansions with an exponential growth.

Now we are ready to state main results of this paper. Put

$$\mathcal{E}_n(x;\beta) = \Big\{(\varepsilon_1,\dots,\varepsilon_n) \in \{0,1,\dots,m-1\}^n \mid \exists(\varepsilon_{n+1},\varepsilon_{n+2},\dots) \in \{0,1,\dots,m-1\}^{\mathbb{N}} : x = \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k}\Big\}$$

and

$$\mathcal{N}_n(x;\beta) = \#\mathcal{E}_n(x;\beta).$$

(We will write simply $\mathcal{N}_n(x)$ if it is clear what $\beta$ is under consideration.) In other words, $\mathcal{N}_n(x)$ counts the number of words of length $n$ in the alphabet $\{0,1,\dots,m-1\}$ which can serve as prefixes of $\beta$-expansions of $x$. We will be interested in the rate of growth of the function $x \mapsto \mathcal{N}_n(x)$.

Let $\beta > 1$ be a *Pisot number* (an algebraic integer whose conjugates are less than 1 in modulus). Our central result is the following

**Theorem 1.1.** *There exists a constant $\gamma = \gamma(\beta, m) > 0$ such that*

$$\lim_{n\to\infty} \frac{\log \mathcal{N}_n(x;\beta)}{n} = \gamma \quad \text{for } \mathcal{L}\text{-a.e. } x \in I_\beta, \tag{1.1}$$

*where $\mathcal{L}$ denotes the Lebesgue measure.*

Let $\mu = \mu_{\beta,m}$ denote the probability measure on $\mathbb{R}$ defined as follows:

$$\mu(E) = \mathbb{P}\left\{(\varepsilon_1,\varepsilon_2,\dots) \in \{0,1,\dots,m-1\}^{\mathbb{N}} : \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k} \in E\right\},$$

where $\mathbb{P} = \prod_1^\infty \{1/m,\dots,1/m\}$, and $E$ is an arbitrary Borel subset of $\mathbb{R}$.

Recall that a Borel probability measure $\nu$ on $\mathbb{R}$ is called *self-similar* if $\nu = \sum_{i=1}^r p_i\, \nu \circ T_i^{-1}$, where $T_1,\dots,T_r$ are linear contractions on $\mathbb{R}$, $p_i \ge 0$ with $\sum_{i=1}^r p_i = 1$. The measure $\mu$ is known to be a self-similar measure supported on $I_\beta$ with $r = m$, $T_i x = (x+i)/\beta$ $(i = 0,1,\dots,m-1)$ and $p_i \equiv 1/m$ ([13]). When $m = 2$, $\mu$ is the so-called *Bernoulli convolution associated with* $\beta$ – see, e.g., [20]. For $x \in I_\beta$, the *local dimension* of $\mu$ at $x$ is defined by

$$d(\mu, x) = \lim_{r\to 0} \frac{\log \mu([x-r, x+r])}{\log r}, \tag{1.2}$$

provided that the limit exists. As an application of Theorem 1.1, we obtain

**Corollary 1.2.** *For $\mathcal{L}$-a.e. $x \in I_\beta$, $d(\mu_{\beta,m}, x) \equiv (\log m - \gamma)/\log \beta$.*

**Theorem 1.3.** *If $\beta$ is an integer such that $\beta$ divides $m$, then $\gamma = \log(m/\beta)$. Otherwise we have $\gamma < \log(m/\beta)$.*

Theorem 1.1, Corollary 1.2 and Theorem 1.3 together yield

**Proposition 1.4.** *We have $d(\mu_{\beta,m}, x) \equiv D_{\beta,m}$ for Lebesgue-a.e. $x \in I_\beta$ with $1 \le D_{\beta,m} < \log_\beta m$. Moreover, $D_{\beta,m} > 1$ unless $\beta$ is an integer dividing $m$.*

In addition to the above results for Pisot $\beta$, we also obtain a general result for all small $\beta$ which holds for all internal $x$. Recall that if $\beta \in \left(1, \frac{1+\sqrt{5}}{2}\right)$ and $m = 2$, then any $x \in \left(0, \frac{1}{\beta-1}\right)$ has a continuum of distinct $\beta$-expansions (see [4, Theorem 3]). We prove a quantitative version of this claim for an arbitrary $m \ge 2$:

**Theorem 1.5.** *Let $\beta$ be an arbitrary number in $\left(1, \frac{1+\sqrt{5}}{2}\right)$. Then there exists $\kappa = \kappa(\beta) > 0$ such that*

$$\varliminf_{n\to\infty} \frac{\log_2 \mathcal{N}_n(x;\beta)}{n} \ge \kappa \quad \text{for any } x \in \left(0, \frac{m-1}{\beta-1}\right). \tag{1.3}$$

**Corollary 1.6.** *For any $\beta \in \left(1, \frac{1+\sqrt{5}}{2}\right)$ and $m = 2$, we have*

$$\overline{d}(\mu, x) \le (1-\kappa)\log_\beta 2$$

*for all $x \in \left(0, \frac{1}{\beta-1}\right)$, where*

$$\overline{d}(\mu, x) = \varlimsup_{r\to 0} \frac{\log \mu([x-r, x+r])}{\log r}.$$

The content of the paper is the following. In Section 2, we prove Theorem 1.1 and Corollary 1.2. Section 3 is devoted to the proof of Theorem 1.3. In Section 4, we consider an important class of examples, namely, the case when $\beta$ is a multinacci number. In Section 5, we prove Theorem 1.5 and give an explicit lower bound for $\kappa$.

## 2. Proof of Theorem 1.1 and Corollary 1.2

First we reformulate our problem in the language of iterated function systems (IFS). Note that

(2.1)
$$\mathcal{E}_n(x;\beta) = \left\{(\varepsilon_1,\dots,\varepsilon_n)\in\{0,1,\dots,m-1\}^n \mid 0\le x-\sum_{k=1}^n \varepsilon_k\beta^{-k}\le \frac{(m-1)\beta^{-n}}{\beta-1}\right\}$$

(see, e.g., [12]). Consider now the following IFS $\Phi=\{S_i\}_{i=1}^m$ on $\mathbb{R}$:

$$S_i(x)=\rho x+(i-1)(1-\rho)/(m-1), \qquad i=1,\dots,m, \tag{2.2}$$

where $\rho=1/\beta\in(0,1)$. Since $m>\beta$, it is clear that $[0,1]$ is the attractor of $\Phi$ (note that $S_m(1)=1$), i.e., $[0,1]=\bigcup_{i=1}^m S_i([0,1])$.

Let $\mathcal{A}$ denote the alphabet $\{1,\dots,m\}$ and $\mathcal{A}_n$ the collection of all words of length $n$ over $\mathcal{A}$, $n\in\mathbb{N}$. For $z\in I_\beta$ it is clear that

$$\mathcal{N}_n\left(\frac{(m-1)z}{\beta-1}\right)=\#\{J=j_1\cdots j_n\in\mathcal{A}_n:\ z\in S_J([0,1])\}, \tag{2.3}$$

where $S_J:=S_{j_1}\circ S_{j_2}\circ\cdots\circ S_{j_n}$. This is because

$$S_J(z)=\frac{1-\rho}{m-1}\sum_{k=1}^n (j_k-1)\rho^{k-1}+\rho^n z,$$

and thus, $S_J([0,1])=\left[\frac{1-\rho}{m-1}\sum_{k=1}^n(j_k-1)\rho^{k-1},\, \frac{1-\rho}{m-1}\sum_{k=1}^n(j_k-1)\rho^{k-1}+\rho^n\right]$, which is none other than a rescaled version of (2.1).

We sketch here the proof of Theorem 1.1: first we encode the interval $[0,1]$ as a cylinder in a subshift space of finite type, and show that $\mathcal{N}_n(\frac{m-1}{\beta-1}z)$ corresponds to the norm of a matrix product which depends on the coding of $z$ and $n$. Next, we construct an irreducible branch of the subshift in question and assign an invariant Markov measure such that its projection under the coding map is equivalent to the Lebesgue measure on a subinterval of $[0,1]$. Then by the subadditive ergodic theorem, $\lim_{n\to\infty}\frac{\log\mathcal{N}_n(\frac{m-1}{\beta-1}z)}{n}$ equals a non-negative constant $\mathcal{L}$-a.e. on this subinterval; in the end we extend the result to the whole interval $[0,1]$.

Finally, we apply the theory of random $\beta$-expansions to show that this constant $\gamma$ is strictly positive.

2.1. **Coding of $[0,1]$ and matrix products.** In this part, we will encode $[0,1]$ via a subshift and show that $\mathcal{N}_n(\frac{m-1}{\beta-1}x)$ can be expressed in terms of matrix products. This approach mainly follows [6].

For $n \in \mathbb{N}$, define

$$P_n = \{S_J(0) :\ J \in \mathcal{A}_n\} \cup \{S_J(1) :\ J \in \mathcal{A}_n\}.$$

The points in $P_n$, written as $h_1, \cdots, h_{s_n}$ (ranked in the increasing order), partition $[0,1]$ into non-overlapping closed intervals which are called *$n$-th net intervals*. Let $\mathcal{F}_n$ denote the collection of $n$-th net intervals, that is,

$$\mathcal{F}_n = \{[h_j, h_{j+1}] :\ j = 1, \ldots, s_n - 1\}.$$

For convenience we write $\mathcal{F}_0 = \{[0,1]\}$. Since $P_n \subset P_{n+1}$, we obtain the following net properties:

(i) $\bigcup_{\Delta \in \mathcal{F}_n} \Delta = [0,1]$ for any $n \geq 0$;
(ii) For any $\Delta_1, \Delta_2 \in \mathcal{F}_n$ with $\Delta_1 \neq \Delta_2$, $\mathrm{int}(\Delta_1) \cap \mathrm{int}(\Delta_2) = \emptyset$;
(iii) For any $\Delta \in \mathcal{F}_n$ ($n \geq 1$), there is a unique element $\widehat{\Delta} \in \mathcal{F}_{n-1}$ such that $\widehat{\Delta} \supset \Delta$.

For $\Delta = [a,b] \in \mathcal{F}_n$, we define

$$\begin{aligned} \mathcal{N}_n(\Delta) &= \#\{J \in \mathcal{A}_n :\ S_J((0,1)) \cap \Delta \neq \emptyset\} \\ &= \#\{J \in \mathcal{A}_n :\ S_J([0,1]) \supset \Delta\}. \end{aligned} \tag{2.4}$$

It is easy to see that

$$\mathcal{N}_n\Big(\frac{m-1}{\beta-1}z\Big) = \mathcal{N}_n(\Delta) \quad \text{for any } \Delta \in \mathcal{F}_n \text{ and each } z \in \mathrm{int}(\Delta), \tag{2.5}$$

where $\mathcal{N}_n(z)$ is defined as in (2.3).

As shown in [6], the interval $[0,1]$ can be coded via a subshift of finite type, and for each $n \geq 1$ and $\Delta \in \mathcal{F}_n$, $\mathcal{N}_n(\Delta)$ corresponds to the norm of certain matrix product which depends on the coding of $\Delta$. More precisely, the following results (C1)-(C4) were obtained in [6, Section 2]:

(C1) There exist a finite alphabet $\Omega = \{1, \ldots, r\}$ with $r \geq 2$ and an $r \times r$ matrix $A = (A_{ij})$ with 0-1 entries such that for each $n \geq 0$, there is a one-to-one surjective map $\phi_n :\ \mathcal{F}_n \to \Omega^{(1)}_{A,n+1}$, where

$$\Omega^{(1)}_{A,n+1} = \big\{x_1 \ldots x_{n+1} \in \Omega^{n+1} :\ x_1 = 1,\ A_{x_i x_{i+1}} = 1 \text{ for } 1 \leq i \leq n\big\}.$$

The map $\phi_n$ is called the *n-th coding map* and for $\Delta \in \mathcal{F}_n$, $\phi_n(\Delta)$ is called *the n-th coding of* $\Delta$.

(C2) The coding maps $\phi_n$ preserve the net structure in the sense that for any $x_1 \dots x_{n+2} \in \Omega^{(1)}_{A,n+2}$,

$$\phi_{n+1}^{-1}(x_1 \dots x_{n+2}) \subseteq \phi_n^{-1}(x_1 \dots x_{n+1}).$$

(C3) There is a family of positive numbers $\ell_i$, $1 \le i \le r$, such that for each $\Delta \in \mathcal{F}_n$ with $\phi_n(\Delta) = x_1 \dots x_{n+1}$,

$$|\Delta| = \ell_{x_{n+1}} \rho^n,$$

where $|\Delta|$ denotes the length of $\Delta$.

(C4) There are a family of positive integers $v_i$, $1 \le i \le r$, with $v_1 = 1$, and a family of non-negative matrices

$$\{T(i,j) : \ 1 \le i,j \le r, \ A_{ij} = 1\}$$

with $T(i,j)$ being a $v_i \times v_j$ matrix, such that for each $n \ge 1$ and $\Delta \in \mathcal{F}_n$,

$$\mathcal{N}_n(\Delta) = \|T(x_1,x_2)\dots T(x_n,x_{n+1})\|, \tag{2.6}$$

where $x_1 \dots x_{n+1} = \phi_n(\Delta)$, $\|M\|$ denotes the sum of the absolute values of entries of $M$. Furthermore, the product $T(x_1,x_2)\dots T(x_n,x_{n+1})$ is a strictly positive $v_{x_{n+1}}$-dimensional row vector.

To prove Theorem 1.1, we still need the following property of $\Omega$, which was proved in [7, Lemma 6.4]):

(C5) There is a non-empty subset $\widehat{\Omega}$ of $\Omega$ satisfying the following properties:
- (i) $\{j \in \Omega : \ A_{ij} = 1\} \subseteq \widehat{\Omega}$ for any $i \in \widehat{\Omega}$.
- (ii) For any $i, j \in \widehat{\Omega}$, there exist $x_1, \dots, x_n \in \widehat{\Omega}$ such that $x_1 = i$, $x_n = j$ and $A_{x_k x_{k+1}} = 1$ for $1 \le k \le n-1$.
- (iii) For any $i \in \Omega$ and $j \in \widehat{\Omega}$, there exist $x_1, \dots, x_n \in \Omega$ such that $x_1 = i$, $x_n = j$ and $A_{x_k x_{k+1}} = 1$ for $1 \le k \le n-1$.

**Remark 2.1.** *Since $\mathcal{F}_n$ has the net structure, we have for each $\Delta \in \mathcal{F}_n$,*

$$|\Delta| = \sum_{\Delta' \in F_{n+1}, \ \Delta' \subseteq \Delta} |\Delta'|,$$

*which together with (C1)-(C3) yields*

$$\ell_i = \rho \sum_{j\in\Omega,\ A_{ij}=1} \ell_j \quad \text{for all } i\in\Omega. \tag{2.7}$$

*By part (i) of (C5), we have also*

$$\ell_i = \rho \sum_{j\in\widehat{\Omega},\ A_{ij}=1} \ell_j \quad \text{for all } i\in\widehat{\Omega}. \tag{2.8}$$

2.2. **Proof of Theorem 1.1.** In this part we prove the following

**Theorem 2.2.** *There exists a constant $\gamma\geq 0$ such that for $\Delta\in\mathcal{F}_k$, if the $k$-th coding $y_1\dots y_{k+1}=\phi_k(\Delta)$ of $\Delta$ satisfies $y_{k+1}\in\widehat{\Omega}$, then*

$$\lim_{n\to\infty}\frac{\log\mathcal{N}_n(\frac{m-1}{\beta-1}x)}{n} = \gamma \text{ for } \mathcal{L}\text{-a.e. } x\in\Delta. \tag{2.9}$$

Let us first show that Theorem 2.2 implies Theorem 1.1. To see it, we say a net interval $\Delta$ is *good* if it satisfies the condition of Theorem 2.2. According to part (iii) of (C5), there is an positive integer $N$ such that for any net interval $\Delta\in\mathcal{F}_n$, there is $k\leq N$ and an $(n+k)$-th net interval which is contained in $\Delta$ and is good. Hence by Theorem 2.2 and (C2)-(C3), there is a constant $c>0$ such that for any net interval $\Delta$, (2.9) holds for a sub-net-interval of $\Delta$ with Lebesgue measure greater than $c|\Delta|$. A recursive argument then shows that (2.9) holds for $[0,1]$.

*Proof of Theorem 2.2.* Consider the one-sided subshift of finite type $(\widehat{\Omega}_A^{\mathbb{N}},\ \sigma)$, where

$$\widehat{\Omega}_A^{\mathbb{N}} = \left\{(x_i)_{i=1}^\infty :\ x_i\in\widehat{\Omega},\ A_{x_ix_{i+1}}=1 \text{ for } i\geq 1\right\},$$

and $\sigma$ is the left shift defined by $(x_i)_{i=1}^\infty\mapsto(x_{i+1})_{i=1}^\infty$. By parts (i)-(ii) of (C5), $(\widehat{\Omega}_A^{\mathbb{N}},\ \sigma)$ is topologically transitive. Define a matrix $P=(P_{ij})_{i,j\in\widehat{\Omega}}$ by

$$P_{ij} = \begin{cases} \rho\ell_j/\ell_i & \text{if } A_{ij}=1,\\ 0 & \text{otherwise.}\end{cases} \tag{2.10}$$

By (2.8) and part (ii) of (C5), $P$ is an irreducible transition matrix. Hence there is a unique $\#(\widehat{\Omega})$-dimensional positive probability vector $\mathbf{p}=(p_i)_{i\in\widehat{\Omega}}$ so that $\mathbf{p}P=\mathbf{p}$. Let $\eta$ be the $(\mathbf{p},P)$-Markov measure on $\widehat{\Omega}_A^{\mathbb{N}}$, i.e.,

$$\eta([x_1\dots x_n]) = p_{x_1}P_{x_1x_2}\dots P_{x_{n-1}x_n}$$

for any cylinder set $[x_1 \dots x_n]$ in $\widehat{\Omega}_A^{\mathbb{N}}$. Since $P$ is irreducible, $\eta$ is ergodic[1]. By the definition of $P$, we can check that

$$\eta([x_1 \dots x_n]) = p_{x_1} \ell_{x_n} \rho^{n-1} \tag{2.11}$$

for any cylinder set $[x_1 \dots x_n]$ in $\widehat{\Omega}_A^{\mathbb{N}}$.

Consider the family of matrices $\{T(i,j) : \ i,j \in \widehat{\Omega}, \ A_{i,j} = 1\}$. Observe that for any $x_1 \dots x_{n+m} \in \widehat{\Omega}_{A,n+m}$,

$$\begin{aligned}
&\|T(x_1,x_2)\dots T(x_{n+m-1},x_{n+m})\| \\
&\quad = \mathbf{e}_{v_{x_1}} T(x_1,x_2)\dots T(x_{n+m-1},x_{n+m}) \mathbf{e}^t_{v_{x_{n+m}}} \\
&\quad \le \mathbf{e}_{v_{x_1}} T(x_1,x_2)\dots T(x_{n-1},x_n) \mathbf{e}^t_{v_{x_n}} \mathbf{e}_{v_{x_n}} T(x_n,x_{n+1})\dots T(x_{n+m-1},x_{n+m}) \mathbf{e}^t_{v_{x_{n+m}}} \\
&\quad = \|T(x_1,x_2)\dots T(x_{n-1},x_n)\| \cdot \|T(x_n,x_{n+1})\dots T(x_{n+m-1},x_{n+m})\|,
\end{aligned}$$

where $\mathbf{e}_k$ denotes the $k$-dimensional row vector $(1,1,\dots,1)$, and $\mathbf{e}_k^t$ denotes the transpose of $\mathbf{e}_k$. By the Kingman subadditive ergodic theorem, there exists a constant $\gamma \ge 0$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|T(x_1,x_2)\dots T(x_{n-1},x_n)\| = \gamma \quad \text{for } \eta\text{-a.e. } x = (x_i)_{i=1}^\infty \in \widehat{\Omega}_A^{\mathbb{N}}. \tag{2.12}$$

Now assume that $\Delta$ is a $k$-th net interval with the coding $\phi_k(\Delta) = y_1 \dots y_{k+1}$ such that $y_{k+1} \in \widehat{\Omega}$. Define the projection map $\pi : \ [y_{k+1}] \to \mathbb{R}$ by

$$\{\pi(x)\} = \bigcap_{n=1}^\infty \phi_{n+k}^{-1}(y_1 \dots y_k x_1 \dots x_{n+1}), \quad x = (x_i)_{i=1}^\infty \in \widehat{\Omega} \text{ with } x_1 = y_{k+1}. \tag{2.13}$$

Since the coding maps preserve the net structure (see (C2)), the projection $\pi$ is well defined and is one-to-one, except for a countable set on which it is is two-to-one. Let $\nu = \eta|_{[y_{k+1}]}$ be the restriction of $\eta$ on the cylinder $[y_{k+1}]$. Let $\nu \circ \pi^{-1}$ be the projection of $\nu$ under $\pi$.

We claim that $\nu \circ \pi^{-1}$ is equivalent to $\mathcal{L}|_\Delta$, the Lebesgue measure restricted on $\Delta$, in the sense that there exists a constant $C \ge 1$ such that $C^{-1}\mathcal{L}|_\Delta \le \nu \circ \pi^{-1} \le C\mathcal{L}|_\Delta$. The claim just follows from the fact that for each sub net interval $\Delta'$ with coding $y_1 \dots y_k x_1 \dots x_{n+1}$,

$$\nu \circ \pi^{-1}(\Delta') = \eta([x_1 \dots x_{n+1}]) = p_{x_1} \ell_{x_{n+1}} \rho^n = p_{y_{k+1}} \rho^{-k} |\Delta'|,$$

[1]The reader may actually check that $\eta$ is the unique invariant measure of maximal entropy, the so-called *Parry measure* on $\widehat{\Omega}_A^{\mathbb{N}}$ – see, e.g., [21] for the definition.

where we use (2.11) and (C3). Since the collection of sub net intervals of $\Delta$ generates the Borel sigma-algebra on $\Delta$, $\nu\circ\pi^{-1}$ only differs from $\mathcal{L}|_\Delta$ by a constant. The claim thus follows.

Now assume that $x=(x_i)_{i=1}^\infty\in[y_{k+1}]$ such that $z=\pi(x)\notin\bigcup_{n\geq 0}P_n$. Then by (2.5),

$$\begin{aligned}\mathcal{N}_{n+k}\Big(\frac{m-1}{\beta-1}z\Big) &= \|T(y_1,y_2)\dots T(y_k,y_{k+1})T(x_1,x_2)\dots T(x_n,x_{n+1})\|\\ &\asymp \|T(x_1,x_2)\dots T(x_n,x_{n+1})\|,\end{aligned}$$

where we use the fact that $T(y_1,y_2)\dots T(y_k,y_{k+1})$ is a strictly positive vector (see (C4)), and the notation $a_n\asymp b_n$ means that $C^{-1}b_n\leq a_n\leq Cb_n$ for a positive constant $C\geq 1$ independent of $n$. This together with (2.12) yields

$$\lim_{n\to\infty}\frac{1}{n}\log\mathcal{N}_n\Big(\frac{m-1}{\beta-1}\pi(x)\Big)=\gamma\qquad\text{for }\eta\text{-a.e. }x\in[y_{k+1}],$$

and hence

$$\lim_{n\to\infty}\frac{1}{n}\log\mathcal{N}_n(z)=\gamma\qquad\text{for }\nu\circ\pi^{-1}\text{-a.e. }z\in\mathbb{R}.$$

Since $\nu\circ\pi^{-1}$ is equivalent to $\mathcal{L}|_\Delta$, we obtain Theorem 2.2 (and thus, Theorem 1.1) with $\gamma\geq 0$. □

2.3. **Proof that $\gamma>0$.** Let us consider first the case of non-integer $\beta$. It is clearly sufficient to prove $\gamma>0$ for $m=\lfloor\beta\rfloor+1$. Following [2], we introduce the random $\beta$-transformation $K_\beta$. Namely, put

$$S_k=\Big[\frac{k}{\beta},\frac{\lfloor\beta\rfloor}{\beta(\beta-1)}+\frac{k-1}{\beta}\Big]\tag{2.14}$$

(the switch regions) and

$$E_k=\Big(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)}+\frac{k-1}{\beta},\frac{k+1}{\beta}\Big),\quad k=1,\dots,\lfloor\beta\rfloor-1,$$

with

$$E_0=\Big[0,\frac{1}{\beta}\Big),\quad E_{\lfloor\beta\rfloor}=\Big(\frac{\lfloor\beta\rfloor}{\beta(\beta-1)}+\frac{\lfloor\beta\rfloor-1}{\beta},\frac{\lfloor\beta\rfloor}{\beta-1}\Big]$$

(the equality regions). Put now $\Omega = \{0,1\}^{\mathbb{N}}$ and the map $K_\beta : \Omega \times I_\beta \to \Omega \times I_\beta$ defined as

$$K_\beta(\omega, x) = \begin{cases} (\omega, \beta x - k), & x \in E_k,\ k = 0, 1, \dots, \lfloor \beta \rfloor, \\ (\sigma(\omega), \beta x - k), & x \in S_k \text{ and } \omega_1 = 1,\ k = 1, \dots, \lfloor \beta \rfloor, \\ (\sigma(\omega), \beta x - k + 1), & x \in S_k \text{ and } \omega_1 = 0,\ k = 1, \dots, \lfloor \beta \rfloor, \end{cases}$$

where $\sigma(\omega_1, \omega_2, \omega_3, \dots) = (\omega_2, \omega_3, \dots)$. The map $K_\beta$ generates all $\beta$-expansions of $x$ by acting as a shift – see [2, p. 159] for more details. More precisely, if $x \in E_k$, then the first digit of its $\beta$-expansion must be $k$; if $x \in S_k$, it can be either $k$ or $k-1$.

It was shown in [2] that there exists a unique probability measure $m_\beta$ on $I_\beta$ such that $m_\beta$ is equivalent to the Lebesgue measure and $\mathbb{P} \otimes m_\beta$ is invariant and ergodic under $K_\beta$, where $\mathbb{P} = \prod_1^\infty \left\{\frac{1}{2}, \frac{1}{2}\right\}$.[2]

The famous Garsia separation lemma ([8, Lemma 1.51]) states that there exists a constant $C = C(\beta, m) > 0$ such that if $\sum_{j=1}^n \varepsilon_j \beta^{-j} \neq \sum_{j=1}^n \varepsilon'_j \beta^{-j}$ for some $\varepsilon_j, \varepsilon'_j \in \{0, 1, \dots, m-1\}$, then $\left|\sum_{j=1}^n (\varepsilon_j - \varepsilon'_j)\beta^{-j}\right| \geq C\beta^{-n}$. Hence

$$\# \left\{ \sum_{j=1}^n \varepsilon_j \beta^{-j} \mid \varepsilon_j \in \{0, 1, \dots, m-1\} \right\} = O(\beta^n). \tag{2.15}$$

In particular, there exist $k \geq 2$ and two words $a_1 \dots a_k$ and $b_1 \dots b_k$ with $a_j, b_j \in \{0, 1, \dots, \lfloor \beta \rfloor\}$ such that $\sum_{j=1}^k a_j \beta^{-j} = \sum_{j=1}^k b_j \beta^{-j}$.

Let $J_{a_1 \dots a_k}$ denote the interval of $x$ which can have $a_1 \dots a_k$ as a prefix of their $\beta$-expansions. (It is obvious that $J_{a_1 \dots a_k} = \left[\sum_1^k a_j \beta^{-j}, \sum_1^k a_j \beta^{-j} + \frac{\lfloor \beta \rfloor}{\beta - 1} \beta^{-k}\right]$.) It follows from the ergodicity of $K_\beta$ and [2, Lemma 8] that for $\mathbb{P} \otimes \mathcal{L}$-a.e. $(\omega, x) \in \Omega \times I_\beta$ the block $a_1 \dots a_k$ appears in the $\beta$-expansion of $x$ (specified by $\omega$) with a limiting frequency $\widetilde{\gamma} > 0$.

In particular, for $\mathcal{L}$-a.e. $x$ there exists a $\beta$-expansion $(\varepsilon_1, \varepsilon_2, \dots)$ which contains the block $a_1 \dots a_k$ with the positive limiting frequency $\widetilde{\gamma}$, i.e.,

$$\lim_{n \to \infty} \frac{1}{n} \#\{j : \varepsilon_j \dots \varepsilon_{j+k-1} = a_1 \dots a_k\} = \widetilde{\gamma}.$$

[2]It should also be noted that $K_\beta$ has a unique ergodic measure of maximal entropy which is singular with respect to $\mathbb{P} \otimes m_\beta$ and whose projection onto the second coordinate is precisely $\mu_{\beta,m}$ – see [1] for more detail.

Since any such block can be replaced with $b_1 \dots b_k$, and the resulting sequence remains a $\beta$-expansion of $x$, we conclude, in view of (1.1), that $\gamma/\log 2 \ge \widetilde{\gamma} > 0$.

Let now $\beta \in \mathbb{N}$, so $m \ge \beta + 1$. In a $\beta$-expansion with digits $\{0, 1, \dots, m-1\}$ one can replace the block 10 with $0\beta$ without altering the rest of the expansion. Since for $\mathcal{L}$-a.e. $x$ its $\beta$-ary expansion (with digits $0, 1, \dots, \beta - 1$) contains the block 01 with the limiting frequency $\beta^{-2} > 0$, we conclude that $\gamma/\log 2 \ge \beta^{-2} > 0$.

The proof of Theorem 1.1 is complete.

The same argument as above proves

**Proposition 2.3.** *If $\beta$ satisfies an algebraic equation with integer coefficients bounded by $m$ in modulus, then there exists $C = C(\beta, m) > 0$ such that*

$$\varliminf_{n\to\infty} \frac{\log \mathcal{N}_n(x;\beta)}{n} \ge C \text{ for } \mathcal{L}\text{-a.e. } x \in I_\beta. \tag{2.16}$$

It is an intriguing open question whether (2.16) holds for all $\beta > 1$. (See also Section 5.)

2.4. **Proof of Corollary 1.2.** Note first that (2.1) can be rewritten as follows:

$$\mathcal{E}_n(x;\beta) = \left\{ (\varepsilon_1, \dots, \varepsilon_n) \in \{0, 1, \dots, m-1\}^n : x - \frac{(m-1)\beta^{-n}}{\beta - 1} \le \sum_{k=1}^n \varepsilon_k \beta^{-k} \le x \right\}.$$

Thus, if $(\varepsilon_1, \dots, \varepsilon_n) \in \mathcal{E}_n(x;\beta)$, then for any $(\varepsilon_{n+1}, \varepsilon_{n+2}, \dots) \in \{0, 1, \dots, m-1\}^{\mathbb{N}}$ we have

$$x - \frac{(m-1)\beta^{-n}}{\beta - 1} \le \sum_{k=1}^\infty \varepsilon_k \beta^{-k} \le x + \frac{(m-1)\beta^{-n}}{\beta - 1}.$$

Hence by definition,

$$\mu\left( x - \frac{(m-1)\beta^{-n}}{\beta - 1}, x + \frac{(m-1)\beta^{-n}}{\beta - 1} \right) \ge m^{-n} \mathcal{N}_n(x;\beta). \tag{2.17}$$

Put now

$$\mathcal{E}'_n(x;\beta) = \Big\{ (\varepsilon_1, \dots, \varepsilon_n) \in \{0, 1, \dots, m-1\}^n : \\ x - \frac{(m-1)\beta^{-n}}{\beta - 1} - \frac{\beta^{-n}}{n^2} \le \sum_{k=1}^n \varepsilon_k \beta^{-k} \le x + \frac{\beta^{-n}}{n^2} \Big\}.$$

We are going to need the following

**Lemma 2.4.** *For $\mathcal{L}$-a.e. $x \in I_\beta$ we have $\mathcal{E}'_n(x;\beta) = \mathcal{E}_n(x;\beta)$ for all $n$, except, possibly, a finite number (depending on $x$).*

*Proof.* We have

$$\mathcal{E}'_n(x;\beta) \setminus \mathcal{E}_n(x;\beta) = \left\{(\varepsilon_1,\dots,\varepsilon_n) : 0 < x - \frac{(m-1)\beta^{-n}}{\beta-1} - \sum_{k=1}^n \varepsilon_k\beta^{-k} \le \frac{\beta^{-n}}{n^2}\right\}$$
$$\cup \left\{(\varepsilon_1,\dots,\varepsilon_n) : 0 < \sum_{k=1}^n \varepsilon_k\beta^{-k} - x \le \frac{\beta^{-n}}{n^2}\right\}.$$

Hence, in view of (2.15),

$$\mathcal{L}\left\{x : \mathcal{E}'_n(x;\beta) \setminus \mathcal{E}_n(x;\beta) \ne \emptyset\right\} = O\left(\frac{1}{n^2}\right),$$

whence by the Borel-Cantelli lemma,

$$\mathcal{L}\Big\{x : \mathcal{E}'_n(x;\beta) \setminus \mathcal{E}_n(x;\beta) \ne \emptyset \text{ for an infinite set of } n\Big\} = 0.$$

□

Return to the proof of the corollary. Put

$$\mathcal{D}'_n(x;\beta) = \left\{(\varepsilon_1,\varepsilon_2,\dots) : x - \frac{\beta^{-n}}{n^2} \le \sum_{k=1}^\infty \varepsilon_k\beta^{-k} \le x + \frac{\beta^{-n}}{n^2}\right\}.$$

Note that if $(\varepsilon_1,\varepsilon_2,\dots) \in \mathcal{D}'_n(x;\beta)$, then $(\varepsilon_1,\dots,\varepsilon_n) \in \mathcal{E}'_n(x;\beta)$, since $\sum_1^n \varepsilon_k\beta^{-k} \ge \sum_1^\infty \varepsilon_k\beta^{-k} - \frac{(m-1)\beta^{-n}}{\beta-1}$. Thus, by Lemma 2.4, for $\mathcal{L}$-a.e. $x$ and all sufficiently large $n$,

$$\mu\left(x - \frac{\beta^{-n}}{n^2}, x + \frac{\beta^{-n}}{n^2}\right) \le m^{-n}\mathcal{N}_n(x;\beta).$$

Together with (2.17), we obtain for $\mathcal{L}$-a.e. $x$,

$$\mu\left(x - \frac{\beta^{-n}}{n^2}, x + \frac{\beta^{-n}}{n^2}\right) \le m^{-n}\mathcal{N}_n(x;\beta) \le \mu\left(x - \frac{(m-1)\beta^{-n}}{\beta-1}, x + \frac{(m-1)\beta^{-n}}{\beta-1}\right).$$

Taking logs, dividing by $n$ and passing to the limit as $n \to \infty$ yields the claim of Corollary 1.2.[3]

---

[3]It is easy to see that $d(\mu, x)$ exists if the limit in (1.2) exists along some exponentially decreasing subsequence of $r$.

## 3. Proof of Theorem 1.3

We first introduce some notation. For $q \in \mathbb{R}$, we use $\tau(q)$ to denote the $L^q$ *spectrum of* $\mu$, which is defined by

$$\tau(q) = \varliminf_{r\to 0+} \frac{\log\left(\sup \sum_i \mu([x_i - r, x_i + r])^q\right)}{\log r},$$

where the supremum is taken over all the disjoint families $\{[x_i - r, x_i + r]\}_i$ of closed intervals with $x_i \in [0, 1]$. It is easily checked that $\tau(q)$ is a concave function of $q$ over $\mathbb{R}$, $\tau(1) = 0$ and $\tau(0) = -1$. For $\alpha \geq 0$, let

$$E(\alpha) = \{x \in [0, 1] : \ d(\mu, x) = \alpha\},$$

where $d(\mu, x)$ is defined as in (1.2). The following lemma is a basic fact in multifractal analysis (see, e.g., [15, Theorem 4.1] for a proof).

**Lemma 3.1.** *Let* $\alpha \geq 0$. *If* $E(\alpha) \neq \emptyset$, *then*

$$\dim_H E(\alpha) \leq \alpha q - \tau(q), \qquad \forall\, q \in \mathbb{R}. \tag{3.1}$$

*Proof of Theorem 1.3.* Set $t = (\log m - \gamma)/\log \beta$. By Corollary 1.2, we have $d(\mu, x) = t$ for $\mathcal{L}$-a.e. $x \in [0, 1]$. It was proved in [3, Proposition 5.3] that $\mu$ is absolutely continuous if and only if $\beta$ is an integer so that $\beta | m$. When $\mu$ is absolutely continuous, $d(\mu, x) = 1$ for $\mathcal{L}$-a.e. $x \in [0, 1]$ and hence $t = 1$, which implies that $\gamma = \log(m/\beta)$.

In the following we assume that $\mu$ is singular. It was proved in [16] that $\dim_H \mu < 1$. Since $d(\mu, x) = t$ for $\mathcal{L}$-a.e. $x \in [0, 1]$, we have $\mathcal{L}(E(t)) = 1$ and hence $\dim_H E(t) = 1$. By (3.1), we have

$$1 \leq tq - \tau(q), \qquad \forall\, q \in \mathbb{R}. \tag{3.2}$$

Taking $q = 1$ in (3.2) and using the fact $\tau(1) = 0$, we have $t \geq 1$. It was proved in [5] that $\tau(q)$ is differentiable for $q > 0$ and $\dim_H \mu = \tau'(1)$. Since $\tau$ is also concave, $\tau'$ is continuous on $(0, +\infty)$. By (3.2) and the fact $\tau(0) = -1$, we have $\tau(q) - \tau(0) \leq tq$ for all $q \in \mathbb{R}$, which implies

$$\tau'(0+) \leq t \leq \tau'(0-). \tag{3.3}$$

Since $\tau$ is concave, it is absolutely continuous on $[0, 1]$ and hence

$$1 = \tau(1) - \tau(0) = \int_{[0,1]} \tau'(x)\, dx. \tag{3.4}$$

Since $\tau'(1) = \dim_H \mu < 1$, and $\tau'$ is non-increasing on $(0,1)$, by (3.4) we must have $\tau'(0+) = \lim_{q\to 0+} \tau'(q) > 1$. This together with (3.3) yields $t > 1$. Hence we have $\gamma < \log(m/\beta)$. $\square$

**Remark 3.2.** (1) *It is interesting to compare Proposition 1.4 with a similar result for a Bernoulli-generic $x$. Let, for simplicity, $m = 2$; then it is known that $d(\mu_{\beta,2}) \equiv H_\beta < 1$ for $\mu_\beta$-a.e. $x$. – see* [14]. *Here $H_\beta$ is Garsia's entropy introduced in* [9] *(see also* [12] *for some lower bounds for $H_\beta$).*

(2) *It was proved in* [5] *that the set of local dimensions of $\mu$ contains the set $\{\tau'(q) : q > 0\}$. In the case that $\mu$ is singular, this set contains a neighborhood of 1. To see it, just note that $\tau'(1) = \dim_H \mu < 1 < \tau'(0+)$.*

(3) *We do not know whether the set of local dimensions of $\mu$,*

$$\{\alpha \geq 0 : E(\alpha) \neq \emptyset\},$$

*is always a closed interval. Nevertheless, it was proved in* [7] *that for each Pisot number $\beta$ and positive integer $m$, there exists an interval $I$ with $\mu(I) > 0$ such that the set of local dimensions of $\mu|_I$ is always a closed interval, where $\mu|_I$ denotes the restriction of $\mu$ on $I$.*

(4) *We conjecture that $\tau'(0)$ exists. If this is true, by (3.3) we have $t = \tau'(0)$.*

(5) *The following result can be proved in a way similar to the proof of Theorem 1.3: assume that $\eta$ is a compactly supported Borel probability measure on $\mathbb{R}^d$ so that $d(\eta, x) = t$ on a set of Hausdorff dimension $d$. Then $t > d$ if $\tau'(1-) < d$. The reader is referred to* [5] *for the definitions of $d(\eta, x)$ and $\tau(q)$ for a measure on $\mathbb{R}^d$.*

## 4. Examples

As we have seen from the proof of Theorem 1.1, the exponent $\gamma$ in (1.1) corresponds to the Lyapunov exponent of certain family of non-negative matrices. In the case when this family contains a rank-one matrix (for instance, this occurs when $v_i = 1$ for some $i \in \widehat{\Omega}$), the corresponding matrix product is degenerate and one may obtain an explicit theoretic formula (via series expansion) for $\gamma$. Let us consider an important family of examples.

**Example 4.1.** Fix an integer $n \geq 2$. Let $\beta_n$ be the positive root of $x^n = x^{n-1} + \ldots + x + 1$ (often called the $n$'th *multinacci number*). Let $m = 2$. The following formula for $\gamma_n = \gamma(\beta_n)$ was obtained in [6, Theorem 1.2]:

$$\gamma_n = \frac{\beta^{-n}\left(1-2\beta^{-n}\right)^2}{2-(n+1)\beta^{-n}} \sum_{k=0}^{\infty} \left( \beta^{-nk} \sum_{J\in\mathcal{A}_k} \log\|M_J\| \right), \tag{4.5}$$

where $\mathcal{A}_0 = \{\emptyset\}$ and $\mathcal{A}_k = \{1,2\}^k$ for $k \geq 1$. $M_\emptyset$ denotes the $2\times 2$ identity matrix, and $M_1, M_2$ are two $2\times 2$ matrices given by

$$M_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

For $J = j_1 \dots j_k \in \mathcal{A}_k$, $M_J$ denotes $M_{j_1}M_{j_2}\dots M_{j_n}$. For any $2\times 2$ non-negative matrix $B$, $\|B\| = (1,1)B(1,1)^t$.

The numerical estimations in Table 1 were given in [6] for $\gamma_n/\log 2$, $n = 2,\dots,10$. We also include in the table the approximate values for $D_\beta = D_{\beta,2}$ (see Corollary 1.2 and Proposition 1.4) and for Garsia's entropy $H_\beta$ for comparison (taken from [11]).

| $n$ | $\beta_n$ | $\gamma_n/\log 2$ | $D_{\beta_n}$ | $H_{\beta_n}$ |
|---|---|---|---|---|
| 2 | 1.618034 | $0.302 \pm 0.001$ | $1.0054 \pm 0.0015$ | 0.995713 |
| 3 | 1.839287 | 0.102500 | 1.028876 | 0.980409 |
| 4 | 1.927562 | 0.041560 | 1.012318 | 0.986926 |
| 5 | 1.965948 | 0.018426 | 1.006510 | 0.992585 |
| 6 | 1.983583 | 0.008590 | 1.003341 | 0.996033 |
| 7 | 1.991964 | 0.004123 | 1.001695 | 0.997937 |
| 8 | 1.996031 | 0.002014 | 1.000854 | 0.998945 |
| 9 | 1.998029 | 0.000993 | 1.000429 | 0.999465 |
| 10 | 1.999019 | 0.000493 | 1.000215 | 0.999731 |

TABLE 1. Approximate values of $\gamma, D_\beta$ and $H_\beta$ for the multinacci family

## 5. Proof of Theorem 1.5 and Corollary 1.6

Let us first observe that without loss of generality we may confine ourselves to the case $m = 2$. Indeed, if $m \geq 3$ and $x \in \left(0, \frac{1}{\beta-1}\right)$, then we can use digits $0, 1$ and apply Theorem 1.5 for $m = 2$. If $x \in \left(\frac{j}{\beta-1}, \frac{j+1}{\beta-1}\right)$ for $1 \leq j \leq m-2$, then we put $y = x - \frac{j}{\beta-1}$ and apply Theorem 1.5 for $m = 2$ to $y$. For the original $x$ the claim will then follow with $\varepsilon_n \in \{j, j+1\}$.

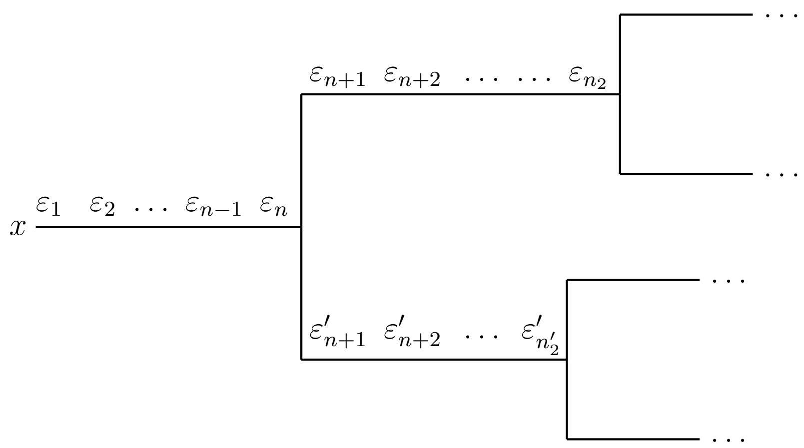


FIGURE 1. Branching and "bifurcations"

If $x = \frac{j}{\beta-1}$ with $1 \le j \le m-2$, then we set $\varepsilon_1 = j-1$ so

$$\beta\left(x - \frac{\varepsilon_1}{\beta}\right) = \frac{\beta+j-1}{\beta-1} = \frac{\varepsilon_2}{\beta} + \frac{\varepsilon_3}{\beta^2} + \cdots$$

It suffices to observe that $\frac{j}{\beta-1} < \frac{\beta+j-1}{\beta-1} < \frac{j+1}{\beta-1}$ and apply the above argument to $(\varepsilon_2, \varepsilon_3, \dots)$.

So, let $m = 2$ and let $x \in I_\beta$ have at least two $\beta$-expansions; then there exists the smallest $n \ge 0$ such that $x \sim (\varepsilon_1, \dots, \varepsilon_n, \varepsilon_{n+1}, \dots)_\beta$ and $x \sim (\varepsilon_1, \dots, \varepsilon_n, \varepsilon'_{n+1}, \dots)_\beta$ with $\varepsilon_{n+1} \ne \varepsilon'_{n+1}$. We may depict this "bifurcation" as is shown in Figure 1.

If $(\varepsilon_{n+1}, \varepsilon_{n+2}, \dots)$ is not a unique expansion, then there exists $n_2 > n$ with the same property, etc. As a result, we obtain a subtree of the binary tree which corresponds to the set of all $\beta$-expansions of $x$, which we call the *branching tree of* $x$ and denote by $\mathcal{T}(x;\beta)$.

The following claim is straightforward:

**Lemma 5.1.** *Suppose $K \in \mathbb{N}$ is such that the length of each branch is at most $K$. Then*

$$\mathcal{N}_n(x;\beta) \ge 2^{n/K-1}.$$

*Proof.* It is obvious that $\mathcal{N}_{Kn}(x;\beta) \ge 2^n$, which yields the claim. □

**Theorem 5.2.** *Suppose* $1 < \beta < \frac{1+\sqrt{5}}{2}$ *and put*

$$\kappa = \begin{cases} \frac{1}{2}\left(\left\lfloor \log_\beta \frac{\beta^2-1}{1+\beta-\beta^2}\right\rfloor + 1\right)^{-1}, & \beta > \sqrt{2} \\ \frac{1}{2}\left(\left\lfloor \log_\beta \frac{1}{\beta-1}\right\rfloor + 1\right)^{-1}, & \beta \le \sqrt{2}. \end{cases} \tag{5.6}$$

*Then for any* $x \in \left[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\right]$ *we have*

$$\mathcal{N}_n(x;\beta) \ge 2^{\kappa n - 1}. \tag{5.7}$$

*Proof.* In view of Lemma 5.1, it suffices to construct a subtree of $\mathcal{T}(x;\beta)$ such that the length of its every branch is at most $1/\kappa$. Put $\Delta_\beta = \left[\frac{1}{\beta}, \frac{1}{\beta(\beta-1)}\right]$; it is easy to check that one can choose different $\varepsilon_1$ for $x$ if and only if $x \in \Delta_\beta$[4].

Put

$$\delta = \min\left(\frac{1+\beta-\beta^2}{\beta^2-1}, \beta - 1\right) = \begin{cases} \frac{1+\beta-\beta^2}{\beta^2-1} & \text{if } \beta > \sqrt{2}, \\ \beta - 1 & \text{if } \beta \le \sqrt{2} \end{cases}. \tag{5.8}$$

Note that $\delta > 0$, in view of $1 < \beta < \frac{1+\sqrt{5}}{2}$. Put $L_\beta(x) = \beta x$, $R_\beta(x) = \beta x - 1$. The maps $L_\beta$ and $R_\beta$ act as shifts on the $\beta$-expansions of $x$, namely, $L_\beta(x)$ shifts a $\beta$-expansion of $x$ if $\varepsilon_1 = 0$ and $R_\beta$ – if $\varepsilon_1 = 1$. Thus, by applying all possible compositions of the two maps we obtain all $\beta$-expansions of $x$. (See subsection 2.3 for more detail.)

Assume first that $x \in \Delta_\beta$. We have two cases.

**Case 1.** $x \in \left[\frac{1+\delta}{\beta}, \frac{1}{\beta(\beta-1)} - \frac{\delta}{\beta}\right]$. It is easy to check, using (5.8), that this is an interval of positive length. Here $L_\beta(x) \in [1+\delta, \frac{1}{\beta-1} - \delta]$ and $R_\beta(x) \in [\delta, \frac{1}{\beta-1} - \delta - 1]$. In either case, the image is at a distance at least $\delta$ from either endpoint of $I_\beta$.

It suffices to estimate the number of iterations one needs to reach the switch region $\Delta_\beta$. In view of symmetry, we can deal with $y \in [\delta, 1/\beta)$; here $L_\beta^k(y) \in \Delta_\beta$ for some $1 \le k \le \lfloor \log_\beta \frac{1}{\delta} \rfloor + 1$.

**Case 2.** $x \in \left(\frac{1}{\beta}, \frac{1+\delta}{\beta}\right)$ or the mirror-symmetric case (which is analogous). Here $R_\beta(x)$ can be very close to 0, so we have no control over its further iterations. Consequently, we remove this branch from $\mathcal{T}(x;\beta)$ and concentrate on the subtree which grows from $L_\beta(x)$.

[4]Notice that $\Delta_\beta$ is none other than $S_1$ given by (2.14) for $m = 2$ and $\lfloor \beta \rfloor = 1$.

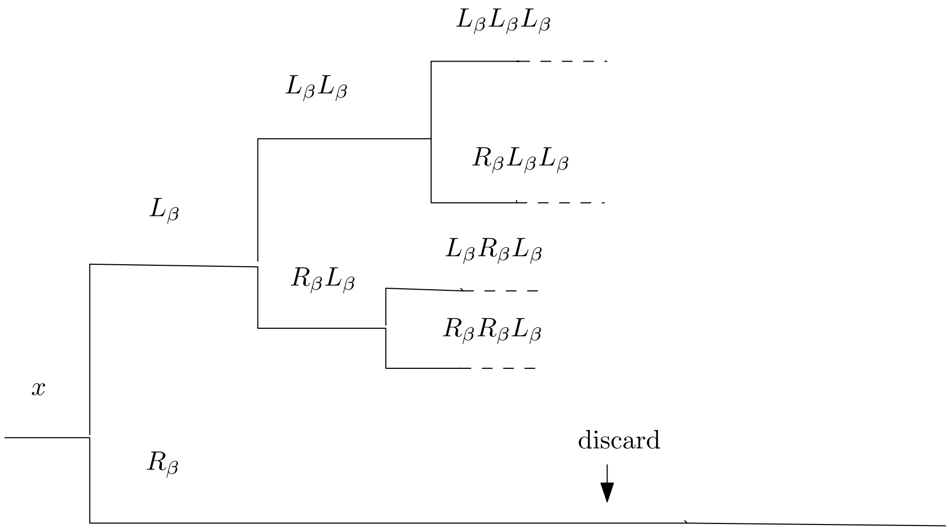


FIGURE 2. Branching for Case 2

We have $L_\beta(x) = \beta x \in (1, 1+\delta)$. Clearly, it lies in $\Delta_\beta$ provided $1+\delta \le \frac{1}{\beta(\beta-1)}$ – which is equivalent to $\delta \le \frac{1+\beta-\beta^2}{\beta(\beta-1)}$, and this is true for $\beta > \sqrt{2}$, in view of (5.8); for $\beta \le \sqrt{2}$ we have $\delta - 1 = \beta \le \frac{1}{\beta(\beta-1)}$. Furthermore,

$$\begin{aligned}
L_\beta L_\beta(x) &= \beta^2 x \in (\beta, (1+\delta)\beta), \\
R_\beta L_\beta(x) &= \beta^2 x - 1 \in (\beta - 1, (1+\delta)\beta - 1).
\end{aligned}$$

Notice that $\beta^2 x \le \frac{1}{\beta-1} - \delta$, because for $\beta > \sqrt{2}$ we have $\beta(1+\delta) = \frac{1}{\beta-1} - \delta$, and for $\beta \le \sqrt{2}$ we have $\beta^2 \le \frac{1}{\beta-1} - \beta + 1$, which is in fact equivalent to $\beta \le \sqrt{2}$. As for $R_\beta L_\beta(x)$, it is clear that it lies in $\left(\delta, \frac{1}{\beta-1} - \delta\right)$ in either case, since $\delta \le \beta - 1$.

We see that the length of each branch of the new tree does not exceed $2(\lfloor \log_\beta \frac{1}{\delta} \rfloor + 1)$ (the factor two appears in the estimate because it may happen that we will have to discard $L_\beta L_\beta L_\beta / R_\beta L_\beta L_\beta$ or $L_\beta R_\beta L_\beta / R_\beta R_\beta L_\beta$), and it suffices to apply Lemma 5.1. $\square$

If $0 < x < 1/\beta$, then there exists a unique $\ell \ge 1$ such that $L_\beta^{\ell-1}(x) < 1/\beta$ but $L_\beta^\ell(x) \ge 1/\beta$. In view of $1 < \frac{1}{\beta(\beta-1)}$, this implies $L_\beta^\ell(x) \in \Delta_\beta$. Similarly, if $\frac{1}{\beta(\beta-1)} < x < \frac{1}{\beta-1}$, then $R_\beta^\ell(x) \in \Delta_\beta$ for some $\ell$. Thus, it takes only a finite number of iterations for any $x \in \left(0, \frac{1}{\beta-1}\right)$ to reach $\Delta_\beta$. Hence follows Theorem 1.5.

**Remark 5.3.** *It would be interesting to determine the sharp analog of the golden ratio in Theorem 1.5 for the case $\beta \in (m-1, m)$ with $m \geq 3$.*

To prove Corollary 1.6, we apply the inequalities (2.17) and (5.7), whence

$$\mu\left(x - \frac{\beta^{-n}}{\beta-1}, x + \frac{\beta^{-n}}{\beta-1}\right) \geq \frac{1}{2} \cdot 2^{(1-\kappa)n},$$

and

$$\overline{\lim_{n\to\infty}} -\frac{1}{n} \log_\beta \mu\left(x - \frac{\beta^{-n}}{\beta-1}, x + \frac{\beta^{-n}}{\beta-1}\right) \leq (1-\kappa)\log_\beta 2.$$

Hence follows the claim of Corollary 1.6.

We define the *minimal growth exponent* as follows:

$$\mathfrak{m}_\beta = \inf_{x\in(0,1/(\beta-1))} \underline{\lim_{n\to\infty}} \sqrt[n]{\mathcal{N}_n(x;\beta)}.$$

**Corollary 5.4.** *For $\beta < \frac{1+\sqrt{5}}{2}$ we have $\mathfrak{m}_\beta \geq 2^\kappa > 1$, where $\kappa$ is given by (5.6).*

The golden ratio in Theorem 5.2 and Corollary 5.4 is a sharp constant in a boring sense, since for $\beta > \frac{1+\sqrt{5}}{2}$ there are always $x$ with a unique $\beta$-expansion (see [10]) and for $\beta = \frac{1+\sqrt{5}}{2}$ there are $x$ with a linear growth of $\mathcal{N}_n(x)$ (see, e.g., [19]). Hence $\mathfrak{m}_\beta = 1$ for $\beta \geq \frac{1+\sqrt{5}}{2}$.

However, it is also a sharp bound in a more interesting sense; let us call the set of $\beta$-expansions of a given $x$ *sparse* if $\lim_{n\to\infty} \frac{1}{n} \log \mathcal{N}_n(x;\beta) = 0$.

**Proposition 5.5.** *For $\beta = \frac{1+\sqrt{5}}{2}$ there exists a continuum of points $x$, each of which has a sparse continuum of $\beta$-expansions.*

*Proof.* Suppose $(m_k)_{k=1}^\infty$ is a strictly increasing sequence of natural numbers. Let $x$ be the number whose $\beta$-expansion is $10^{2m_1}10^{2m_2}10^{2m_3}\dots$ We claim that such an $x$ has a required property.

Indeed, as was shown in [19], the set of all $\beta$-expansions in this case is the Cartesian product $\mathfrak{X}_{m_1} \times \mathfrak{X}_{m_2} \times \dots$, where

$$\mathfrak{X}_{m_k} = \left\{ (\varepsilon_1, \dots, \varepsilon_{2m_k+1}) : \sum_{j=1}^{2m_k+1} \varepsilon_j \beta^{-j} = \frac{1}{\beta} \right\}.$$

It follows from [19, Lemma 2.1] that $\#\mathfrak{X}_{m_k} = m_k$, whence by [19, Lemma 2.2],

$$\#\mathcal{D}_\beta(10^{2m_1}\dots 10^{2m_k}) = \prod_{j=1}^k m_j,$$

where $\mathcal{D}_\beta(\cdot)$ is given by

$$\mathcal{D}_\beta(\varepsilon_1,\dots,\varepsilon_n) = \left\{(\varepsilon'_1,\dots,\varepsilon'_n)\in\{0,1\}^n : \sum_{k=1}^n \varepsilon_k\beta^{-k} = \sum_{k=1}^n \varepsilon'_k\beta^{-k}\right\}.$$

Hence for $n=\sum_1^k(2m_j+1)$,

$$\frac{\log\mathcal{N}_n(x;\beta)}{n} \sim \frac{\sum_{j=1}^k \log m_j}{2\sum_1^k m_j+1} \to 0, \quad k\to+\infty,$$

since $m_k \nearrow +\infty$. Therefore, $\lim_n \sqrt[n]{\mathcal{N}_n(x;\beta)} = 1$. It suffices to observe that there exists a continuum strictly increasing sequences of natural numbers – for instance, one can always choose $m_k\in\{2k-1,2k\}$. □

A similar proof works for the multinacci $\beta$. It is an open question whether given $\beta>\frac{1+\sqrt5}{2}$, it is always possible to find $x$ with a sparse continuum of $\beta$-expansions.

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