TYPICAL SELF-AFFINE SETS WITH NON-EMPTY INTERIOR

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Dedicated to the memory of Professor Ka-Sing Lau

Abstract. Let $T_1, \ldots, T_m$ be a family of $d \times d$ invertible real matrices with $\|T_i\| < 1/2$ for $1 \le i \le m$. We provide some sufficient conditions on these matrices such that the self-affine set generated by the iterated function system $\{T_i x + a_i\}_{i=1}^m$ on $\mathbb{R}^d$ has non-empty interior for almost all $(a_1, \ldots, a_m) \in \mathbb{R}^{md}$.

1. Introduction

In this paper, we provide some sufficient conditions for a typical self-affine set to have non-empty interior.

Let us first introduce some necessary notation and definitions. By an *affine iterated function system* on $\mathbb{R}^d$ we mean a finite family $\mathcal{F} = \{f_i\}_{i=1}^m$ of affine mappings from $\mathbb{R}^d$ to $\mathbb{R}^d$, taking the form

$$f_i(x) = T_i x + a_i, \qquad i = 1, \ldots, m,$$

where T_i are contracting $d \times d$ invertible real matrices and $a_i \in \mathbb{R}^d$. It is well known [10] that there exists a unique non-empty compact set $K \subset \mathbb{R}^d$ such that

$$K = \bigcup_{i=1}^m f_i(K).$$

We call K the *attractor* of $\mathcal{F}$, or the *self-affine set* generated by $\mathcal{F}$.

In what follows, let $T_1, \ldots, T_m$ be a fixed family of contracting $d \times d$ invertible real matrices. Let $\Sigma = \{1, \ldots, m\}^{\mathbb{N}}$ denote the symbolic space over the alphabet $\{1, \ldots, m\}$. Endow Σ with the product topology and let $\mathcal{P}(\Sigma)$ denote the space of Borel probability measures on Σ.

For $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^{md}$, let $\pi^{\mathbf{a}} : \Sigma \to \mathbb{R}^d$ be the coding map associated with the IFS $\{f_i^{\mathbf{a}}(x) = T_i x + a_i\}_{i=1}^m$, here we write $f_i^{\mathbf{a}}$ instead of f_i to emphasize its dependence of $\mathbf{a}$. That is,

$$\pi^{\mathbf{a}}(\mathbf{i}) = \lim_{n \to \infty} f_{i_1}^{\mathbf{a}} \circ \cdots \circ f_{i_n}^{\mathbf{a}}(0) \tag{1.1}$$

for $\mathbf{i} = (i_n)_{n=1}^\infty \in \Sigma$. Set $K^{\mathbf{a}} = \pi^{\mathbf{a}}(\Sigma)$. It is well known [10] that $K^{\mathbf{a}}$ is the attractor of $\{f_i^{\mathbf{a}}\}_{i=1}^m$.

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In his seminal work [5], Falconer introduced a quantity associated to the matrices $T_1, \ldots, T_m$, nowadays usually called the *affinity dimension* $\dim_{\mathrm{AFF}}(T_1, \ldots, T_m)$ (see Definition 2.3), which is always an upper bound for the upper box-counting dimension of $K^{\mathbf{a}}$, and such that when $\|T_i\| < 1/2$ for all $1 \le i \le m$, then for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$,

$$\dim_{\mathrm{H}} K^{\mathbf{a}} = \dim_{\mathrm{B}} K^{\mathbf{a}} = \min\{d, \dim_{\mathrm{AFF}}(T_1, \ldots, T_m)\}.$$

where $\dim_{\mathrm{H}}$ and $\dim_{\mathrm{B}}$ stand for the Hausdorff and box-counting dimensions respectively (see e.g. [6] for the definitions). In fact, Falconer proved this with $1/3$ as the upper bound on the norms; it was subsequently shown by Solomyak [16] that $1/2$ suffices. Later, Jordan, Pollicott and Simon [11] further showed that if $\|T_i\| < 1/2$ for all i and $\dim_{\mathrm{AFF}}(T_1, \ldots, T_m) > d$, then $K^{\mathbf{a}}$ has positive Lebesgue measure for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$. We remark that the condition $\dim_{\mathrm{AFF}}(T_1, \ldots, T_m) > d$ is equivalent to $\sum_{i=1}^m |\det(T_i)| > 1$, where $\det(T_i)$ denotes the determinant of T_i.

A question arises naturally that under which conditions on $T_1, \ldots, T_m$, $K^{\mathbf{a}}$ has non-empty interior for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$. Although this seems a rather fundamental question, it has hardly been studied.

In this paper, we study the above question. For a $d \times d$ real matrix A, let $\alpha_1(A) \ge \cdots \ge \alpha_d(A)$ denote the singular values of A, that is, $\alpha_1(A), \ldots, \alpha_d(A)$ are square roots of the eigenvalues of A^*A. Here A^* stands for the transpose of A. Write $\Sigma_n = \{1, \ldots, m\}^n$ for $n \in \mathbb{N}$ and set $T_I = T_{i_1} \cdots T_{i_n}$ for $I = i_1 \ldots i_n \in \Sigma_n$. Define

$$t(T_1, \ldots, T_m) = \inf \left\{ t \ge 0 \colon \sup_{n \ge 1} \sum_{I \in \Sigma_n} \alpha_d(T_I)^t |\det(T_I)| \le 1 \right\}. \tag{1.2}$$

The first result of the paper is the following.

Theorem 1.1. *Assume that $\|T_i\| < 1/2$ for $1 \le i \le m$. Suppose $t(T_1, \ldots, T_m) > d$. Then $K^{\mathbf{a}}$ has non-empty interior for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$.*

It is easy to see that $t(T_1, \ldots, T_m) > d$ if and only if $\sum_{I \in \Sigma_n} \alpha_d(T_I)^d |\det(T_I)| > 1$ for some $n \in \mathbb{N}$. As a direct corollary of Theorem 1.1, we have the following.

Corollary 1.2. *Assume that $\|T_i\| < 1/2$ for $1 \le i \le m$. Then $K^{\mathbf{a}}$ has non-empty interior for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$ provided that one of the following two conditions fulfills:*

(i) $\sum_{i=1}^m \alpha_d(T_i)^d |\det(T_i)| > 1$.

(ii) *All T_i are scalar multiples of orthogonal matrices, and* $\sum_{i=1}^m |\det(T_i)|^2 > 1$.

Next we provide an improvement of Theorem 1.1 in the special case when the matrices $T_1, \ldots, T_m$ commute.

Theorem 1.3. *Assume that $\|T_i\| < 1/2$ for $1 \le i \le m$. Moreover, suppose that $T_iT_j = T_jT_i$ for all $1 \le i, j \le m$, and $\sum_{i=1}^m |\det(T_i)|^2 > 1$. Then $K^{\mathbf{a}}$ has non-empty interior for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$.*

The above results provide some sufficient conditions on $(T_1, \ldots, T_m)$ such that $K^{\mathbf{a}}$ has non-empty interior for almost all $\mathbf{a}$. We don't know whether these conditions are sharp. A natural conjecture is that the conditions of $\|T_i\| < 1/2$ for $1 \le i \le m$ and $\sum_{i=1}^m |\det(T_i)| > 1$ would suffice for $K^{\mathbf{a}}$ to have the non-empty interior for almost all $\mathbf{a}$.

Now we address some related works in the literature. In [15] Shmerkin investigated a special class of overlapping affine IFSs $\Phi_{\alpha,\beta} = \{\phi_i(x,y) = (\alpha x + d_i, \beta y + d_i)\}_{i=1}^2$ on the plane with $d_1 = 0$ and $d_2 = 1$. He proved, among other things, that there is an open region $V \subset \{(\alpha,\beta) \colon 0 < \alpha < \beta < 1,\ 2\alpha\beta > 1\}$ such that for almost all $(\alpha,\beta) \in V$, the attractor $K_{\alpha,\beta}$ of $\Phi_{\alpha,\beta}$ has non-empty interior. Later Dajani, Jiang and Kempton [4] showed that there exists a number $C \approx 1.05^{-1}$ such that $K_{\alpha,\beta}$ has non-empty interior for each pair (α,β) satisfying $C < \alpha < \beta < 1$. The result of Dajani et al. was subsequently improved and extended in [7, 8, 1]. We remark that the interior problem has also been extensively studied for integral self-affine sets (see e.g. [2, 12, 9] and the references therein). Recall that an integral self-affine set is the attractor of an affine IFS $\{Ax + a_i\}_{i=1}^m$ on $\mathbb{R}^d$ in which A^{-1} is an integral expanding $d \times d$ matrix and $a_i \in \mathbb{Z}^d$ for all $1 \le i \le m$. In [9], He, Lau and Rao produced, among other things, a finite algorithm to determine whether a given integral self-affine set has non-empty interior. It is worth pointing out that there exist self-affine sets of positive Lebesgue measure which have empty interior; see [3] for such examples in the self-similar setting.

For the convenience of the readers, we illustrate the rough ideas in the proofs of Theorems 1.1 and 1.3. Under the assumptions of Theorem 1.1, we first show that there exist a Borel probability measure μ on Σ, $C > 0$, $t > d$ and $r \in (0,1)$ such that

$$\mu([I]) \le C\alpha_d(T_I)^t |\det(T_I)| r^n$$

for all $n \in \mathbb{N}$ and $I \in \Sigma_n$, where $[I] := \{x = (x_i)_{i=1}^\infty \in \Sigma \colon x_1 \cdots x_n = I\}$; see Lemma 4.2. Write $\mu^{\mathbf{a}} := \mu \circ (\pi^{\mathbf{a}})^{-1}$ for $\mathbf{a} \in \mathbb{R}^{md}$. Clearly $\mu^{\mathbf{a}}$ is supported on $K^{\mathbf{a}}$. Let $\widehat{\mu^{\mathbf{a}}}$ denote the Fourier transform of $\mu^{\mathbf{a}}$; see Section 2.1. We manage to prove that

$$\int_{B(0,\rho)} \int_{\mathbb{R}^d} |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 \|\xi\|^t \, d\xi d\mathbf{a} < \infty \tag{1.3}$$

for each $\rho > 0$, where $B(0,\rho)$ stands for the closed ball in $\mathbb{R}^{md}$ of radius ρ centred at the origin. The proof of (1.3) is based on some key inequalities (see Propositions 3.3 and 3.4). By (1.3), for almost all $\mathbf{a}$, $\int_{\mathbb{R}^d} |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 \|\xi\|^t \, d\xi < \infty$; which implies that the Sobolev dimension of $\mu^{\mathbf{a}}$ is larger than $2d$, hence $K^{\mathbf{a}}$ has non-empty interior (see Definition 2.1 and Lemma 2.2). This concludes Theorem 1.1. To prove Theorem 1.3, our main idea is to construct two self-affine sets $E^{\mathbf{a}}, F^{\mathbf{a}} \subset \mathbb{R}^d$ for each $\mathbf{a} \in \mathbb{R}^{md}$ such that $K^{\mathbf{a}}$ contains a translation of the sum set $E^{\mathbf{a}} + F^{\mathbf{a}}$; and moreover for almost all $\mathbf{a}$, $E^{\mathbf{a}}$ and $F^{\mathbf{a}}$ have positive Lebesgue measure. By the Steinhaus theorem, $K^{\mathbf{a}}$ has non-empty interior for almost all $\mathbf{a}$. In this approach, the commutative assumption on $T_1, \ldots, T_m$ plays a significant role.

It is worth pointing out that by adapting the proof of Theorem 1.1 we can give an alternative proof of a known result (see Proposition 5.1) on the Hausdorff dimension and the absolute continuity of projections of measures under the coding map $\pi^{\mathbf{a}}$. This will be illustrated in Section 6.

The paper is organized as follows. In Section 2 we give some preliminaries. In Section 3 we prove two key inequalities that are needed in the proof of Theorem 1.1. The proofs of Theorems 1.1 and 1.3 are given in Sections 4 and 5 respectively. In Section 6 we give an alternative proof of Proposition 5.1.

2. Preliminaries

2.1. **Fourier transform, Soblev energy and Soblev dimension.** Recall that the Fourier transform $\widehat{f}$ of a Lebesgue integrable function $f \in L^1(\mathbb{R}^d)$ is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-i\langle \xi, x\rangle} f(x)\, dx, \qquad \xi \in \mathbb{R}^d,$$

where $\langle \cdot, \cdot \rangle$ stands for the usual inner product in $\mathbb{R}^d$. Similarly, for a finite measure μ on $\mathbb{R}^d$ with compact support, the Fourier transform of μ is defined by

$$\widehat{\mu}(\xi) = \int e^{-i\langle \xi, x\rangle}\, d\mu(x) \qquad \xi \in \mathbb{R}^d.$$

Let $\mathcal{S}(\mathbb{R}^d)$ denote the Schwartz class of rapidly decreasing functions on $\mathbb{R}^d$. It consists of infinitely differentiable functions on $\mathbb{R}^d$ all of whose derivatives remain bounded when multiplied by any polynomial. A basic fact in Fourier analysis is that $f \in \mathcal{S}(\mathbb{R}^d)$ if and only if $\widehat{f} \in \mathcal{S}(\mathbb{R}^d)$. Let $C_0^\infty(\mathbb{R}^d)$ denote the collection of infinitely differentiable functions on $\mathbb{R}^d$ with compact support. Clearly $C_0^\infty(\mathbb{R}^d) \subset \mathcal{S}(\mathbb{R}^d)$.

Let $\mathcal{M}(\mathbb{R}^d)$ denote the collection of finite Borel measures on $\mathbb{R}^d$ with compact support. Following Mattila [13] and Peres and Schlag [14], we introduce the following.

Definition 2.1. *The Sobolev energy of degree $s \in \mathbb{R}$ of a measure $\mu \in \mathcal{M}(\mathbb{R}^d)$ is*

$$\mathcal{I}_s(\mu) = \int_{\mathbb{R}^d} |\widehat{\mu}(x)|^2 \|x\|^{s-d}\, dx < \infty,$$

and the Sobolev dimension of $\mu \in \mathcal{M}(\mathbb{R}^d)$ is

$$\dim_S \mu = \sup\left\{s \in \mathbb{R} \colon \mathcal{I}_s(\mu) < \infty\right\},$$

where we take the convention that $\sup \emptyset = 0$.

The following result is needed in the proof of Theorem 1.1.

Lemma 2.2 ([13, Theorem 5.4])**.** *Let $\mu \in \mathcal{M}(\mathbb{R}^d)$ with $\mu \neq 0$. Suppose that $\dim_S \mu > 2d$. Then μ is absolutely continuous with a continuous density, so its support has non-empty interior.*

2.2. **Singular value function and affinity dimension.** Let $\mathrm{Mat}_d(\mathbb{R})$ denote the set of $d \times d$ real matrices. For $A \in \mathrm{Mat}_d(\mathbb{R})$, the singular values $\alpha_1(A) \geq \cdots \geq \alpha_d(A)$ are the square roots of the eigenvalues of A^*A. Alternatively, they are the lengths of the semi-axes of the ellipsoid $A(B(0,1))$, where $B(0,1)$ is the unit ball in $\mathbb{R}^d$.

For $s \geq 0$, we define the singular value function $\phi^s \colon \mathrm{Mat}_d(\mathbb{R}) \to [0,\infty)$ by

$$\phi^s(A) = \begin{cases} \alpha_1(A) \cdots \alpha_{\lfloor s \rfloor}(A) \alpha_k^{s-\lfloor s \rfloor} & \text{if } 0 \leq s \leq d, \\ |\det(A)|^{s/d} & \text{if } s > d, \end{cases} \tag{2.1}$$

where $\lfloor s \rfloor$ is the integral part of s. Here we make the convention $0^0 = 1$.

Definition 2.3. *Let $(T_1, \ldots, T_m)$ be a tuple of $d \times d$ real matrices. The affinity dimension of $(T_1, \ldots, T_m)$ is defined by*

$$\dim_{\mathrm{AFF}}(T_1, \ldots, T_m) = \inf \left\{ s \geq 0 \colon \sum_{n=1}^{\infty} \sum_{I \in \{1,\ldots,m\}^n} \phi^s(T_I) < \infty \right\}.$$

3. Useful inequalities

In this section we establish several inequalities (Propositions 3.3, 3.4 and 3.5), of which the first two are needed in the proof of Theorem 1.1, and the third one is needed in the proof of Proposition 5.1(i).

For $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^{md}$, let $\pi^{\mathbf{a}}$ be the coding map associated with the IFS $\{T_i x + a_i\}_{i=1}^m$; see (1.1). For a differentiable function $\phi \colon \mathbb{R}^d \to \mathbb{R}$ and $x \in \mathbb{R}^d$, let $\nabla_x \phi$ denote the gradient of ϕ at x. We begin with a simple lemma.

Lemma 3.1. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be a linear function and $\psi \in C_0^\infty(\mathbb{R}^d)$. For $\lambda > 0$, let*

$$I(\lambda) = \int_{\mathbb{R}^d} e^{-i\lambda\phi(x)} \psi(x)\, dx.$$

Then for each $N \in \mathbb{N}$, there exists $C = C(\psi, N) > 0$ such that

$$|I(\lambda)| \leq \frac{C}{\min\{1, \|\nabla \phi\|\}^N} (1+\lambda)^{-N} \quad \text{for all } \lambda > 0.$$

Proof. Let $N \in \mathbb{N}$. Since $\psi \in C_0^\infty(\mathbb{R}^d)$, there exists $C = C(\psi, N) > 0$ such that

$$|\widehat{\psi}(\xi)| \leq C(1 + \|\xi\|)^{-N}$$

for $\xi \in \mathbb{R}^d$. Since $\phi \colon \mathbb{R}^d \to \mathbb{R}$ is linear, there exists $u \in \mathbb{R}^d$ such that $\phi(x) = \langle u, x \rangle$ for $x \in \mathbb{R}^d$. Hence $I(\lambda) = \widehat{\psi}(\lambda u)$ for $\lambda > 0$. It follows that for each $\lambda > 0$,

$$|I(\lambda)| \leq C(1 + \|\lambda u\|)^{-N} \leq C\left(1 + \lambda \min\{1, \|u\|\}\right)^{-N} \leq \frac{C}{\min\{1, \|u\|\}^N} (1+\lambda)^{-N}.$$

Clearly $\nabla_x \phi = u$ for $x \in \mathbb{R}^d$. This proves the lemma. □

Lemma 3.2. *Assume that* $\delta := \max_{1\le i\le m} \|T_i\| < 1/2$. *Then*

$$\|\nabla_{\mathbf{a}}\langle v, \pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)\rangle\| \geq \frac{1-2\delta}{1-\delta} \tag{3.1}$$

for all $\mathbf{a} \in \mathbb{R}^{md}$, $x = (x_k)_{k=1}^{\infty}$, $y = (y_k)_{k=1}^{\infty} \in \Sigma$ *with* $x_1 \neq y_1$ *and* $v \in \mathbb{R}^d$ *with* $\|v\| = 1$.

Proof. The core of our proof follows closely the proof of [16, Proposition 3.1]. Let $x = (x_k)_{k=1}^{\infty}$, $y = (y_k)_{k=1}^{\infty} \in \Sigma$ with $x_1 \neq y_1$, $v \in \mathbb{R}^d$ with $\|v\| = 1$ and $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^{md}$. Without loss of generality we may assume that $x_1 = 1$ and $y_1 = 2$. Write $\mathbf{I}_d := \mathrm{diag}(\underbrace{1, \ldots, 1}_{d})$. Then

$$\pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y) = a_1 - a_2 + \sum_{k=1}^{\infty} T_{x|k} a_{x_{k+1}} - \sum_{k=1}^{\infty} T_{y|k} a_{y_{k+1}} = \sum_{j=1}^{m} U_j a_j, \tag{3.2}$$

where $U_1, \ldots, U_m$ are $d \times d$ matrices defined by

$$\begin{aligned} U_1 &= \mathbf{I}_d + \sum_{k\ge1:\, x_{k+1}=1} T_{x|k} - \sum_{p\ge1:\, y_{p+1}=1} T_{y|p+1}, \\ U_2 &= -\mathbf{I}_d + \sum_{k\ge1:\, x_{k+1}=2} T_{x|k} - \sum_{p\ge1:\, y_{p+1}=2} T_{y|p+1}, \\ U_j &= \sum_{k\ge1:\, x_{k+1}=j} T_{x|k} - \sum_{p\ge1:\, y_{p+1}=j} T_{y|p+1} \quad \text{for } 3 \le j \le m. \end{aligned} \tag{3.3}$$

Set $A = U_1 - \mathbf{I}_d$ and $B = U_2 + \mathbf{I}_d$. By (3.3),

$$\|A\| + \|B\| + \sum_{j=3}^{m} \|U_j\| \le \sum_{k=1}^{\infty} \|T_{x|k}\| + \sum_{p=1}^{\infty} \|T_{y|p}\| \le 2\sum_{k=1}^{\infty} \delta^k = \frac{2\delta}{1-\delta} < 2.$$

Hence either $\|A\| \le \delta/(1-\delta)$ or $\|B\| \le \delta/(1-\delta)$. Suppose that $\|A\| \le \delta/(1-\delta)$ whilst the other case follows from the same argument. Then

$$\|U_1^{-1}\| = \|(\mathbf{I}_d - A)^{-1}\| \le \sum_{k=0}^{\infty} \|A\|^k \le \sum_{k=0}^{\infty} \left(\frac{\delta}{1-\delta}\right)^k = \frac{1-\delta}{1-2\delta}. \tag{3.4}$$

By (3.2),

$$\nabla_{\mathbf{a}}\langle v, \pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)\rangle = \nabla_{\mathbf{a}} \left\langle v,\ \sum_{j=1}^{m} U_j a_j \right\rangle = (v^t U_1, \ldots, v^t U_m),$$

where v^t denotes the transpose of v. Since $\|v\| = 1$, it follows that

$$\|\nabla_{\mathbf{a}}\langle v, \pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)\rangle\| \ge \|v^t U_1\| \ge \alpha_d(U_1) = \|U_1^{-1}\|^{-1} \ge \frac{1-2\delta}{1-\delta},$$

as desired. □

Now we are ready to deduce an important integral estimate. For $x, y \in \Sigma$, let $x \wedge y$ denote the common initial segment of x and y.

Proposition 3.3. *Assume that* $\delta := \max_{1\le i\le m}\|T_i\| < 1/2$. *Let* $\psi \in C_0^\infty(\mathbb{R}^{md})$ *and* $N \in \mathbb{N}$. *Then there exists* $C = C(\psi, N, \delta) > 0$ *such that*

$$\left|\int_{\mathbb{R}^{md}} \psi(\mathbf{a}) e^{-i\langle \xi, \pi^{\mathbf{a}}(x)-\pi^{\mathbf{a}}(y)\rangle}\, d\mathbf{a}\right| \le C\left(1+\|T^*_{x\wedge y}\xi\|\right)^{-N} \tag{3.5}$$

for all $\xi \in \mathbb{R}^d$ *and* $x, y \in \Sigma$ *with* $x \ne y$, *where* $T^*_{x\wedge y}$ *stands for the transpose of* $T_{x\wedge y}$.

Proof. Let $\xi \in \mathbb{R}^d\backslash\{0\}$ and $x, y \in \Sigma$ with $x \ne y$. Let $n = |x\wedge y|$ be the word length of $x \wedge y$. Write

$$v_{\xi,x,y} = \frac{T^*_{x\wedge y}\xi}{\|T^*_{x\wedge y}\xi\|}.$$

Then

$$\begin{aligned}\langle \xi, \pi^{\mathbf{a}}(x)-\pi^{\mathbf{a}}(y)\rangle &= \langle \xi, T_{x\wedge y}(\pi^{\mathbf{a}}(\sigma^n x)-\pi^{\mathbf{a}}(\sigma^n y))\rangle\\ &= \langle T^*_{x\wedge y}\xi, \pi^{\mathbf{a}}(\sigma^n x)-\pi^{\mathbf{a}}(\sigma^n y)\rangle\\ &= \|T^*_{x\wedge y}\xi\|\langle v_{\xi,x,y}, \pi^{\mathbf{a}}(\sigma^n x)-\pi^{\mathbf{a}}(\sigma^n y)\rangle.\end{aligned}$$

Defining $\phi\colon \mathbb{R}^{md}\to\mathbb{R}$ by $\phi(\mathbf{a}) = \langle v_{\xi,x,y}, \pi^{\mathbf{a}}(\sigma^n x)-\pi^{\mathbf{a}}(\sigma^n y)\rangle$, we get

$$\int_{\mathbb{R}^{md}} \psi(\mathbf{a}) e^{-i\langle \xi, \pi^{\mathbf{a}}(x)-\pi^{\mathbf{a}}(y)\rangle}\, d\mathbf{a} = \int_{\mathbb{R}^{md}} \psi(\mathbf{a}) e^{-i\|T^*_{x\wedge y}\xi\|\phi(\mathbf{a})}\, d\mathbf{a}. \tag{3.6}$$

Notice that ϕ is linear (cf. (3.2)). So by Lemma 3.2, $\|\nabla_{\mathbf{a}}\phi\| \ge (1-2\delta)/(1-\delta)$. Applying Lemma 3.1 (in which we take $\lambda = \|T^*_{x\wedge y}\xi\|$) yields that for each $N\in\mathbb{N}$, there exists $C = C(N,\psi,\delta) > 0$ so that

$$\left|\int_{\mathbb{R}^{md}} \psi(\mathbf{a}) e^{-i\|T^*_{x\wedge y}\xi\|\phi(\mathbf{a})}\, d\mathbf{a}\right| \le C(1+\|T^*_{x\wedge y}\xi\|)^{-N}.$$

Combining it with (3.6) completes the proof. □

In the remaining part of this section, we shall mean by $a \lesssim_\varepsilon b$ that $a \le Cb$ for some positive constant C depending on ε. We write $a \approx_\varepsilon b$ if $a \lesssim_\varepsilon b$ and $b \lesssim_\varepsilon a$.

For $d \in \mathbb{N}$, let $\mathrm{GL}(d,\mathbb{R})$ denote the collection of $d\times d$ invertible real matrices.

Proposition 3.4. *Let* $d\in\mathbb{N}$, $t \ge 0$ *and* $N > t + d$. *Then*

$$\int_{\mathbb{R}^d} (1+\|Tx\|)^{-N}\|x\|^t\, dx \approx_{N,d,t} \frac{1}{\alpha_d(T)^t|\det(T)|}$$

for $T \in \mathrm{GL}(d,\mathbb{R})$.

Proof. Substituting $y = Tx$ gives

$$\int_{\mathbb{R}^d} (1+\|Tx\|)^{-N}\|x\|^t\, dx = \frac{1}{|\det(T)|}\int_{\mathbb{R}^d} (1+\|y\|)^{-N}\|T^{-1}y\|^t\, dy. \tag{3.7}$$

Let $\beta_i = 1/\alpha_i(T)$ for $i = 1, \dots, d$. Then $\beta_1^2, \dots, \beta_d^2$ are the eigenvalues of $(T^{-1})^*(T^{-1})$. Choosing coordinate axes in the directions of the eigenvectors of $(T^{-1})^*(T^{-1})$ corresponding to $\beta_1^2, \dots, \beta_d^2$, we obtain

(3.8)

$$\begin{aligned}\int_{\mathbb{R}^d}(1+\|y\|)^{-N}\|T^{-1}y\|^t\,dy &= \int\cdots\int_{\mathbb{R}^d}(1+\|y\|)^{-N}\left(\beta_1^2y_1^2+\cdots+\beta_d^2y_d^2\right)^{t/2}\,dy_1\cdots dy_d\\ &\approx_{d,t}\int\cdots\int_{\mathbb{R}^d}(1+\|y\|)^{-N}\left(\beta_1^ty_1^t+\cdots+\beta_d^ty_d^t\right)\,dy_1\cdots dy_d\\ &= \sum_{i=1}^d \beta_i^t c_i,\end{aligned}$$

where $c_i := \int_{\mathbb{R}^d}(1+\|y\|)^{-N}|y_i|^t\,dy_1\cdots dy_d$. Since $c_i \approx_{N,d,t} 1$ by $N > d+t$, it follows from (3.7), (3.8) that

$$\begin{aligned}\int_{\mathbb{R}^d}(1+\|Tx\|)^{-N}\|x\|^t\,dx &\approx_{d,t}\frac{1}{|\det(T)|}\sum_{i=1}^d\beta_i^tc_i\\ &\approx_{N,d,t}\frac{1}{|\det(T)|}\max_{i\in\{1,\dots,d\}}\beta_i^t\\ &= \frac{1}{\alpha_d(T)^t|\det(T)|},\end{aligned}$$

which completes the proof of the proposition. □

As a complement of Proposition 3.4, we have the following.

Proposition 3.5. *Let $d \in \mathbb{N}$, $t \in (0,d)\backslash\mathbb{Z}$ and $N > t$. Then*

$$\int_{\mathbb{R}^d}(1+\|Tx\|)^{-N}\|x\|^{t-d}\,dx \approx_{N,d,t}\frac{1}{\phi^t(T)}$$

for $T \in \mathrm{GL}(d,\mathbb{R})$.

The proof of the above proposition is based a simple lemma.

Lemma 3.6. *Let $d \in \mathbb{N}$ and $s > 1$. Then for $(x_1, \dots, x_d) \in \mathbb{R}^d\backslash\{0\}$,*

$$\int_{\mathbb{R}}\frac{1}{\left(\sum_{i=1}^d|x_i|^s\right)+|y|^s}\,dy \approx_{d,s}\frac{1}{\sum_{i=1}^d|x_i|^{s-1}}.$$

Proof. Set $A = \sum_{i=1}^d|x_i|^s$. Then

$$\begin{aligned}\int_{\mathbb{R}}\frac{1}{\left(\sum_{i=1}^d|x_i|^s\right)+|y|^s}\,dy = 2\int_0^\infty\frac{1}{A+y^s}\,dy &\approx \int_0^{A^{1/s}}\frac{1}{A}\,dy + 2\int_{A^{1/s}}^\infty\frac{1}{y^s}\,dy\\ &\approx\frac{1}{A^{(s-1)/s}} = \frac{1}{\left(\sum_{i=1}^d|x_i|^s\right)^{(s-1)/s}}\approx_{d,s}\frac{1}{\sum_{i=1}^d|x_i|^{s-1}},\end{aligned}$$

as desired. ∎

Proof of Proposition 3.5. Suppose $k < t < k+1$ for some $k \in \{0, \dots, d-1\}$. Let $\alpha_1 \geq \cdots \geq \alpha_d$ be the singular values of T. Choosing coordinate axes in the directions of the eigenvectors of T^*T corresponding to $\alpha_1^2, \dots, \alpha_d^2$, we obtain

(3.9)
$$\int_{\mathbb{R}^d} (1 + \|Tx\|)^{-N} \|x\|^{t-d}\, dx = \int \cdots \int_{\mathbb{R}^d} \frac{dx_1 \cdots dx_d}{\left(1 + \sqrt{\sum_{i=1}^d |\alpha_i x_i|^2}\right)^N \left(\sum_{j=1}^d |x_j|^2\right)^{(d-t)/2}}$$
$$\approx_{N,d,t} \int \cdots \int_{\mathbb{R}^d} \frac{dx_1 \cdots dx_d}{\left(1 + \sum_{i=1}^d |\alpha_i x_i|^N\right) \left(\sum_{j=1}^d |x_j|^{d-t}\right)}.$$

Since $d - t > d - (k+1)$, applying Lemma 3.6 repeatedly yields

$$\int \cdots \int_{\mathbb{R}^{d-(k+1)}} \frac{1}{\sum_{i=1}^d |x_i|^{d-t}}\, dx_{k+2} \cdots dx_d \approx_{d,t} \frac{1}{\sum_{i=1}^{k+1} |x_i|^{k+1-t}}. \tag{3.10}$$

We make a convention that $\alpha_1 \cdots \alpha_k = 1$ if $k = 0$. Then

$$\begin{aligned}
&\int_{\mathbb{R}^d} (1 + \|Tx\|)^{-N} \|x\|^{t-d}\, dx \\
&\lesssim_{N,d,t} \int \cdots \int_{\mathbb{R}^d} \frac{dx_1 \cdots dx_d}{\left(1 + \sum_{i=1}^{k+1} |\alpha_i x_i|^N\right)\left(\sum_{j=1}^d |x_j|^{d-t}\right)} && \text{(by (3.9))} \\
&\approx_{N,d,t} \int \cdots \int_{\mathbb{R}^{k+1}} \frac{dx_1 \cdots dx_{k+1}}{\left(1 + \sum_{i=1}^{k+1} |\alpha_i x_i|^N\right)\left(\sum_{j=1}^{k+1} |x_j|^{k+1-t}\right)} && \text{(by (3.10))} \\
&\leq \int \cdots \int_{\mathbb{R}^{k+1}} \frac{dx_1 \cdots dx_{k+1}}{\left(1 + \sum_{i=1}^{k+1} |\alpha_i x_i|^N\right) |x_{k+1}|^{k+1-t}} \\
&= \frac{1}{\alpha_1 \cdots \alpha_k \alpha_{k+1}^{t-k}} \int \cdots \int_{\mathbb{R}^{k+1}} \frac{dy_1 \cdots dy_{k+1}}{\left(1 + \sum_{i=1}^{k+1} |y_i|^N\right) |y_{k+1}|^{k+1-t}} \\
&= \frac{1}{\phi^t(T)} \int \cdots \int_{\mathbb{R}^{k+1}} \frac{dy_1 \cdots dy_{k+1}}{\left(1 + \sum_{i=1}^{k+1} |y_i|^N\right) |y_{k+1}|^{k+1-t}}
\end{aligned} \tag{3.11}$$

where in the second last equality we took a change of variables via $y_i = \alpha_i x$ for $1 \leq i \leq k+1$. Since $N > t > k$, applying Lemma 3.6 repeatedly to the variables $y_1, \dots, y_k$ yields

(3.12)
$$\int \cdots \int_{\mathbb{R}^{k+1}} \frac{dy_1 \cdots dy_{k+1}}{\left(1 + \sum_{i=1}^{k+1} |y_i|^N\right) |y_{k+1}|^{k+1-t}} \lesssim_{N,d,t} \int_{\mathbb{R}} \frac{1}{(1 + |y_{k+1}|^{N-k})\, |y_{k+1}|^{k+1-t}}\, dy_{k+1}$$
$$\lesssim_{N,d,t} 1$$

where in the last inequality we used $N - k + (k+1) - t = N - t + 1 > 1$ and $0 < k + 1 - t < 1$. Combining (3.11) with (3.12) yields the upper bound

$$\int_{\mathbb{R}^d} (1 + \|Tx\|)^{-N} \|x\|^{t-d}\, dx \lesssim_{N,d,t} \frac{1}{\phi^t(T)}.$$

Next we prove the lower bound. Let Ω denote the set of $(x_1, \ldots, x_d) \in \mathbb{R}^d$ such that

$$\begin{aligned}
&|x_i| \le \frac{1}{\alpha_i} \quad \text{ for } 1 \le i \le k,\\
&\frac{1}{\alpha_{k+1}} \le |x_{k+1}| \le \frac{2}{\alpha_{k+1}} \quad \text{ and}\\
&|x_j| \le \frac{1}{\alpha_{k+1}} \quad \text{ for } k+2 \le j \le d.
\end{aligned}$$

Then for each $(x_1, \ldots, x_d) \in \Omega$, we have

$$\sum_{i=1}^{d} |x_i|^{d-t} \le 2^{d-t} d \alpha_{k+1}^{t-d} \lesssim_{d,t} \alpha_{k+1}^{t-d}$$

and $|\alpha_i x_i| \le 2$ for $1 \le i \le d$ which implies

$$\frac{1}{1 + \sum_{i=1}^{d} |\alpha_i x_i|^N} \ge \frac{1}{1 + 2^N d} \gtrsim_{N,d} 1.$$

Hence by (3.9),

$$\begin{aligned}
\int_{\mathbb{R}^d} (1 + \|Tx\|)^{-N} \|x\|^{t-d}\, dx &\gtrsim_{N,d,t} \int \cdots \int_{\Omega} \frac{dx_1 \cdots dx_d}{\left(1 + \sum_{i=1}^{d} |\alpha_i x_i|^N\right)\left(\sum_{j=1}^{d} |x_j|^{d-t}\right)}\\
&\gtrsim_{N,d,t} \int \cdots \int_{\Omega} \alpha_{k+1}^{d-t}\, dx\\
&= \mathcal{L}^d(\Omega) \alpha_{k+1}^{d-t}\\
&\gtrsim_d \frac{1}{\alpha_1 \cdots \alpha_k \alpha_{k+1}^{d-k} \alpha_{k+1}^{t-d}} = \frac{1}{\phi^t(T)}.
\end{aligned}$$

This finishes the proof of the lower bound. □

4. The proofs of Theorem 1.1 and Corollary 1.2

We first introduce some notation. For $n \in \mathbb{N}$ write $\Sigma_n = \{1, \ldots, m\}^n$. Set $\Sigma_0 = \{\varnothing\}$ where $\varnothing$ stands for the empty word. Write $\Sigma^* = \bigcup_{n=0}^{\infty} \Sigma_n$. Set $|I| = n$ for every $I \in \Sigma_n$. For each $t \ge 0$, we define $g_t : \Sigma^* \to (0, \infty)$ by

$$g_t(I) = \alpha_d(T_I)^t |\det(T_I)|, \tag{4.1}$$

where we take the convention that $g_t(\varnothing) = 1$.

Lemma 4.1. *Let $t \geq 0$. Then g_t is super-multiplicative on Σ^* in the sense that*

$$g_t(IJ) \geq g_t(I)g_t(J) \quad \text{for all } I, J \in \Sigma^*. \tag{4.2}$$

Consequently,

$$\lim_{n\to\infty} \left(\sum_{I \in \Sigma_n} g_t(I) \right)^{1/n} = \sup_{n \in \mathbb{N}} \left(\sum_{I \in \Sigma_n} g_t(I) \right)^{1/n}. \tag{4.3}$$

Proof. Notice that for $I, J \in \Sigma^*$,

$$\alpha_d(T_{IJ}) = \|T_{IJ}^{-1}\|^{-1} \geq \|T_I^{-1}\|^{-1}\|T_J^{-1}\|^{-1} = \alpha_d(T_I)\alpha_d(T_J).$$

It follows that

$$g_t(IJ) = \alpha_d(T_{IJ})^t |\det(T_{IJ})| \geq \alpha_d(T_I)^t \alpha_d(T_J)^t |\det(T_I)||\det(T_J)| = g_t(I)g_t(J).$$

Hence (4.2) holds. Set $a_n := \sum_{I \in \Sigma_n} g_t(I)$ for $n \in \mathbb{N}$. By (4.2), $a_{n+m} \geq a_n a_m$ for all $n, m \in \mathbb{N}$, from which (4.3) follows. □

The next lemma allows us to construct a certain regular measure on Σ under the assumptions of Theorem 1.1.

Lemma 4.2. *Suppose $t(T_1, \dots, T_m) > d$. Then for every t with $d < t < t(T_1, \dots, T_m)$, there exist a Borel probability measure μ on Σ, $r \in (0, 1)$ and $C > 0$ such that*

$$\mu([I]) \leq C g_t(I) r^{|I|} \quad \text{for all } I \in \Sigma^*, \tag{4.4}$$

where $[I] := \{x = (x_n)_{n=1}^\infty \in \Sigma : x_1 \cdots x_k = I\}$ for $I \in \Sigma_k$.

Proof. Let $d < t < t(T_1, \dots, T_m)$. By the definition of $t(T_1, \dots, T_m)$, there exists $N \in \mathbb{N}$ such that

$$\lambda := \sum_{I \in \Sigma_N} g_t(I) > 1.$$

Define a probability vector $p = \{p_I\}_{I \in \Sigma_N}$ by

$$p_I = g_t(I)\lambda^{-1}, \quad I \in \Sigma_N.$$

Let μ be the Bernoulli product measure on $\Sigma = (\Sigma_N)^{\mathbb{N}}$ associated with p. That is,

$$\mu([I_1 \dots I_k]) = \prod_{\ell=1}^{k} p_{I_\ell} \tag{4.5}$$

for every $k \in \mathbb{N}$ and $I_1, \dots I_k \in \Sigma_N$.

Set $r = \lambda^{-1/N}$. Then $0 < r < 1$. Next we show that (4.4) holds for some $C > 0$. To this end, let $I \in \Sigma^*$. Then I can be written as $I = I_1 \dots I_k W$ with $I_1, \dots, I_k \in \Sigma_N$ and $1 \leq |W| \leq N$. It may happen that $k = 0$ and in that case $I_1 \dots I_k$ should be viewed as the empty word $\varnothing$. It follows from (4.2) that

$$\frac{g_t(I_1 \dots I_k)}{g_t(I)} = \frac{g_t(I_1 \dots I_k)}{g_t(I_1 \dots I_k W)} \leq \frac{1}{g_t(W)} \leq \gamma \tag{4.6}$$

where $\gamma := \max\{1/g_t(J)\colon |J| \le N\} < \infty$. By (4.5), (4.2) and (4.6),

$$\mu([I]) \le \mu([I_1 \dots I_k]) = \prod_{\ell=1}^{k} \frac{g_t(I_\ell)}{\lambda} \le g_t(I_1 \dots I_k)\lambda^{-k} \le \gamma g_t(I)\lambda^{-k}.$$

Letting $C = \gamma\lambda$ and using $k \ge |I|/N - 1$, we see that $\gamma\lambda^{-k} \le C(\lambda^{-1/N})^{|I|} = Cr^{|I|}$, so $\mu(I) \le Cg_t(I)r^{|I|}$. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Fix t so that $d < t < t(T_1, \dots, T_m)$. By Lemma 4.2, there exist a Borel probability measure μ on Σ, $r \in (0,1)$ and $C > 0$ such that

$$\mu([I]) \le Cg_t(I)r^{|I|} \quad \text{for all } I \in \Sigma^*. \tag{4.7}$$

Notice that $\mu([I]) \to 0$ as $|I| \to \infty$. So μ has no atoms.

For brevity we write $\mu^{\mathbf{a}} = \mu \circ (\pi^{\mathbf{a}})^{-1}$ for $\mathbf{a} \in \mathbb{R}^{md}$. Clearly, $\mu^{\mathbf{a}}$ is supported on $K^{\mathbf{a}}$ for each $\mathbf{a}$. For $\rho > 0$, let $B(0,\rho)$ denote the closed ball in $\mathbb{R}^{md}$ of radius ρ centred at the origin. We claim that for each $\rho > 0$,

$$\int_{B(0,\rho)} \int_{\mathbb{R}^d} |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 \|\xi\|^t \, d\xi \, d\mathbf{a} < \infty. \tag{4.8}$$

Clearly, (4.8) implies that for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in B(0,\rho)$,

$$\int_{\mathbb{R}^d} |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 \|\xi\|^t \, d\xi < \infty;$$

so $\dim_S \mu^{\mathbf{a}} \ge t + d > 2d$ by Definition 2.1. By Lemma 2.2, $K^{\mathbf{a}}$ has non-empty interior for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in B(0,\rho)$.

In what follows we prove (4.8). Fix $\rho > 0$. Take $\psi \in C_0^\infty(\mathbb{R}^{md})$ such that $0 \le \psi \le 1$ and $\psi(x) = 1$ for all $x \in B(0,\rho)$. Applying Fubini's theorem,

$$\begin{aligned}
&\int_{B(0,\rho)} \int_{\mathbb{R}^d} |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 \|\xi\|^t \, d\xi \, d\mathbf{a} \\
&\quad \le \int_{\mathbb{R}^{md}} \int_{\mathbb{R}^d} \psi(\mathbf{a}) |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 \|\xi\|^t \, d\mathbf{a} \, d\xi \\
&\quad = \int_{\mathbb{R}^{md}} \int_{\mathbb{R}^d} \psi(\mathbf{a}) \|\xi\|^t \int_\Sigma \int_\Sigma e^{-i\langle \xi, \pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)\rangle} \, d\mu(x) d\mu(y) \, d\mathbf{a} \, d\xi \\
&\quad = \int_\Sigma \int_\Sigma \int_{\mathbb{R}^d} \|\xi\|^t \int_{\mathbb{R}^{md}} \psi(\mathbf{a}) e^{-i\langle \xi, \pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)\rangle} \, d\mathbf{a} \, d\xi \, d\mu(x) d\mu(y).
\end{aligned} \tag{4.9}$$

Take $N > t + d$. By Proposition 3.3, there exists $\widetilde{C} > 0$ such that

$$\left| \int_{\mathbb{R}^{md}} \psi(\mathbf{a}) e^{-i\langle \xi, \pi^{\mathbf{a}}(x) - \pi^{\mathbf{a}}(y)\rangle} \, d\mathbf{a} \right| \le \widetilde{C}(1 + \|T^*_{x \wedge y}\xi\|)^{-N} \tag{4.10}$$

for all $\xi \in \mathbb{R}^d$ and $x, y \in \Sigma$ with $x \neq y$. Since μ has no atoms, $\mu \times \mu$ is fully supported on $\{(x,y) \in \Sigma \times \Sigma : \ x \neq y\}$. To see this, simply notice that

$$\mu \times \mu\{(x,x) : \ x \in \Sigma\} \leq \sum_{I \in \Sigma_n} \mu([I])^2 \leq \sup_{I \in \Sigma_n} \mu([I]) \to 0 \ \text{ as } n \to \infty.$$

Hence, by (4.9) and (4.10),

$$\begin{aligned}
&\int_{B(0,\rho)} \int_{\mathbb{R}^d} |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 \|\xi\|^t \, d\xi \, d\mathbf{a} \\
&\quad \leq \widetilde{C} \int_\Sigma \int_\Sigma \int_{\mathbb{R}^d} (1 + \|T^*_{x \wedge y} \xi\|)^{-N} \|\xi\|^t \, d\xi \, d\mu(x) d\mu(y) \\
&\quad \leq C' \int_\Sigma \int_\Sigma g_t(x \wedge y)^{-1} \, d\mu(x) d\mu(y) && \text{(by Proposition 3.4)} \\
&\quad \leq C' \sum_{n=0}^{\infty} \sum_{I \in \Sigma_n} g_t(I)^{-1} \mu([I])^2 \\
&\quad \leq C'' \sum_{n=0}^{\infty} \sum_{I \in \Sigma_n} r^n \mu([I]) && \text{(by (4.4))} \\
&\quad = \frac{C''}{1-r} < \infty,
\end{aligned}$$

where C', C'' are two positive constants. This proves (4.8). □

Proof of Corollary 1.2. Clearly the condition $\sum_{i=1}^m \alpha_d(T_i)^d |\det(T_i)| > 1$ implies that $t(T_1, \ldots, T_m) > d$. Hence by Theorem 1.1, $K^{\mathbf{a}}$ has non-empty interior for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$ if condition (i) holds.

Notice that whenever T_i $(i = 1, \ldots, m)$ are scalar multiples of orthogonal matrices,

$$\sum_{i=1}^m \alpha_d(T_i)^d |\det(T_i)| = \sum_{i=1}^m |\det(T_i)|^2.$$

Hence condition (ii) implies condition (i). □

5. The proof of Theorem 1.3

In this section we prove Theorem 1.3. For $\mathbf{a} = (a_1, \ldots, a_m) \in \mathbb{R}^{md}$, let $\pi^{\mathbf{a}}$ be the coding map associated with the IFS $\{T_i x + a_i\}_{i=1}^m$; see (1.1). Let Σ^* be defined as in the beginning part of Section 4. Recall that for a Borel probability measure η on $\mathbb{R}^d$, its Hausdorff dimension $\dim_{\mathrm{H}} \eta$ is the smallest Hausdorff dimension of a Borel set F of positive η measure. Part (ii) of the following result is needed in our proof.

Proposition 5.1. [11, Proposition 4.4] *Assume that $\|T_i\| < 1/2$ for $1 \leq i \leq m$. Let μ be a Borel probability measure on Σ. Suppose that there exist $s > 0$ and $C > 0$ such that*

$$\mu([I]) \leq C \phi^s(T_I) \quad \textit{for } I \in \Sigma^*,$$

where ϕ^s is the singular value function defined as in (2.1). Then the following properties hold:

(i) *If $0 < s \le d$, then $\dim_{\rm H} \mu^{\mathbf{a}} \ge s$ for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$, where $\mu^{\mathbf{a}} := \mu \circ (\pi^{\mathbf{a}})^{-1}$.*
(ii) *If $s > d$, then $\mu^{\mathbf{a}} \ll \mathcal{L}^d$ for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$.*

We remark that part (i) of the above proposition was also implicitly proved in [5]. In Section 6, we will provide an alternative proof of Proposition 5.1 by adapting the proof of Theorem 1.1.

Proof of Theorem 1.3. The proof is conducted as follows. For each $\mathbf{a} \in \mathbb{R}^{md}$, we will construct two compact sets $E^{\mathbf{a}}, F^{\mathbf{a}} \subset \mathbb{R}^d$ and a vector $v^{\mathbf{a}} \in \mathbb{R}^d$ such that

$$K^{\mathbf{a}} \supset E^{\mathbf{a}} + F^{\mathbf{a}} + v^{\mathbf{a}} := \{x + y + v^{\mathbf{a}} : \ x \in E^{\mathbf{a}}, \, y \in F^{\mathbf{a}}\}.$$

Then we will show that both $E^{\mathbf{a}}$ and $F^{\mathbf{a}}$ have positive d-dimensional Lebesgue measure for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$. Clearly by the Steinhaus theorem (see e.g. [17]), $K^{\mathbf{a}}$ has nonempty interior for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$.

Before giving our constructions of $E^{\mathbf{a}}$, $F^{\mathbf{a}}$ and $v^{\mathbf{a}}$ for $\mathbf{a} \in \mathbb{R}^{md}$, we first make some preparation. Write

$$\mathcal{T}_n := \{T_I \colon I \in \Sigma_n\}, \quad n \in \mathbb{N}.$$

Since $T_1, \ldots, T_m$ commute, each element in $\mathcal{T}_n$ is of the form $T_1^{p_1} \cdots T_m^{p_m}$, with $p_1, \ldots, p_m$ being nonnegative integers so that $p_1 + \cdots + p_m = n$. It follows that

$$\#\mathcal{T}_n \le (n+1)^m \quad \text{for every } n \in \mathbb{N}, \tag{5.1}$$

where $\#$ stands for cardinality.

Since $\sum_{i=1}^m |\det(T_i)|^2 > 1$, by continuity we can choose $t > 2$ such that

$$\lambda := \sum_{i=1}^m |\det(T_i)|^t > 1.$$

Then for $n \in \mathbb{N}$,

$$\begin{aligned} \lambda^n &= \sum_{I \in \Sigma_n} |\det(T_I)|^t = \sum_{A \in \mathcal{T}_n} \sum_{I \in \Sigma_n \colon T_I = A} |\det(T_I)|^t \\ &= \sum_{A \in \mathcal{T}_n} \#\{I \in \Sigma_n \colon T_I = A\} \cdot |\det(A)|^t. \end{aligned} \tag{5.2}$$

Since $\lambda > 1$, we can choose a large positive integer N such that $\lambda^N > (N+1)^m$. Then by (5.1),

$$\lambda^N > \#\mathcal{T}_N.$$

Applying this to (5.2) (in which we take $n = N$) yields that there exists $A \in \mathcal{T}_N$ such that

$$\#\{I \in \Sigma_N \colon T_I = A\} \cdot |\det(A)|^t > 1.$$

Setting $\mathcal{A} = \{I \in \Sigma_N \colon T_I = A\}$, we obtain

$$(\#\mathcal{A}) \cdot |\det A|^t > 1. \tag{5.3}$$

Fix an element $J \in \mathcal{A}$.

Let $\mathbf{a} = (a_1, \dots, a_m) \in \mathbb{R}^{md}$. For $I = i_1 \dots i_N \in \Sigma_N$, define $a_I = \sum_{k=0}^{N-1} T_{i_1 \dots i_k} a_{i_{k+1}}$. Then it is easily checked that

$$f_I^{\mathbf{a}}(x) := f_{i_1}^{\mathbf{a}} \circ \cdots \circ f_{i_N}^{\mathbf{a}}(x) = T_I x + a_I, \quad I \in \Sigma_N.$$

Hence by the definition of $\mathcal{A}$, $f_I^{\mathbf{a}}(x) = Ax + a_I$ for each $I \in \mathcal{A}$. It follows that $\{Ax + a_I\}_{I \in \mathcal{A}}$ is a sub-family of the IFS $\{f_I^{\mathbf{a}}\}_{I \in \Sigma_N}$. Therefore, letting $G^{\mathbf{a}}$ be the attractor of $\{Ax + a_I\}_{I \in \mathcal{A}}$, we have

$$K^{\mathbf{a}} \supset G^{\mathbf{a}} = \left\{ \sum_{k=0}^{\infty} A^k a_{I_{k+1}} : (I_k)_{k=1}^{\infty} \in \mathcal{A}^{\mathbb{N}} \right\}. \tag{5.4}$$

Notice that for each $(I_k)_{k=1}^{\infty} \in \mathcal{A}^{\mathbb{N}}$,

$$\begin{aligned} \sum_{k=0}^{\infty} A^k a_{I_{k+1}} &= \sum_{k=0}^{\infty} \left(A^{2k} a_{I_{2k+1}} + A^{2k+1} a_{I_{2k+2}} \right) \\ &= \left(\sum_{k=0}^{\infty} \left(A^{2k} a_{I_{2k+1}} + A^{2k+1} a_J \right) \right) + \\ &\quad \left(\sum_{k=0}^{\infty} \left(A^{2k+1} a_{I_{2k+2}} + A^{2k} a_J \right) \right) - \sum_{k=0}^{\infty} A^k a_J. \end{aligned} \tag{5.5}$$

(Recall that J is a fixed element in $\mathcal{A}$.) Define $v^{\mathbf{a}} = -\sum_{k=0}^{\infty} A^k a_J$ and

$$E^{\mathbf{a}} = \left\{ \sum_{k=0}^{\infty} A^k a_{I_{k+1}} : I_{2n+1} \in \mathcal{A} \text{ and } I_{2n+2} = J \text{ for all } n \geq 0 \right\},$$

$$F^{\mathbf{a}} = \left\{ \sum_{k=0}^{\infty} A^k a_{I_{k+1}} : I_{2n+1} = J \text{ and } I_{2n+2} \in \mathcal{A} \text{ for all } n \geq 0 \right\}.$$

By (5.4) and (5.5),

$$K^{\mathbf{a}} \supset G^{\mathbf{a}} = E^{\mathbf{a}} + F^{\mathbf{a}} + v^{\mathbf{a}}.$$

Next we show that for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$, $\mathcal{L}^d(E^{\mathbf{a}}) > 0$ and $\mathcal{L}^d(F^{\mathbf{a}}) > 0$. Noticing that $F^{\mathbf{a}} = AE^{\mathbf{a}} + a_J$ with A being invertible, so we only need to show that $\mathcal{L}^d(E^{\mathbf{a}}) > 0$ for $\mathcal{L}^{md}$-a.e. $\mathbf{a}$.

Define $\Lambda = \{IJ : I \in \mathcal{A}\}$. Then Λ is a subset of Σ_{2N}, so $\Lambda^{\mathbb{N}}$ is a compact subset of Σ since $\Sigma = (\Sigma_{2N})^{\mathbb{N}}$. By the definition of $E^{\mathbf{a}}$, we see that $E^{\mathbf{a}} = \pi^{\mathbf{a}}(\Lambda^{\mathbb{N}})$. Let μ be the Bernoulli product measure on $\Lambda^{\mathbb{N}}$ associated the uniform probability vector $(1/\#\mathcal{A}, \dots, 1/\#\mathcal{A})$. That is,

$$\mu([\omega_1 \cdots \omega_n]) = \left(\frac{1}{\#\mathcal{A}} \right)^n \quad \text{for all } n \in \mathbb{N} \text{ and } \omega_1, \dots, \omega_n \in \Lambda. \tag{5.6}$$

Since $\Lambda^{\mathbb{N}}$ is a compact subset of Σ, μ can be viewed as a Borel probability measure on Σ. In particular, $\pi_*^{\mathbf{a}} \mu = \mu \circ (\pi^{\mathbf{a}})^{-1}$ is supported on $E^{\mathbf{a}}$ for each $\mathbf{a} \in \mathbb{R}^{md}$.

Now we claim that there exists $C > 0$ such that

$$\mu([I]) \leq C \phi^{t/2}(T_I) \quad \text{for all } I \in \Sigma^*, \tag{5.7}$$

where ϕ^s denotes the singular value function defined as in (2.1). To prove the claim, let $I \in \Sigma^*$. Then there is a unique integer $k \geq 0$ such that $2kN \leq |I| < 2(k+1)N$. Write $I = I_1 I_2$ with $|I_1| = 2kN$. Clearly, $\mu([I]) \leq \mu([I_1])$. If $I_1 \notin \Lambda^k$, then $\mu([I_1]) = 0$ since μ is supported on $\Lambda^{\mathbb{N}}$. Otherwise if $I_1 \in \Lambda^k$, by (5.6) and (5.3),

$$\mu([I]) \leq \mu([I_1]) = \left(\frac{1}{\#\mathcal{A}}\right)^k < |\det(A)|^{kt} = |\det(T_{I_1})|^{t/2}, \tag{5.8}$$

where in the last equality we have used the fact that $I_1 \in \mathcal{A}^{2k}$ which implies $T_{I_1} = A^{2k}$. Since

$$|\det(T_I)| = |\det(T_{I_1}) \det(T_{I_2})| \geq |\det(T_{I_1})| \left(\min_{1 \leq i \leq m} |\det(T_i)|\right)^{2N},$$

it follows from (5.8) that

$$\mu([I]) \leq |\det(T_{I_1})|^{t/2} \leq C |\det(T_I)|^{t/2} = C\phi^{dt/2}(T_I),$$

where $C := (\min_{1 \leq i \leq m} |\det(T_i)|)^{-tN}$, and in the last equality we have used that $t > 2$. This completes the proof of (5.7).

Finally by (5.7) and Proposition 5.1(ii), $\mu^{\mathbf{a}} \ll \mathcal{L}^d$ for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$, where $\mu^{\mathbf{a}} := \mu \circ (\pi^{\mathbf{a}})^{-1}$. Since $\mu^{\mathbf{a}}$ is supported on $E^{\mathbf{a}}$ for each $\mathbf{a}$, it follows that $\mathcal{L}^d(E^{\mathbf{a}}) > 0$ for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$. □

6. An alternative proof of Proposition 5.1

We remark that Proposition 5.1 can be alternatively proved by estimating the Sobolev energies and Sobolev dimension of $\mu^{\mathbf{a}}$. Below we give a sketched proof.

Sketched proof of Proposition 5.1. We first prove (i). Take any $t \in (0, s) \backslash \mathbb{Z}$. Then there exists $r \in (0, 1)$ such that

$$\mu([I]) \leq C r^{|I|} \phi^t(T_I) \quad \text{for } I \in \Sigma^*. \tag{6.1}$$

Take $N > t$. For each $\rho > 0$, following the proof of Theorem 1.1 with minor changes, we obtain

$$\begin{aligned}
&\int_{B(0,\rho)} \int_{\mathbb{R}^d} |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 \|\xi\|^{t-d} \, d\xi \, d\mathbf{a} \\
&\quad \leq \widetilde{C} \int_{\Sigma} \int_{\Sigma} \int_{\mathbb{R}^d} (1 + \|T^*_{x\wedge y} \xi\|)^{-N} \|\xi\|^{t-d} \, d\xi \, d\mu(x) d\mu(y) \\
&\quad \leq C' \int_{\Sigma} \int_{\Sigma} \left(\phi^t(T_{x\wedge y})\right)^{-1} \, d\mu(x) d\mu(y) && \text{(by Proposition 3.5)} \\
&\quad \leq C' \sum_{n=0}^{\infty} \sum_{I \in \Sigma_n} \left(\phi^t(T_I)\right)^{-1} \mu([I])^2 \\
&\quad \leq C'' \sum_{n=0}^{\infty} \sum_{I \in \Sigma_n} r^n \mu([I]) = \frac{C''}{1-r} < \infty, && \text{(by (6.1))}
\end{aligned}$$

where $\widetilde{C}, C', C''$ are positive constants. This implies that $\mathcal{I}_t(\mu^{\mathbf{a}}) < \infty$ and hence $\dim_S \mu^{\mathbf{a}} \geq t$ for almost all $\mathbf{a}$. It is known that $\dim_{\mathrm{H}} \eta \geq \min\{\dim_S \eta, d\}$ for each Borel probability measure η on $\mathbb{R}^d$ (see e.g. [14, p. 199]). Hence $\dim_{\mathrm{H}} \mu^{\mathbf{a}} \geq t$ for almost all $\mathbf{a}$. Since t is arbitrarily taken from $(0, s)\backslash\mathbb{Z}$, we obtain $\dim_{\mathrm{H}} \mu^{\mathbf{a}} \geq s$ for almost all $\mathbf{a}$.

To prove (ii), notice that there is $r \in (0, 1)$ such that

$$\mu([I]) \leq C r^{|I|} |\det(T_I)| = C r^{|I|} g_0(T_I) \quad \text{for } I \in \Sigma^*, \tag{6.2}$$

where g_0 is defined as in (4.1). Take $N > d$. For each $\rho > 0$, following the proof of Theorem 1.1 (in which take $t = 0$) yields

$$\begin{aligned}
&\int_{B(0,\rho)} \int_{\mathbb{R}^d} |\widehat{\mu^{\mathbf{a}}}(\xi)|^2 \, d\xi \, d\mathbf{a} \\
&\quad \leq \widetilde{C} \int_\Sigma \int_\Sigma \int_{\mathbb{R}^d} (1 + \|T^*_{x\wedge y}\xi\|)^{-N} \, d\xi \, d\mu(x) d\mu(y) \\
&\quad \leq C' \int_\Sigma \int_\Sigma g_0(x \wedge y)^{-1} \, d\mu(x) d\mu(y) && \text{(by Proposition 3.4)} \\
&\quad \leq C' \sum_{n=0}^{\infty} \sum_{I \in \Sigma_n} g_0(I)^{-1} \mu([I])^2 \\
&\quad \leq C'' \sum_{n=0}^{\infty} \sum_{I \in \Sigma_n} r^n \mu([I]) = \frac{C''}{1-r} < \infty. && \text{(by (6.2))}
\end{aligned}$$

It implies that for $\mathcal{L}^{md}$-a.e. $\mathbf{a} \in \mathbb{R}^{md}$, $\widehat{\mu^{\mathbf{a}}} \in L^2(\mathbb{R}^d)$ and thus $\mu^{\mathbf{a}}$ is absolutely continuous with an L^2-density. □

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