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Multifractal analysis for disintegrations of Gibbs measures and conditional Birkhoff averages

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Abstract. The paper is devoted to the study of the multifractal structure of disintegrations of Gibbs measures and conditional (random) Birkhoff averages. Our approach is based on the relativized thermodynamic formalism, convex analysis and especially, the delicate constructions of Moran-like subsets of level sets.

1. *Introduction*

The present paper is devoted to the study of the multifractal structure of disintegrations of Gibbs measures and the conditional level sets of Birkhoff averages.

Before formulating our results, we first give some notation and backgrounds about the multifractal analysis. Let η be a compactly supported Borel probability measure on $\mathbb{R}^d$ (or on a symbolic space). For $x \in \mathbb{R}^d$, the *local dimension* of η at x is defined by

$$d(\eta, x) = \lim_{r \to 0+} \frac{\log \eta(B(x,r))}{\log r}$$

provided the limit exists, where $B(x,r)$ stands for the closed ball in $\mathbb{R}^d$ of radius r centered at x. For $\alpha \geq 0$, define

$$E_\eta(\alpha) = \{x \in \mathbb{R}^d : \quad d(\eta, x) = \alpha\}.$$

The sets $E_\eta(\alpha)$ are called *the level sets* of η, and $\dim_H E_\eta(\alpha)$ are the *dimension spectra* of η (where $\dim_H$ denotes the Hausdorff dimension). For $q \in \mathbb{R}$, the *L^q-spectrum* of η is defined as

$$\tau(\eta, q) = \liminf_{r \to 0+} \frac{\log\left(\sup \sum_i \eta(B(x_i, r))^q\right)}{\log r},$$

where the supremum is taken over all the disjoint families $\{B(x_i, r)\}_i$ of closed balls with x_i in the support of η. It is easy to check that $\tau(\eta, q)$ is a concave function of q over $\mathbb{R}$.

For a given measure, it is usually very hard or impossible to calculate the corresponding dimension spectra directly. The celebrated heuristic principle known as the *multifractal formalism*, which was first introduced in [**29, 30, 31**], states that the dimension spectra $\dim_H E_\eta(\alpha)$ and the L^q-spectra $\tau(\eta, q)$ form a Legendre-transform pair, i.e.,

$$\dim_H E_\eta(\alpha) = \tau^*(\alpha) := \inf\{\alpha q - \tau(\eta, q) \colon q \in \mathbb{R}\}.$$

Although false in general, the multifractal formalism has been verified for many interesting measures (see, e.g., [**4, 14, 19, 25, 47, 50, 52**] and references therein). It still remains open to which extent the multifractal formalism could hold.

For a given measure on $\mathbb{R}^d$, it is interesting to study the possible finer version of the multifractal formalism. To be more precise, suppose μ is a Borel probability measure on $\mathbb{R}^d$ (or on a symbolic space) and let ξ be a Borel measurable partition of $\mathbb{R}^d$ in the sense of Rohlin [**54**]. Let $\{\mu_C\}_{C \in \xi}$ be the corresponding disintegration of μ with respect to ξ (see, e.g., [**54**] or [**49**, Chapter IV] for the theory about measurable partitions and disintegrations). A problem arises naturally: if μ is a measure having some *good* dynamical properties and satisfying the multifractal formalism, and ξ is a *natural* Borel partition, would μ_C satisfy the multifractal formalism for typical C in some good situations?

For the above problem, a simple and nontrivial model is the disintegration of Gibbs measures on symbolic product spaces. Let (X, T) and (Σ, σ) be two one-sided full shift spaces, over the alphabets $\{1, \ldots, l\}$ and $\{1, \ldots, m\}$ respectively. Let $(X \times \Sigma, T \times \sigma)$ be the product of (X, T) and (Σ, σ). Endow $X \times \Sigma$ with the metric

$$d((x, y), (\tilde{x}, \tilde{y})) = m^{-\inf\{i \in \mathbb{N} \colon\ (x_i, y_i) \neq (\tilde{x}_i, \tilde{y}_i)\}},$$

where $x = (x_i)_{i=1}^\infty, \tilde{x} = (\tilde{x}_i)_{i=1}^\infty \in X$ and $y = (y_i)_{i=1}^\infty, \tilde{y} = (\tilde{y}_i)_{i=1}^\infty \in \Sigma$. Let ϕ be a real-valued Hölder continuous function on $X \times \Sigma$ and let $\mu = \mu_\phi$ denote the Gibbs measure associated with ϕ (see [**13**]). Consider the partition $\xi = \{\pi^{-1}(x) : \ x \in X\}$ of $X \times \Sigma$, where π is the canonical projection from $X \times \Sigma$ to X given by $(x, y) \mapsto x$.

For brevity, we write $\{\mu_x\}$ for the disintegration $\{\mu_{\pi^{-1}(x)}\}_{x\in X}$. Let $\nu = \mu\circ\pi^{-1}$ be the projection of μ under π. The family $\{\mu_x\}$ satisfies the following properties: (i) for each $x \in X$, μ_x is a Borel probability measure supported on $\pi^{-1}(x)$; (ii) for each Borel set $A \subset X\times\Sigma$, $\mu_x(A)$ is Borel measurable and $\mu(A) = \int \mu_x(A)d\nu(x)$ (see [**54**]).

To study the multifractal property of $\{\mu_x\}$, we write $\tau_x(q) = \tau(\mu_x, q)$ for $x \in X$ and $q \in \mathbb{R}$. Our first result is the following theorem.

THEOREM 1.1. *There is a Borel set $\Gamma \subset X$ with $\nu(\Gamma) = 1$ such that for each $x \in \Gamma$,*

(i) $\tau_x(q) = \tau(q)$ for any $q \in \mathbb{R}$, here τ is a real-valued concave function satisfying

$$\tau(q) = -\frac{1}{\log m}\left(qh_\nu(T) - qP(T\times\sigma,\phi) + P_\nu(q\phi)\right),$$

where $h_\nu(T)$ denotes the measure-theoretic entropy of ν, $P(T\times\sigma,\phi)$ the topological pressure of ϕ (see (4.2)), and $P_\nu(q\phi)$ is the relativized topological pressure of $q\phi$ (see §2).

(ii) $E_{\mu_x}(\beta) \neq \emptyset$ if and only if $\beta \in [\beta_{\min}, \beta_{\max}]$, where $\beta_{\min} = \lim_{q\to\infty}\tau(q)/q$ and $\beta_{\max} = \lim_{q\to-\infty}\tau(q)/q$. Furthermore for all $\beta \in [\beta_{\min}, \beta_{\max}]$, we have

$$\begin{aligned}\dim_H E_{\mu_x}(\beta) &= \inf_{q\in\mathbb{R}}\{\beta q - \tau(q)\} \\ &= \frac{1}{\log m}\sup\left\{h_{\widetilde{\mu}}(T\times\sigma) - h_\nu(T)\right\},\end{aligned} \tag{1.1}$$

where the supremum is taken over the set of $T\times\sigma$-invariant measures $\widetilde{\mu}$ satisfying $\widetilde{\mu}\circ\pi^{-1} = \nu$ and $\int\phi\,d\widetilde{\mu} = P(T\times\sigma,\phi) - h_\nu(T) - \beta\log m$.

The above theorem shows that for ν-a.e. $x \in X$, the measure μ_x satisfies the multifractal formalism. Our proof of Theorem 1.1 is based on the study of the conditional Birkhoff average of ϕ. For $x \in X$ and $\alpha \in \mathbb{R}$, we define

$$E_x(\alpha) = \left\{y \in \Sigma:\ \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\phi(T^i x, \sigma^i y) = \alpha\right\}.$$

Clearly, $E_x(\alpha)$ is the x-section of the level set

$$E(\alpha) = \left\{(x', y) \in X\times\Sigma:\ \lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\phi(T^i x', \sigma^i y) = \alpha\right\}$$

of the classical Birkhoff average of ϕ. There is a simple relation between $E_x(\alpha)$ and $E_{\mu_x}(\beta)$ (see Lemma 4.2(ii)). Hence to study the dimension spectra $\dim_H E_{\mu_x}(\alpha)$ of μ_x, it suffices to study $\dim_H E_x(\alpha)$. Set

$$\Delta_\nu := \left\{\alpha \in \mathbb{R}:\ \alpha = \int\phi\,d\widetilde{\mu}\ \text{for some}\ \widetilde{\mu} \in \mathcal{M}_\nu(X\times\Sigma)\right\},$$

where $\mathcal{M}_\nu(X\times\Sigma)$ denotes the set of all $T\times\sigma$-invariant Borel probability measures $\widetilde{\mu}$ on $X\times\Sigma$ such that $\widetilde{\mu}\circ\pi^{-1} = \nu$. We have the following result about the structure and dimension of $E_x(\alpha)$,.

THEOREM 1.2. *There exists a Borel set $\Gamma \subseteq X$ with $\nu(\Gamma) = 1$ such that for any $x \in \Gamma$,*

(i) $\{\alpha \in \mathbb{R} : E_x(\alpha) \neq \emptyset\} = \Delta_\nu$*;*

(ii) for any $\alpha \in \Delta_\nu$*,*

$$\dim_H E_x(\alpha) = \frac{1}{\log m} \inf_{q \in \mathbb{R}} \{P_\nu(q\phi) - \alpha q\} = \frac{1}{\log m} \sup_{\widetilde{\mu}} \{h_{\widetilde{\mu}}(T \times \sigma) - h_\nu(T)\},$$

where the supremum is taken over the set of all $T \times \sigma$-invariant measures $\widetilde{\mu}$ such that $\widetilde{\mu} \circ \pi^{-1} = \nu$ and $\int \phi \, d\widetilde{\mu} = \alpha$.

Theorem 1.1 is deduced from Theorem 1.2 and a variational principle between $\tau(q)$ and the relative entropies (see Proposition 4.4 and Corollary 4.5). It has some natural geometric realizations (see §5).

The main purpose of this paper is to generalize Theorem 1.2 to random and high dimensional cases and to remove the regularity assumption of ϕ. For this purpose, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Lebesgue space with an ergodic transformation ϑ on Ω. Let Φ be a bounded $\mathcal{F} \otimes \mathcal{B}(\Sigma)$-measurable $\mathbb{R}^d$-valued function on $\Omega \times \Sigma$, where $\mathcal{B}(\Sigma)$ denotes the Borel σ-algebra on Σ. Assume that Φ is equi-continuous in the sense that for any $\epsilon > 0$, there exists $\delta > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$|\Phi(\omega, y) - \Phi(\omega, y')| < \epsilon \quad \text{if } d(y, y') < \delta.$$

Let $\mathcal{M}_\mathbb{P}(\Omega \times \Sigma)$ denote the collection of all probability measures $\widetilde{\mu}$ on the measurable space $(\Omega \times \Sigma, \mathcal{F} \otimes \mathcal{B}(\Sigma))$ such that $\widetilde{\mu}$ is $\vartheta \times \sigma$-invariant and $\widetilde{\mu} \circ \pi^{-1} = \mathbb{P}$, where π denotes the projection $(\omega, y) \mapsto \omega$ from $\Omega \times \Sigma$ to Ω. Now we define

$$\Delta_\mathbb{P} = \left\{ \int \Phi \, d\widetilde{\mu} : \ \widetilde{\mu} \in \mathcal{M}_\mathbb{P}(\Omega \times \Sigma) \right\}.$$

For $\omega \in \Omega$ and $\alpha \in \mathbb{R}^d$, we denote

$$E_\omega(\alpha) = \left\{ y \in \Sigma : \ \lim_{n \to \infty} \frac{1}{n} S_n \Phi(\omega, y) = \alpha \right\},$$

where $S_n \Phi(\omega, y) := \sum_{i=0}^{n-1} \Phi(\vartheta^i \omega, \sigma^i y)$. Under the above setting, we have the following general result.

THEOREM 1.3. *There exists $\Gamma \in \mathcal{F}$ with $\mathbb{P}(\Gamma) = 1$ such that for any $\omega \in \Gamma$,*

(i) $\{\alpha \in \mathbb{R}^d : E_\omega(\alpha) \neq \emptyset\} = \Delta_\mathbb{P}$*;*

(ii) for any $\alpha \in \Delta_\mathbb{P}$*,*

$$\begin{aligned} \dim_H E_\omega(\alpha) &= \frac{1}{\log m} \inf_{q \in \mathbb{R}^d} \{P_\mathbb{P}(q) - \langle \alpha, q \rangle\} \\ &= \frac{1}{\log m} \sup \{h(\widetilde{\mu}|\mathbb{P}) : \ \widetilde{\mu} \in \mathcal{G}(\alpha)\}, \end{aligned}$$

where $P_{\mathbb{P}}(q) := P_{\mathbb{P}}(\langle q, \Phi\rangle)$ *denotes the relativized topological pressure of* $\psi_q = \langle q, \Phi\rangle$*, here* $\langle \cdot \rangle$ *is the inner product on* $\mathbb{R}^d$ *and* $\psi_q(\omega, y) = \langle q, \Phi(\omega, y)\rangle$*,* $h(\widetilde{\mu}|\mathbb{P})$ *denotes the relativized entropy of* $\widetilde{\mu}$ *and* $\mathcal{G}(\alpha)$ *is defined by*

$$\mathcal{G}(\alpha) := \left\{\widetilde{\mu} \in \mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma) : \int \Phi \, d\widetilde{\mu} = \alpha\right\}.$$

The reader may see §2 for the definitions of relativized topological pressure and relativized entropy. We remark that in part (ii) of Theorem 1.1-1.3, the supremum is always attained at some $\widetilde{\mu}$. This fact is due to the upper semi-continuity of $h_{\widetilde{\mu}}(T \times \sigma)$ and $h(\widetilde{\mu}|\mathcal{P})$ in our settings. However the infimum may be not attained for boundary points α.

It is worth pointing out that rather than random (conditional) Birkhoff averages, the multifractal analysis of classical Birkhoff averages has been studied intensively in a recent decade (see, e.g., [**8, 22, 36, 45, 51, 55**] and also [**6, 9, 15, 26, 37, 48**]). The multi-dimensional case was first studied in [**21**] for Hölder continuous potentials and was further developed for arbitrary continuous functions in [**22**] for symbolic spaces, in [**26**] for conformal repellers, and in [**55**] for dynamical systems satisfying the specification condition. For instance, for an arbitrary $\mathbb{R}^d$-valued continuous function Φ on symbolic product spaces, we have

$$\dim_H E(\alpha) = \frac{1}{\log m} \sup_{\widetilde{\mu}} h_{\widetilde{\mu}}(T \times \sigma), \tag{1.2}$$

where the supremum is taken over the collection of $T\times\sigma$-invariant Borel probability measures $\widetilde{\mu}$ with $\int \Phi d\widetilde{\mu} = \alpha$ (see e.g., [**22**, Theorem A]). Rather than considering the Birkhoff average $S_n\phi/n$, Barreira, Saussol and Schmeling [**7, 8**] studied the multifractal structure of the more general average $S_n\phi/S_n\psi$ and its multi-dimensional version.

Theorem 1.3 provides a finer and random version of the variational principle (1.2). One of the main difficulties for studying $E_\omega(\alpha)$ rather than $E(\alpha)$ comes from the fact that $E_\omega(\alpha)$ is much sensitive to ω and is not σ-invariant.

The reduction of Theorem 1.3(ii) to the deterministic case strengthens (1.2). Let $\Delta := \{\int \Phi d\mu : \ \mu - T \times \sigma \text{ invariant}\}$. When $d = 1$, it is known [**45**] that for $\alpha \in \Delta$,

$$\inf_{q\in\mathbb{R}^d} \{P(\langle q, \Phi\rangle) - \langle \alpha, q\rangle\} = \sup_{\widetilde{\mu}} h_{\widetilde{\mu}}(T \times \sigma),$$

in (1.2), where $P(\cdot)$ is the usual pressure function. When $d \geq 2$, the equality is only known (cf. [**21**]) to hold for those points α in the range of gradients of $P(\cdot)$. By using a technique from convex analysis, we are able to set up the variational principle for all α including the boundary points.

We point out that Theorem 1.1 strengthens a previous result of Kifer ([**39**, Theorem 5.1]) who proved, under a more general setting of random Gibbs measure, that $\tau(q)$ is analytic over $\mathbb{R}$; and for any given $\beta = \tau'(q)$, (1.1) holds for a.e. x.

Kifer took a direct approach by the thermodynamic formalism for random shifts, which is not enough to deal with the boundary points $\beta_{\min}$ and $\beta_{\max}$ whenever they are not included in the range of τ'. We remark that an analogue of Kifer's result was also obtained by Fan [**20**] (see also Fan and Shieh [**24**]) in the setting of infinite products through a large deviation approach, and some further study was given by Barral, Coppens and Mandelbrot in [**5**] to the multiplicative martingale measures, for which the potential can have a dense countable set of discontinuities.

We remark that under the setting of Theorem 1.3, the relativized topological pressure function $P_{\mathbb{P}}(q)$ may be not differentiable. For those $\alpha \in \Delta_{\mathbb{P}}$ not corresponding to the gradients of $P_{\mathbb{P}}$, one can not prove the lower bound of $\dim_H E_\omega(\alpha)$ directly through the classical approach using the relativized thermodynamic formalism or the large deviation principle. Hence some new ideas are needed to overcome this difficulty. In the following we sketch our main steps and key ideas for the proof of Theorem 1.3.

The proof of part (i) of Theorem 1.3 is based on the relativized thermodynamic formalism and the construction of Moran-like subsets of level sets of random Birkhoff averages. For the construction of Moran-like subsets, we extend an idea used in [**21, 22**]. Nevertheless our construction depends on the recurrence and ergodic properties of the random transformation and is much more subtle. To prove (ii), we define

$$f_\omega(\alpha; n, \epsilon) = \#\{y_1 \dots y_n : \quad |S_n\Phi(\omega, y) - n\alpha| < n\epsilon \text{ for some } y = (y_i)_{i=1}^\infty \in \Sigma\}$$

for $\omega \in \Omega$, $\alpha \in \Delta_{\mathbb{P}}$, $n \in \mathbb{N}$ and $\epsilon > 0$, where $\#A$ denotes the cardinality of A. We first prove that there is an upper semi-continuous and concave function Λ on $\Delta_{\mathbb{P}}$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$\lim_{\epsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log f_\omega(\alpha; n, \epsilon) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log f_\omega(\alpha; n, \epsilon) = \Lambda(\alpha), \quad \forall\, \alpha \in \Delta_{\mathbb{P}}.$$

A delicate Moran construction (depending on ω) is also used in the proof of the concavity of Λ. Then we show that for $\mathbb{P}$-a.e. ω, $\dim_H E_x(\alpha) = \frac{1}{\log m}\Lambda(\alpha)$ for all $\alpha \in \Delta_{\mathbb{P}}$. In this step, the proof of the lower bound is crucial and the main idea is to construct Moran-like subsets of $E_\omega(\alpha)$ with the Hausdorff dimension equal to $\frac{1}{\log m}\Lambda(\alpha)$. Our next step is to prove a duality principle between $P_{\mathbb{P}}(q)$ and $h(\widetilde{\mu}|\mathbb{P})$ (i.e. the second equality in (ii)) by convex analysis. In the last step, we show that

$$\sup\{h(\widetilde{\mu}|\mathbb{P})\colon \widetilde{\mu} \in \mathcal{G}(\alpha)\} \le \Lambda(\alpha) \le \frac{1}{\log m}\inf_{q\in\mathbb{R}^d}\{P_{\mathbb{P}}(q) - \langle \alpha, q\rangle\}.$$

The second inequality just follows from a box principle, whilst the first inequality is derived from a relativized version of Shannon-Mcmillian-Brieman theorem [**10**], using the concavity and upper semi-continuity of Λ.

The paper is arranged in the following way: in §2, we give some preliminaries about the relativized thermodynamic formalism for random shifts. In §3, we prove Theorem 1.3. In §4, we prove Theorem 1.1. In §5, we give some geometric realizations of theorem 1.2 and some remarks.

2. *Preliminaries*

In this section, we outline the classical relativized thermodynamical formalism for random shift which is needed in the proofs of our main theorems. The reader is referred to [**10, 11, 38, 40**] for more details.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with an ergodic $\mathbb{P}$-preserving transformation ϑ on it. Furthermore, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a Lebesgue space, i.e., it is measurably isomorphic to an interval (maybe empty) with the completion of the Borel σ-algebra and the Lebesgue measure on it together, maybe, with countably many atoms (cf. [**54**]). Fix an integer $m \geq 2$. Let $\Sigma = \{1, 2, \ldots, m\}^{\mathbb{N}}$ be the product space endowed with the metric

$$d(x, y) = m^{-\min\{i:\, x_i \neq y_i\}} \quad \text{for } x = (x_i)_{i=1}^{\infty},\ y = (y_i)_{i=1}^{\infty} \in \Sigma. \tag{2.1}$$

It is known (see [**13**]) that Σ is compact. Consider the shift map $\sigma\colon (x_i)_{i=1}^{\infty} \mapsto (x_{i+1})_{i=1}^{\infty}$ on Σ. The dynamical system (Σ, σ) is called the *one-sided full shift on m symbols.* Let $\mathcal{B}$ denote the Borel σ-algebra on Σ. Our target system is the product space $(\Omega \times \Sigma, \mathcal{F} \otimes \mathcal{B})$ with the measurable transformation $\Theta := \vartheta \times \sigma$, which can be viewed as a special random dynamical system (RDS) over Σ.

2.1. *Invariant measures for RDS*

Let $\pi : \Omega \times \Sigma \to \Omega$ be the canonical projection $(\omega, x) \mapsto \omega$. A measure μ on the measurable space $(\Omega \times \Sigma, \mathcal{F} \otimes \mathcal{B})$ is said to have *marginal* $\mathbb{P}$ *on* Ω if $\mu \circ \pi^{-1} = \mathbb{P}$. Denote by $\mathcal{P}_{\mathbb{P}}(\Omega \times \Sigma)$ the space of probability measures on $\Omega \times \Sigma$ having marginal $\mathbb{P}$ on Ω. Let $\mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma)$ denote the set of Θ-invariant elements of $\mathcal{P}_{\mathbb{P}}(\Omega \times \Sigma)$. It is clear that $\mathcal{P}_{\mathbb{P}}(\Omega \times \Sigma)$ and $\mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma)$ are convex. Let $\mathcal{E}_{\mathbb{P}}(\Omega \times \Sigma)$ denote the set of ergodic measures in $\mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma)$ with respect to Θ.

Denote by $L^1(\Omega, C(\Sigma))$ the space of measurable in ω and continuous in x functions $\phi(\omega, x)$ on $\Omega \times \Sigma$ such that

$$\|\phi\| = \int \sup_{x \in \Sigma} |\phi(\omega, x)| d\mathbb{P}(\omega) < \infty.$$

For $\mu, \mu_n \in \mathcal{P}_{\mathbb{P}}(\Omega \times \Sigma)$, $n = 1, 2, \ldots$, we say that μ_n converge to μ if $\int \phi \, d\mu_n \to \int \phi \, d\mu$ as $n \to \infty$ for any $\phi \in L^1(\Omega, C(\Sigma))$. This convergence introduces a weak* topology in $\mathcal{P}_{\mathbb{P}}(\Omega \times \Sigma)$.

Proposition 2.1. *(i) $\mathcal{P}_{\mathbb{P}}(\Omega \times \Sigma)$ is compact in this weak* topology, and $\mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma)$ is a non-empty compact convex subset of $\mathcal{P}_{\mathbb{P}}(\Omega \times \Sigma)$;*

(ii) $\mathcal{E}_{\mathbb{P}}(\Omega \times \Sigma)$ coincides with the set of extreme points of $\mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma)$;

(iii) For any $\mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma)$, there is a unique probability measure Q_μ on $\mathcal{E}_{\mathbb{P}}(\Omega \times \Sigma)$ such that

$$\int_{\Omega \times \Sigma} \phi \, d\mu = \int_{\mathcal{E}_{\mathbb{P}}(\Omega \times \Sigma)} \int_{\Omega \times \Sigma} \phi \, d\eta \, dQ_\mu(\eta), \quad \forall\, \phi \in L^1(\Omega, C(\Sigma)).$$

Proof. See [**40**, Lemma 2.1(i)] for a proof of (i), and see [**17**, Lemma 6.19] for (ii). Part (iii) follows from (i), (ii) and Choquet's representation theorem (cf. [**16**]). □

2.2. *Disintegrations of measures* A map $\mu\colon \Omega \times \mathcal{B} \to [0,1]$, $(\omega, B) \mapsto \mu_\omega(B)$, is said to be a *random probability measure on* Σ if it satisfies (i) for each $B \in \mathcal{B}$, $\omega \mapsto \mu_\omega(B)$ is measurable, (ii) for $\mathbb{P}$-almost every $\omega \in \Omega$, $B \mapsto \mu_\omega(B)$ is a Borel probability measure. The connection between $\mathcal{P}_\mathbb{P}(\Omega \times \Sigma)$ and random measures is given by the following proposition (see [**17**, Proposition 3.6] for a proof).

Proposition 2.2. *For each $\mu \in \mathcal{P}_\mathbb{P}(\Omega \times \Sigma)$, there exists a random measure $\omega \mapsto \mu_\omega$ such that*

$$\int_{\Omega\times\Sigma} \phi(\omega, x)\, d\mu(\omega, x) = \int_\Omega \int_\Sigma \phi(\omega, x)\, d\mu_\omega(x)\, d\mathbb{P}(\omega)$$

for every bounded measurable $\phi : \Omega \times \Sigma \to \mathbb{R}$. The random measure $\omega \mapsto \mu_\omega$ is unique $\mathbb{P}$-a.e.

The random measure $\omega \mapsto \mu_\omega$ in the above proposition is often named as the *disintegration* of μ.

2.3. *Relativized topological pressure and relativized entropy* Let $\phi \in L^1(\Omega, C(\Sigma))$. For $\omega \in \Omega$ and $n \in \mathbb{N}$, define

$$P_\mathbb{P}(\phi)(\omega, n) = \sum_{A\in\xi^n} \sup_{x\in A} \exp(S_n\phi(\omega, x)),$$

where $S_n\phi(\omega, x) = \sum_{i=0}^{n-1} \phi \circ \Theta^i(\omega, x)$, ξ^n denotes the partition $\{[i_1 \dots i_n] : i_1 \dots i_n \in \{1, \dots, m\}^n\}$ of Σ, and $[i_1 \dots i_n]$ is the n-cylinder $\{(x_i)_{i=1}^\infty \in \Sigma\colon x_k = i_k \text{ for } 1 \le k \le n\}$. The *relativized topological pressure* of ϕ for the RDS is defined by

$$P_\mathbb{P}(\phi) = \limsup_{n\to\infty} \frac{1}{n} \int \log P_\mathbb{P}(\phi)(\omega, n)\, d\mathbb{P}(\omega).$$

Since $\mathbb{P}$ is ergodic, we have (see, e.g., [**40**, Proposition 1.6])

$$\lim_{n\to\infty} \frac{1}{n} \log P_\mathbb{P}(\phi)(\omega, n) = P_\mathbb{P}(\phi) \qquad \mathbb{P}\text{-a.e.} \tag{2.2}$$

Now let $\mathcal{R}^n$ denote the partition $\{\Omega \times A\colon A \in \xi^n\}$ of $\Omega \times \Sigma$. For given $\mu \in \mathcal{M}_\mathbb{P}(\Omega \times \Sigma)$, the conditional entropy of $\mathcal{R}^n$ given the σ-algebra $\pi^{-1}(\mathcal{F})$ is defined by

$$H_\mu(\mathcal{R}^n|\pi^{-1}(\mathcal{F})) = \int H_{\mu_w}(\xi^n)\, d\mathbb{P}(\omega),$$

where $H_{\mu_\omega}(\xi^n) := -\sum_{A\in\xi^n} \mu_\omega(A) \log \mu_\omega(A)$ denotes the usual entropy of the partition ξ^n and $\omega \mapsto \mu_\omega$ is the random measure corresponding to μ as in Proposition 2.2. The *relativized entropy* of μ for the RDS is defined by

$$h(\mu|\mathbb{P}) = \lim_{n\to\infty} \frac{1}{n} H_\mu(\mathcal{R}^n|\pi^{-1}(\mathcal{F})).$$

The above limit exists by the subadditivity of the conditional entropy. Thus

$$h(\mu|\mathbb{P}) = \inf_n \frac{1}{n} H_\mu(\mathcal{R}^n|\pi^{-1}(\mathcal{F})). \tag{2.3}$$

(cf. [**38**, Theorem 1.1, p. 40]). Moreover, if μ is ergodic with respect to Θ, then

$$\lim_{n\to\infty} \frac{1}{n} \log \mu_\omega(\xi^n(y)) = h(\mu|\mathbb{P}) \quad \text{for } \mu\text{-a.e. } (\omega, y) \tag{2.4}$$

(cf. [**10**, Theorem 4.2]), where $\xi^n(y)$ denotes the member in ξ^n that contains y. The Abramov-Rohlin formula states that $h_\mu(\Theta) = h(\mu|\mathbb{P}) + h_{\mathbb{P}}(\vartheta)$ (see [**2**]), where $h_\mu(\Theta)$ and $h_{\mathbb{P}}(\vartheta)$ are the ordinary entropies of the corresponding measure preserving transformations.

The following variational principle, connecting the relativized topological pressure and the relativized entropy, was proved by Bogenschütz in [**10**, Theorem 6.1]. It is a generalization of the (deterministic) relativized variational principle of Ledrappier and Walters [**42**].

Proposition 2.3. $P_{\mathbb{P}}(\phi) = \sup\left\{ h(\mu|\mathbb{P}) + \int \phi \, d\mu \colon \mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma) \right\}.$

We point out that the relativized entropy map $\mu \to h(\mu|\mathbb{P})$ is affine and upper semi-continuous on $\mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma)$. (The proof of the affinity is similar to that for the usual entropy map (cf. [**56**, Theorem 8.1]), while the upper semi-continuity follows from (2.3) and Lemma 2.1(iii) in [**40**]. The reader is referred to [**43**] for details.) Hence the supremum in the above variational formula is always attained at some member of $\mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma)$. Furthermore, as an application of Choquet theorem (cf. [**53**] or [**56**, p. 186]), we have

$$h(\mu|\mathbb{P}) = \int h(\eta|\mathbb{P}) dQ_\mu(\eta), \tag{2.5}$$

where $\mu = \int \eta \, dQ_\mu(\eta)$ is the ergodic decomposition of μ.

3. *The proof of Theorem 1.3*

In this section, we provide a full proof of Theorem 1.3. For the convenience of the reader, we recall some basic notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a Lebesgue space with an ergodic transformation ϑ on Ω. Let (Σ, σ) be the one-sided full shift space over m symbols. Denote $\Theta = \vartheta \times \sigma$. Fix $d \in \mathbb{N}$. Denote by $C(\Sigma, \mathbb{R}^d)$ the set of $\mathbb{R}^d$-valued continuous functions on Σ. Let Φ be a bounded $\mathcal{F} \otimes \mathcal{B}(\Sigma)$-measurable function taking values in $\mathbb{R}^d$ such that (i) $\Phi(\omega, \cdot) \in C(\Sigma, \mathbb{R}^d)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$; (ii) $\omega \mapsto \Phi(\omega, \cdot)$ is *equicontinuous*, i.e., for any $\epsilon > 0$, there exists $\delta > 0$ such that for $\mathbb{P}$-a.e. $\omega \in \Omega$,

$$|\Phi(\omega, y) - \Phi(\omega, y')| < \epsilon \quad \text{whenever } d(y, y') < \delta. \tag{3.1}$$

It is clear that for any $q \in \mathbb{R}^d$, $\langle q, \Phi \rangle \in L^1(\Omega, C(\Sigma))$ (see §2.1). Here $\langle \cdot, \cdot \rangle$ is the inner product on $\mathbb{R}^d$. Now define

$$\Delta_{\mathbb{P}} = \left\{ \int \Phi \, d\mu : \ \mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma) \right\}, \tag{3.2}$$

where $\mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma)$ is defined as in §2.1. For $\omega \in \Omega$ and $\alpha \in \mathbb{R}^d$, denote

$$E_\omega(\alpha) = \left\{ y \in \Sigma : \ \lim_{n\to\infty} \frac{1}{n} S_n \Phi(\omega, y) = \alpha \right\},$$

where $S_n\Phi(\omega, y) := \sum_{i=0}^{n-1} \Phi \circ \Theta^i(\omega, y)$. Write

$$\mathcal{G}(\alpha) := \left\{ \mu \in \mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma) : \int \Phi \, d\mu = \alpha \right\}. \tag{3.3}$$

The proof of Theorem 1.3 is rather long and will be divided into a sequence of lemmas and propositions in the remainder of this section.

3.1. *The set* $\{\alpha \in \mathbb{R}^d : \ E_\omega(\alpha) \neq \emptyset\}$ In this subsection, we prove the following proposition.

PROPOSITION 3.1. *There exists a measurable set $H \subset \Omega$ with $\mathbb{P}(H) = 1$ such that $\{\alpha \in \mathbb{R}^d : \ E_\omega(\alpha) \neq \emptyset\} = \Delta_{\mathbb{P}}$ for all $\omega \in H$, where $\Delta_{\mathbb{P}}$ is defined by (3.2).*

We divide the proof into several lemmas.

LEMMA 3.2. *For $\mathbb{P}$-a.e. $\omega \in \Omega$ we have $\{\alpha \in \mathbb{R}^d : \ E_\omega(\alpha) \neq \emptyset\} \subseteq \Delta_{\mathbb{P}}$.*

Proof. By Proposition 2.3, $P_{\mathbb{P}}(q) = P_{\mathbb{P}}(\langle q, \Phi \rangle)$ is a real convex function of q over $\mathbb{R}^d$. Hence it is continuous on $\mathbb{R}^d$. According to (2.2), we have for any $q \in \mathbb{R}^d$,

$$\lim_{n\to\infty} \frac{1}{n} \log P_{\mathbb{P}}(\langle q, \Phi \rangle)(\omega, n) = P_{\mathbb{P}}(q) \qquad \mathbb{P}\text{-a.e.} \tag{3.4}$$

Let $\{q_i\}_{i=1}^\infty$ be a countable sequence dense in $\mathbb{R}^d$ and let Γ be the set of points ω in Ω such that the equality in (3.4) holds for all q_i. Clearly $\mathbb{P}(\Gamma) = 1$. We show below that $\{\alpha \in \mathbb{R}^d : \ E_\omega(\alpha) \neq \emptyset\} \subseteq \Delta_{\mathbb{P}}$ for $\omega \in \Gamma$.

We first show that for any $\omega \in \Gamma$,

$$\lim_{n\to\infty} \frac{1}{n} \log P_{\mathbb{P}}(\langle q, \Phi \rangle)(\omega, n) = P_{\mathbb{P}}(q), \qquad \forall \, q \in \mathbb{R}^d.$$

Fix $\omega \in \Gamma$ and $q \in \mathbb{R}^d$. There exists a subsequence $\{q_{i_k}\}$ converging to q. Observe that

$$\left| \frac{1}{n} \log P_{\mathbb{P}}(\langle q_{i_k}, \Phi \rangle)(\omega, n) - \frac{1}{n} \log P_{\mathbb{P}}(\langle q, \Phi \rangle)(\omega, n) \right| \leq |\langle q_{i_k} - q, \Phi \rangle| \leq |q_{i_k} - q| \cdot \|\Phi\|_\infty$$

for all $n, k \in \mathbb{N}$, where $\|\Phi\|_\infty = \sup_{(\omega,y)\in\Omega\times\Sigma} |\Phi(\omega, y)|$. This implies that

$$\lim_{n\to\infty} \frac{1}{n} \log P_{\mathbb{P}}(\langle q, \Phi\rangle)(\omega, n) = \lim_{k\to\infty} P_{\mathbb{P}}(q_{i_k}) = P_{\mathbb{P}}(q),$$

and we are done.

Next, fix $\omega \in \Gamma$. Assume that $\alpha \in \mathbb{R}^d$ satisfies $E_\omega(\alpha) \neq \emptyset$, we show $\alpha \in \Delta_{\mathbb{P}}$. By the assumption on α, there is $y \in \Sigma$ such that

$$\lim_{n\to\infty} \frac{1}{n} S_n\Phi(\omega, y) = \alpha$$

and hence

$$\lim_{n\to\infty} \frac{1}{n} S_n\langle q, \Phi\rangle(\omega, y) = \langle \alpha, q\rangle, \quad \forall\, q \in \mathbb{R}^d.$$

By the definition of $P_{\mathbb{P}}(\langle q, \Phi\rangle)(\omega, n)$, we have $\limsup_{n\to\infty} \frac{1}{n} \log P_{\mathbb{P}}(\langle q, \Phi\rangle)(\omega, n) \geq \langle \alpha, q\rangle$, and hence

$$P_{\mathbb{P}}(q) \geq \langle \alpha, q\rangle, \qquad \forall\, q \in \mathbb{R}^d. \tag{3.5}$$

Suppose $\alpha \notin \Delta_{\mathbb{P}}$. Since $\Delta_{\mathbb{P}}$ is a compact convex subset of $\mathbb{R}^d$, there must exist $e \in \mathbb{R}^d$ such that

$$\langle \alpha, e\rangle > \sup_{\beta\in\Delta_{\mathbb{P}}} \langle \beta, e\rangle.$$

That is, there exists a hyperplane separating α and $\Delta_{\mathbb{P}}$ (cf. [**32**, Theorem 4.1.1]). Take $q = te$ ($t \in \mathbb{R}^+$). Then for sufficiently large t,

$$\langle \alpha, q\rangle > \sup_{\beta\in\Delta_{\mathbb{P}}} \langle \beta, q\rangle + 2\log m. \tag{3.6}$$

However by Proposition 2.3,

$$\begin{aligned} P_{\mathbb{P}}(q) &\leq \log m + \sup\left\{\int \langle q, \Phi\rangle \, d\mu : \quad \mu \in \mathcal{M}_{\mathbb{P}}(\Omega\times\Sigma)\right\} \\ &= \log m + \sup_{\beta\in\Delta_{\mathbb{P}}} \langle \beta, q\rangle. \end{aligned}$$

This together with (3.6) yields $\langle \alpha, q\rangle > P_{\mathbb{P}}(q) + \log m$, which contradicts (3.5). Hence $\alpha \in \Delta_{\mathbb{P}}$. □

To prove the other direction of Proposition 3.1, we need a few more lemmas.

LEMMA 3.3. *Let $\alpha \in \Delta_{\mathbb{P}}$ and $\mu \in \mathcal{G}(\alpha)$ (see (3.3)). For any $\epsilon > 0$, there exists $k \in \mathbb{N}$, $p_1, \ldots, p_k \geq 0$ with $\sum_{i=1}^k p_i = 1$ and ergodic measures $\mu_1, \ldots, \mu_k \in \mathcal{E}_{\mathbb{P}}(\Omega\times\Sigma)$ such that*

$$\left|\int \Phi \, d\widetilde{\mu} - \alpha\right| < \epsilon \quad \text{and} \quad h(\widetilde{\mu}|\mathbb{P}) \geq h(\mu|\mathbb{P}) - \epsilon,$$

where $\widetilde{\mu} = \sum_{i=1}^k p_i\mu_i$.

Proof. By Proposition 2.1(iii), there is a probability measure Q_μ on $\mathcal{E}_{\mathbb{P}}(\Omega\times\Sigma)$ such that
$$\int \Phi\ d\mu = \int_{\mathcal{E}_{\mathbb{P}}(\Omega\times\Sigma)} \int \Phi\ d\eta\ dQ_\mu(\eta).$$
Recall that $\eta \mapsto \int \Phi\ d\eta$ is continuous on $\mathcal{E}_{\mathbb{P}}(\Omega\times\Sigma)$ under the weak* topology, and $\mathcal{E}_{\mathbb{P}}(\Omega\times\Sigma)$ is compact (see Proposition 2.1(ii)). Hence by the open covering theorem, there exist $k\in\mathbb{N}$ and a Borel partition $\{\mathcal{E}_1,\ldots,\mathcal{E}_k\}$ of $\mathcal{E}_{\mathbb{P}}(\Omega\times\Sigma)$ such that
$$\left|\int \Phi\ d\eta - \int \Phi\ d\eta'\right| < \epsilon, \quad \forall\ \eta,\ \eta' \in \mathcal{E}_i.$$
For $i=1,\ldots,k$, choose $\mu_i\in\mathcal{E}_i$ such that $Q_\mu(\mathcal{E}_i)h(\mu_i|\mathbb{P}) \geq \int_{\mathcal{E}_i}(h(\eta|\mathbb{P})-\epsilon)dQ_\mu(\eta)$, and put $p_i = Q_\mu(\mathcal{E}_i)$. Then by (2.5), $\widetilde{\mu} = \sum_{i=1}^k p_i\mu_i$ satisfies our requirement. □

LEMMA 3.4. *There exists $A\in\mathcal{F}$ with $\mathbb{P}(A)=1$ such that $\lim_{n\to\infty} V_n(A,\Phi)/n = 0$, where $V_n(A,\Phi)$ is defined by*
$$V_n(A,\Phi) = \sup\{|S_n\Phi(\omega,y) - S_n\Phi(\omega,y')| :\ \omega\in A,\ y,\ y'\in\Sigma \text{ with } y|_n = y'|_n\}$$
with $y|_n := y_1\ldots y_n$.

Proof. It follows directly from the assumption (3.1). □

Let A be a set such that Lemma 3.4 holds. Since $\mathbb{P}$ is ϑ-invariant, we have $\mathbb{P}(\bigcap_{i\in\mathbb{N}}\vartheta^{-i}(A)) = 1$, i.e., the set of points whose forward orbits are contained in A has full measure. Hence it is of no harm to assume that Lemma 3.4 holds for $A=\Omega$ in the sequel since we are concerning $\mathbb{P}$-a.e. conclusions. We simply write $V_n(\Phi)$ for $V_n(\Omega,\Phi)$.

For $\omega\in\Omega$, $\alpha\in\mathbb{R}^d$, $n\in\mathbb{N}$ and $\epsilon>0$, denote
$$F_\omega(\alpha;n,\epsilon) = \{I\in\Sigma_n :\ |S_n\Phi(\omega,y) - n\alpha| < n\epsilon \text{ for some } y\in[I]\}$$
and
$$f_\omega(\alpha;n,\epsilon) = \#F_\omega(\alpha;n,\epsilon). \tag{3.7}$$
The following lemma plays a key role in the proof of Proposition 3.1.

LEMMA 3.5. *Let $\alpha_1,\alpha_2\in\mathbb{R}^d$. Assume that $\mathbb{P}(\{\omega\in\Omega:\ E_\omega(\alpha_i)\neq\emptyset\}) = 1$ for $i=1,2$. Then $\mathbb{P}(\{\omega\in\Omega:\ E_\omega((\alpha_1+\alpha_2)/2)\neq\emptyset\}) = 1$.*

Proof. For $i\in\{1,2\}$ and $k,j\in\mathbb{N}$, denote
$$A_{i,k,j} := \{\omega\in\Omega:\ f_\omega(\alpha_i;n,1/k)\geq 1 \text{ for all } n\geq j\}.$$
Let H denote the set of all points ω satisfying
$$\lim_{n\to\infty}\frac{1}{n}\sum_{s=0}^{n-1}\chi_{A_{i,k,j}}(\vartheta^{ps+q}\omega) = \mathbb{P}(A_{i,k,j}), \quad \forall\ i\in\{1,2\},\ k,j,p,q\in\mathbb{N}. \tag{3.8}$$

(Here χ_A denotes the characteristic function of the set A.) By Birkhoff ergodic theorem, $\mathbb{P}(H) = 1$. In the following we show that $E_\omega((\alpha_1 + \alpha_2)/2) \neq \emptyset$ for all $\omega \in H$.

Construct a sequence $\{\epsilon_k\}_{k=1}^{\infty}$ by $\epsilon_k = 1/k$. By the assumption of the lemma, we have for each $i \in \{1, 2\}$ and $\epsilon > 0$,

$$\mathbb{P}(\{\omega \in \Omega : \liminf_{n\to\infty} f_\omega(\alpha_i; n, \epsilon) \geq 1\}) = 1.$$

As a consequence, we can choose a sequence of integers $\{n_k\} \uparrow \infty$ such that for any $i \in \{1, 2\}$ and $k \in \mathbb{N}$, the set

$$G_{i,k} := A_{i,k,n_k} = \{\omega \in \Omega : \ f_\omega(\alpha_i; n, \epsilon_k) \geq 1 \text{ for } n \geq n_k\} \tag{3.9}$$

has measure $\mathbb{P}(G_{i,k}) > 1 - 2^{-k}$.

Fix $\widetilde{\omega} \in H$. By (3.8) and (3.9), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} \chi_{G_{i,k}}(\vartheta^{n_k s+q}\widetilde{\omega}) = \mathbb{P}(G_{i,k}) > 1 - 2^{-k}, \quad \forall\, i \in \{1, 2\},\ k, q \in \mathbb{N}. \tag{3.10}$$

In the following we show that $E_{\widetilde{\omega}}((\alpha_1 + \alpha_2)/2)$ contains a non-empty Moran-like subset of Σ. First we construct inductively a sequence of even integers $\{m_k\}_{k=1}^{\infty} \uparrow \infty$ (depending on $\widetilde{\omega}$) as follows. By (3.10), we can choose an even integer m_1 large enough such that $m_1 \geq 2^{n_2}$ and for $i \in \{1, 2\}$,

$$\begin{aligned}
&\frac{1}{m_1} \sum_{s=0}^{m_1-1} \chi_{G_{i,1}}(\vartheta^{n_1 s}\widetilde{\omega}) > 1 - 2^{-1} \qquad \text{and} \\
&\frac{1}{\ell} \sum_{s=0}^{\ell-1} \chi_{G_{i,2}}(\vartheta^{n_2 s+q}\widetilde{\omega}) \geq 1 - 2^{-2}, \quad \forall\, \ell \geq m_1/n_2,\ 0 \leq q \leq n_2 - 1.
\end{aligned}$$

Suppose $m_1, \ldots, m_{k-1}$ have been constructed. By (3.10) again, we choose an even number m_k large enough such that

$$m_k \geq \max\{2^{m_{k-1}}, 2^{n_{k+1}}\} \qquad \text{and for } i \in \{1, 2\}, \tag{3.11}$$

$$\frac{1}{m_k} \sum_{s=0}^{m_k-1} \chi_{G_{i,k}}(\vartheta^{n_k s+\sum_{j=1}^{k-1} n_j m_j}\widetilde{\omega}) > 1 - 2^{-k}, \qquad \text{and} \tag{3.12}$$

$$\frac{1}{\ell} \sum_{s=0}^{\ell-1} \chi_{G_{i,k+1}}(\vartheta^{n_{k+1} s+q}\widetilde{\omega}) \geq 1 - 2^{-k-1}, \quad \forall\, \ell \geq \frac{m_k}{n_{k+1}},\ 0 \leq q \leq n_{k+1} - 1. \tag{3.13}$$

In this way we obtain a sequence $\{m_k\}_{k=1}^{\infty}$. Now for any $k \in \mathbb{N}$ and $1 \leq j \leq m_k$, we denote

$$N(k, j) = \begin{cases} (j - 1)n_k & \text{if } k = 1, \\ \sum_{s=1}^{k-1} n_s m_s + (j - 1)n_k & \text{if } k \geq 2 \end{cases} \tag{3.14}$$

and construct a subset $\Upsilon_{k,j}$ of Σ_{n_k} by

$$\Upsilon_{k,j} = \begin{cases} F_{\vartheta^{N(k,j)}\widetilde{\omega}}(\alpha_{t_j}; n_k, \epsilon_k) & \text{if } \vartheta^{N(k,j)}\widetilde{\omega} \in G_{t_j,k}, \\ \Sigma_{n_k} & \text{otherwise,} \end{cases}$$

where t_j is defined by

$$t_j = \begin{cases} 1 & \text{if } j \text{ is odd,} \\ 2 & \text{if } j \text{ is even.} \end{cases}$$

By the definition of $G_{i,k}$, we have $\Upsilon_{k,j} \neq \emptyset$.

Define $\Upsilon = \prod_{k=1}^{\infty} \prod_{j=1}^{m_k} \Upsilon_{k,j}$. It is the subset of Σ consisting of the points y of the form

$$y = I_{1,1} \ldots I_{1,m_1}\ I_{2,1} \ldots I_{2,m_2} \ldots I_{k,1} \ldots I_{k,m_k} \ldots, \tag{3.15}$$

where $I_{k,j} \in \Upsilon_{k,j}$. By the definition of $\Upsilon_{k,j}$ and Lemma 3.4, we have for $y \in \Upsilon$,

$$\left| S_{n_k}\Phi\left(\vartheta^{N(k,j)}(\widetilde{\omega}), \sigma^{N(k,j)}y\right) - n_k\alpha_{t_j}\right| \leq n_k\epsilon_k + V_{n_k}(\Phi) \ \text{ if } \ \vartheta^{N(k,j)}\widetilde{\omega} \in G_{t_j,k}. \tag{3.16}$$

Let $\alpha = (\alpha_1 + \alpha_2)/2$. We show $E_{\widetilde{\omega}}(\alpha) \supset \Upsilon$. To see this, take $y \in \Upsilon$ and write y in the form (3.15). We will show below that $|S_n\Phi(\widetilde{\omega}, y) - n\alpha| = o(n)$.

Given $n \in \mathbb{N}$ with $n > m_1n_1 + m_2n_2$, let $k = k_n$ and $j = j_n \in [1, m_{k+1}]$ be the integers such that $N(k+1, j) < n \leq N(k+1, j+1)$, where we adopt the convention $N(k+1, m_{k+1}+1) := N(k+2, 1)$. We have

$$\begin{aligned}
|S_n\Phi(\widetilde{\omega}, y) - n\alpha| \leq & \left|S_{N(k,1)}\Phi(\widetilde{\omega}, y) - N(k,1)\alpha\right| \\
& + \left|\left(\sum_{p=1}^{m_k} S_{n_k}\Phi\left(\vartheta^{N(k,p)}\widetilde{\omega}, \sigma^{N(k,p)}y\right)\right) - m_kn_k\alpha\right| \\
& + \left|\left(\sum_{p=1}^{j} S_{n_{k+1}}\Phi\left(\vartheta^{N(k+1,p)}\widetilde{\omega}, \sigma^{N(k+1,p)}y\right)\right) - jn_{k+1}\alpha\right| \\
& + n_{k+1}(\|\Phi\|_\infty + |\alpha|) \qquad (\text{where } \|\Phi\|_\infty := \sup\nolimits_{u \in \Omega\times\Sigma} |\Phi(u)|) \\
:= & (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}).
\end{aligned}$$

By (3.11), we have $(\mathrm{I}) = O(N(k,1)) = o(N(k+1,1)) = o(n)$ and $(\mathrm{IV}) = O(n_{k+1}) = o(m_k) = o(n)$. According to (3.16) and (3.12), we have

$$\begin{aligned}
(\mathrm{II}) = & \left|\sum_{p=1}^{m_k}\left(S_{n_k}\Phi\left(\vartheta^{N(k,p)}\widetilde{\omega}, \sigma^{N(k,p)}y\right) - n_k\alpha_{t_p}\right)\right| \\
\leq & m_kn_k\epsilon_k + m_kV_{n_k}(\Phi) \\
& + 2\#\{1 \leq p \leq m_k :\ \vartheta^{N(k,p)}\widetilde{\omega} \notin G_{t_p,k}\} \cdot n_k(\|\Phi\|_\infty + |\alpha|) \\
\leq & m_kn_k\epsilon_k + m_kV_{n_k}(\Phi) \\
& + 2\sum_{i=1}^{2} \#\{1 \leq p \leq m_k :\ \vartheta^{N(k,p)}\widetilde{\omega} \notin G_{i,k}\} \cdot n_k(\|\Phi\|_\infty + |\alpha|) \\
\leq & m_kn_k\epsilon_k + m_kV_{n_k}(\Phi) \\
& + 4 \cdot 2^{-k}m_kn_k(\|\Phi\|_\infty + |\alpha|) \\
= & o(n).
\end{aligned}$$

Similarly by (3.16) and (3.13), we have

$$
\begin{aligned}
(\mathrm{III}) \leq & \left| \sum_{p=1}^{j} \left(S_{n_{k+1}} \Phi \left(\vartheta^{N(k+1,p)} \widetilde{\omega}, \sigma^{N(k+1,p)} y \right) - n_{k+1} \alpha_{t_p} \right) \right| + n_{k+1} |\alpha_2 - \alpha_1| \\
\leq & j n_{k+1} \epsilon_{k+1} + j V_{n_{k+1}}(\Phi) + n_{k+1} |\alpha_2 - \alpha_1| \\
& + 2\#\{1 \leq p \leq j : \ \vartheta^{N(k+1,p)} \widetilde{\omega} \notin G_{t_p,k+1}\} \cdot n_{k+1}(\|\Phi\|_\infty + |\alpha|) \\
\leq & j n_{k+1} \epsilon_{k+1} + j V_{n_{k+1}}(\Phi) + n_{k+1} |\alpha_2 - \alpha_1| \\
& + 2 \sum_{i=1}^{2} \#\{1 \leq p \leq j : \ \vartheta^{N(k+1,p)} \widetilde{\omega} \notin G_{i,k+1}\} \cdot n_{k+1}(\|\Phi\|_\infty + |\alpha|) \\
\leq & j n_{k+1} \epsilon_{k+1} + j V_{n_{k+1}}(\Phi) + n_{k+1} |\alpha_2 - \alpha_1| + 4 \cdot 2^{-k-1} \frac{n}{n_{k+1}} \cdot n_{k+1}(\|\Phi\|_\infty + |\alpha|) \\
= & o(n),
\end{aligned}
$$

where we have used the inequality $\#\{1 \leq p \leq j : \ \vartheta^{N(k+1,p)} \widetilde{\omega} \notin G_{i,k+1}\} \leq 2^{-k-1} \frac{n}{n_{k+1}}$. To see it, in (3.13) we take $\ell = \left[\frac{n}{n_{k+1}} \right]$ and $q \in [0, n_{k+1})$ so that $q \equiv N(k+1,1) (\mod n_{k+1})$. (Here $[a]$ denotes the integer part of a for $a \in \mathbb{R}$.)

Therefore we have shown that $|S_n \Phi(\widetilde{\omega}, y) - n\alpha| = o(n)$, i.e., $y \in E_{\widetilde{\omega}}(\alpha)$. Since y is taken from Υ arbitrarily, we have $E_{\widetilde{\omega}}(\alpha) \supset \Upsilon \neq \emptyset$. The lemma is proved. □

We remark that the proof of the above lemma involves the construction of Moran-like subsets of $E_\omega(\alpha)$ (depending on ω), which is a key technique in this paper. Indeed it will be used several times more in this section.

LEMMA 3.6. *There exists a countable subset $\widetilde{\Delta}$ of $\Delta_{\mathbb{P}}$ such that $\widetilde{\Delta}$ is dense in $\Delta_{\mathbb{P}}$ and $\mathbb{P}(\{\omega \in \Omega \colon E_\omega(\alpha) \neq \emptyset$ for all $\alpha \in \widetilde{\Delta}\}) = 1$.*

Proof. By Lemma 3.3, there exists a sequence of ergodic measures $\{\mu_i\}_{i=1}^\infty$ on $\Omega \times \Sigma$ with $\mu_i \circ \pi^{-1} = \mathbb{P}$ such that the set of all finite rational convex combinations of $\alpha_i := \int \Phi \, d\mu_i$ is dense in $\Delta_{\mathbb{P}}$.

By Birkhoff ergodic theorem, for each $i \in \mathbb{N}$ the set

$$\left\{ (\omega, y) \in \Omega \times \Sigma : \lim_{n \to \infty} \frac{1}{n} S_n \Phi(\omega, y) = \alpha_i \right\}$$

has full μ_i measure. Since $\mu_i \circ \pi^{-1} = \mathbb{P}$, we have $\mathbb{P}(\{\omega \in \Omega : \ E_\omega(\alpha_i) \neq \emptyset\}) = 1$. Hence by Lemma 3.5, we have

$$\mathbb{P}(\{\omega \in \Omega : \ E_\omega(\alpha) \neq \emptyset\}) = 1$$

for any α in the following set

$$\widetilde{\Delta} := \bigcup_{n=1}^{\infty} \left\{ \frac{b_1 + \ldots + b_{2^n}}{2^n} : \ b_1, \ldots, b_{2^n} \in \{\alpha_i : i \in \mathbb{N}\} \right\}.$$

Clearly, $\widetilde{\Delta}$ is dense in the set of all rational convex combinations of $\{\alpha_i : i \in \mathbb{N}\}$, and thus dense in $\Delta_{\mathbb{P}}$. Since $\widetilde{\Delta}$ is countable, we have $\mathbb{P}(\{\omega \in \Omega \colon E_\omega(\alpha) \neq \emptyset$ for all $\alpha \in \widetilde{\Delta}\}) = 1$. $\square$

Lemma 3.7. *There exists a measurable set $H \subset \Omega$ with $\mathbb{P}(H) = 1$ such that $\{\alpha \in \mathbb{R}^d : E_\omega(\alpha) \neq \emptyset\} \supseteq \Delta_{\mathbb{P}}$ for each $\omega \in H$.*

Proof. The lemma will be proved in a way similar to that of Lemma 3.5. Let Δ be constructed as in Lemma 3.6. For $\alpha \in \Delta$ and $k, j \in \mathbb{N}$, denote

$$A_{\alpha,k,j} := \{\omega \in \Omega : f_\omega(\alpha; n, 1/k) \geq 1 \text{ for all } n \geq j\}.$$

Let H denote the set of all points ω such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} \chi_{A_{\alpha,k,j}}(\vartheta^{ps+q}\omega) = \mathbb{P}(A_{\alpha,k,j}), \quad \forall\, \alpha \in \Delta,\ k, j, p, q \in \mathbb{N}. \tag{3.17}$$

Then $\mathbb{P}(H) = 1$ by Birkhoff ergodic theorem. Thus to prove the lemma, it suffices to show that $E_\omega(\beta) \neq \emptyset$ for all $\omega \in H$ and $\beta \in \Delta_{\mathbb{P}}$.

Fix $\beta \in \Delta_{\mathbb{P}}$. Take a sequence $\{\alpha_k\}_{k=1}^\infty \subset \Delta$ such that $\lim_{k\to\infty} \alpha_k = \beta$. Define $\epsilon_k = 1/k$ for $k \in \mathbb{N}$. By Lemma 3.6,

$$\mathbb{P}(\{\omega \in \Omega : \liminf_{n\to\infty} f_\omega(\alpha_k; n, \epsilon_k) \geq 1\}) = 1.$$

Therefore we can choose a sequence of integers $\{n_k\} \uparrow \infty$ such that for any $k \in \mathbb{N}$, the set

$$G_k := A_{\alpha_k,k,n_k} = \{\omega \in \Omega : f_\omega(\alpha_k; n, \epsilon_k) \geq 1 \text{ for } n \geq n_k\} \tag{3.18}$$

has measure $\mathbb{P}(G_k) > 1 - 2^{-k}$.

Fix $\widetilde{\omega} \in H$. By (3.17) and (3.18), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} \chi_{G_k}(\vartheta^{n_k s+q}\widetilde{\omega}) = \mathbb{P}(G_k) > 1 - 2^{-k}, \quad k, q \in \mathbb{N}. \tag{3.19}$$

We construct inductively a sequence of integers $\{m_k\}_{k=1}^\infty \uparrow \infty$ (depending on $\widetilde{\omega}$) as follows. By (3.19), we can choose an integer m_1 large enough such that $m_1 \geq 2^{n_2}$ and

$$\begin{aligned} &\frac{1}{m_1} \sum_{s=0}^{m_1-1} \chi_{G_1}(\vartheta^{n_1 s}\widetilde{\omega}) > 1 - 2^{-1}, \quad \text{and} \\ &\frac{1}{\ell} \sum_{s=0}^{\ell-1} \chi_{G_2}(\vartheta^{n_2 s+q}\widetilde{\omega}) \geq 1 - 2^{-2}, \quad \forall\, \ell \geq \frac{m_1}{n_2},\ 0 \leq q \leq n_2 - 1. \end{aligned}$$

Suppose $m_1, \ldots, m_{k-1}$ have been constructed. By (3.19) again, we choose m_k large enough such that

$$
\begin{aligned}
& m_k \geq \max\{2^{m_{k-1}}, 2^{n_{k+1}}\}, \\
& \frac{1}{m_k} \sum_{s=0}^{m_k-1} \chi_{G_k}\left(\vartheta^{n_k s + \sum_{j=1}^{k-1} n_j m_j} \widetilde{\omega}\right) > 1 - 2^{-k}, \quad \text{ and} \\
& \frac{1}{\ell} \sum_{s=0}^{\ell-1} \chi_{G_{k+1}}(\vartheta^{n_{k+1}s+q} \widetilde{\omega}) \geq 1 - 2^{-k-1}, \quad \forall\, \ell \geq \frac{m_k}{n_{k+1}},\ 0 \leq q \leq n_{k+1} - 1.
\end{aligned}
$$

In this way we obtain a sequence $\{m_k\}_{k=1}^{\infty}$. Now for any $k \in \mathbb{N}$ and $1 \leq j \leq m_k$, define $N(k,j)$ the same as in (3.14) and construct $\Upsilon_{k,j} \subset \Sigma_{n_k}$ by

$$
\Upsilon_{k,j} = \begin{cases} F_{\vartheta^{N(k,j)}\widetilde{\omega}}(\alpha_k; n_k, \epsilon_k) & \text{if } \vartheta^{N(k,j)}\widetilde{\omega} \in G_k, \\ \Sigma_{n_k} & \text{otherwise.} \end{cases}
$$

By the definition of G_k, we have $\Upsilon_{k,j} \neq \emptyset$.

Define $\Upsilon = \prod_{k=1}^{\infty} \prod_{j=1}^{m_k} \Upsilon_{k,j}$. We can show that $E_{\widetilde{\omega}}(\beta) \supset \Upsilon$ by an estimation analogous to that in the proof of Lemma 3.5. This finishes the proof. □

Proof of Proposition 3.1. It follows directly from Lemma 3.2 and Lemma 3.7. □

3.2. *A formal formula for* $\dim_H E_\omega(\alpha)$ For $\omega \in \Omega$ and $\alpha \in \Delta_{\mathbb{P}}$, we define

$$
\underline{\Lambda}_\omega(\alpha) = \lim_{\epsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log f_\omega(\alpha; n, \epsilon)
$$

and $\overline{\Lambda}_\omega(\alpha)$ by taking the upper limit, where $f_\omega(\alpha; n, \epsilon)$ is defined as in (3.7). By Proposition 3.1, we have $\underline{\Lambda}_\omega(\alpha) \geq 0$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. In this subsection, we prove the following two propositions.

Proposition 3.8. *There is a function $\Lambda : \Delta_{\mathbb{P}} \to [0, \infty)$ such that for any $\alpha \in \Delta_{\mathbb{P}}$, we have $\underline{\Lambda}_\omega(\alpha) = \overline{\Lambda}_\omega(\alpha) = \Lambda(\alpha)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Furthermore the function Λ is concave and upper semi-continuous on $\Delta_{\mathbb{P}}$.*

Proposition 3.9. *There exists a measurable set $H \subset \Omega$ with $\mathbb{P}(H) = 1$ such that*

$$
\dim_H E_\omega(\alpha) = \frac{1}{\log m} \Lambda(\alpha), \quad \forall\, \omega \in H,\ \alpha \in \Delta_{\mathbb{P}}.
$$

The proposition 3.8 just follows from the following three lemmas.

Lemma 3.10. *There are two functions $\underline{\Lambda}, \overline{\Lambda}$ from $\Delta_{\mathbb{P}}$ to $[0, \infty)$ such that for any $\alpha \in \Delta_{\mathbb{P}}$,*

$$
\underline{\Lambda}_\omega(\alpha) = \underline{\Lambda}(\alpha) \quad \textit{and} \quad \overline{\Lambda}_\omega(\alpha) = \overline{\Lambda}(\alpha) \quad \textit{for} \ \ \mathbb{P}\textit{-a.e.} \ \ \omega \in \Omega.
$$

Proof. Let $\omega \in \Omega$, $\alpha \in \Delta_{\mathbb{P}}$ and $\epsilon > 0$. Let n be an integer larger than $(\|\Phi\|_\infty + |\alpha|)/\epsilon$, where $\|\Phi\|_\infty = \sup_{(\omega,y)} |\Phi(\omega, y)|$. Suppose that $|S_{n-1}\Phi(\vartheta\omega, y) - (n-1)\alpha| < (n-1)\epsilon$ for some $y \in \Sigma$. Then for $z \in \sigma^{-1}(y)$, we have

$$|S_n\Phi(\omega, z) - n\alpha| \le |S_{n-1}\phi(\vartheta\omega, y) - (n-1)\alpha| + \|\Phi\|_\infty + |\alpha| < 2n\epsilon.$$

It follows that $f_\omega(\alpha; n, 2\epsilon) \ge f_{\vartheta\omega}(\alpha; n-1, \epsilon)$. Letting $n \to \infty$ and $\epsilon \to 0$, we have

$$\underline{\Lambda}_\omega(\alpha) \ge \underline{\Lambda}_{\vartheta\omega}(\alpha), \quad \overline{\Lambda}_\omega(\alpha) \ge \overline{\Lambda}_{\vartheta\omega}(\alpha).$$

This combining with Birkhoff ergodic theorem yields the desired result. $\square$

LEMMA 3.11. *The functions $\underline{\Lambda}$, $\overline{\Lambda}$ in the above lemma coincide on $\Delta_{\mathbb{P}}$.*

Proof. We only need to show that $\underline{\Lambda} \ge \overline{\Lambda}$ on $\Delta_{\mathbb{P}}$. Fix $\alpha \in \Delta_{\mathbb{P}}$ and $\gamma > 0$, we show below that $\underline{\Lambda}(\alpha) \ge \overline{\Lambda}(\alpha) - 2\gamma$.

Without loss of generality we assume $\overline{\Lambda}(\alpha) - 2\gamma > 0$. Let $\epsilon > 0$. Take $\delta > 0$ such that

$$(1 - 2\delta)(\overline{\Lambda}(\alpha) - \gamma) \ge \overline{\Lambda}(\alpha) - 2\gamma \quad \text{and} \quad \delta(\|\Phi\|_\infty + |\alpha|) \le \epsilon. \tag{3.20}$$

By Lemma 3.4, there exists $N_0 \in \mathbb{N}$ such that $V_n(\Phi) \le n\epsilon$ for all $n \ge N_0$. Since $\overline{\Lambda}_\omega(\alpha) = \overline{\Lambda}(\alpha)$ for $\mathbb{P}$-a.e. $\omega \in \Omega$, there exist two integers N_1, N_2 with $N_2 > N_1 > N_0$ such that the set

$$A := \left\{\omega \in \Omega : \text{ there is } \ell \in [N_1, N_2] \text{ such that } f_\omega(\alpha; \ell, \epsilon) \ge e^{\ell(\overline{\Lambda}(\alpha) - \gamma)}\right\}$$

has measure $\mathbb{P}(A) > 1 - \delta$. Now denote

$$H := \left\{\omega \in \Omega : \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(\vartheta^i\omega) = \mathbb{P}(A)\right\}.$$

Then $\mathbb{P}(H) = 1$ by Birkhoff ergodic theorem.

Fix $\widetilde{\omega} \in H$. We construct a sequence of integers $\{\ell_k\}_{k=1}^\infty$ inductively as follows. First we determine ℓ_1. If $\widetilde{\omega} \notin A$, we let $\ell_1 = 1$; otherwise we let ℓ_1 be an integer between N_1 and N_2 such that $f_{\widetilde{\omega}}(\alpha; \ell_1, \epsilon) \ge e^{\ell_1(\overline{\Lambda}(\alpha) - \gamma)}$. Now, suppose $\ell_1, \ldots, \ell_{k-1}$ have been constructed for some $k \ge 2$. We determine ℓ_k in the following way: if $\vartheta^{\ell_1 + \ldots + \ell_{k-1}}\widetilde{\omega} \notin A$, we let $\ell_k = 1$; otherwise, we let ℓ_k be an integer between N_1 and N_2 such that

$$f_{\vartheta^{\ell_1+\ldots+\ell_{k-1}}\widetilde{\omega}}(\alpha; \ell_k, \epsilon) \ge e^{\ell_k(\overline{\Lambda}(\alpha) - \gamma)}.$$

In this way, we can construct the sequence $\{\ell_k\}_{k=1}^\infty$ well.

For $j \in \mathbb{N}$, we define $\Omega_j \subset \Sigma_{\ell_j}$ by

$$\Omega_j = \begin{cases} \Sigma_1 & \text{if } \ell_j = 1, \\ F_{\vartheta^{\ell_1+\ldots+\ell_{j-1}}\widetilde{\omega}}(\alpha; \ell_j, \epsilon) & \text{otherwise.} \end{cases}$$

Since $\widetilde{\omega} \in H$ and $\mathbb{P}(A) > 1-\delta$, there exists $N_3 \in \mathbb{N}$ such that $\frac{1}{n}\sum_{i=0}^{n-1}\chi_A(\vartheta^i\widetilde{\omega}) > 1-\delta$ for $n \geq N_3$.

Now fix n such that

$$n \geq \max\{N_3,\ N_2(\|\Phi\|_\infty + |\alpha|)/\epsilon\}. \tag{3.21}$$

Let $k = k_n$ be the unique integer such that

$$\ell_1 + \ldots + \ell_k \leq n < \ell_1 + \ldots + \ell_{k+1}.$$

Then

$$\begin{aligned}\#\{1 \leq j \leq k :\ \ell_j = 1\} &= \#\{1 \leq j \leq k :\ \vartheta^{\ell_1+\ldots+\ell_{j-1}}\widetilde{\omega} \notin A\} \\ &\leq \#\{0 \leq i \leq n-1 :\ \vartheta^i\widetilde{\omega} \notin A\} \\ &\leq \delta n.\end{aligned} \tag{3.22}$$

Hence

$$\begin{aligned}\sum_{1\leq j\leq k,\ \ell_j>1} \ell_j &\geq n - \ell_{k+1} - \sum_{1\leq j\leq k,\ \ell_j=1} 1 \\ &\geq n - N_2 - n\delta = n(1-\delta) - N_2.\end{aligned} \tag{3.23}$$

Now let us estimate $f_{\widetilde{\omega}}(\alpha; n, 4\epsilon)$. To do this, consider $\prod_{j=1}^k \Omega_j$. For each $I \in \prod_{j=1}^k \Omega_j$ and $y \in [I]$, by the definition of Ω_j, we have

$$\begin{aligned}&\left|S_{\ell_j}\Phi(\vartheta^{\ell_1+\ldots+\ell_{j-1}}\widetilde{\omega}, \sigma^{\ell_1+\ldots+\ell_{j-1}}y) - \ell_j\alpha\right| \\ \leq\ &\ell_j\epsilon + V_{\ell_j}(\phi) \leq 2\ell_j\epsilon \quad \text{whenever } 1 \leq j \leq k \text{ and } \ell_j > 1.\end{aligned}$$

Hence

$$\begin{aligned}|S_n\Phi(\omega, y) - n\alpha| \ &\leq \sum_{1\leq j\leq k,\ \ell_j>1}\left|S_{\ell_j}\Phi(\vartheta^{\ell_1+\ldots+\ell_{j-1}}\widetilde{\omega}, \sigma^{\ell_1+\ldots+\ell_{j-1}}y) - \ell_j\alpha\right| \\ &\quad + \sum_{1\leq j\leq k,\ \ell_j=1}\left|S_{\ell_j}\Phi(\vartheta^{\ell_1+\ldots+\ell_{j-1}}\widetilde{\omega}, \sigma^{\ell_1+\ldots+\ell_{j-1}}y) - \ell_j\alpha\right| \\ &\quad + \left|S_{n-\ell_1-\ldots-\ell_k}\Phi(\vartheta^{\ell_1+\ldots+\ell_k}\widetilde{\omega}, \sigma^{\ell_1+\ldots+\ell_k}y) - (n - \ell_1 - \ldots - \ell_k)\alpha\right| \\ &\leq \left(\sum_{1\leq j\leq k,\ \ell_j>1} 2\ell_j\epsilon\right) + \left(\#\{1 \leq j \leq k,\ \ell_j = 1\} + N_2\right)\left(\|\Phi\|_\infty + |\alpha|\right) \\ &\leq 2n\epsilon + (n\delta + N_2)(\|\Phi\|_\infty + |\alpha|) \leq 4n\epsilon. \qquad \text{(by (3.22), (3.20), (3.21))}\end{aligned}$$

It follows that

$$\begin{aligned}f_{\widetilde{\omega}}(\alpha; n, 4\epsilon) \ &\geq \ \#\prod_{j=1}^k \Omega_j \geq \prod_{1\leq j\leq k,\ \ell_j>1} e^{\ell_j(\overline{\Lambda}(\alpha)-\gamma)} \\ &\geq \ e^{(n(1-\delta)-N_2)(\overline{\Lambda}(\alpha)-\gamma)} \qquad \text{(by (3.23))} \\ &\geq \ e^{-N_2(\overline{\Lambda}(\alpha)-\gamma)} \cdot e^{n(\overline{\Lambda}(\alpha)-2\gamma)}. \qquad \text{(by (3.20))}\end{aligned}$$

Letting $n \to \infty$, we have $\liminf_{n\to\infty} \frac{1}{n} \log f_{\widetilde{\omega}}(\alpha; n, 4\epsilon) \geq \overline{\Lambda}(\alpha) - 2\gamma$.

Since $\widetilde{\omega}$ is taken from H arbitrarily and $\mathbb{P}(H) = 1$, we have $\liminf_{n\to\infty} \frac{1}{n} \log f_{\omega}(\alpha; n, 4\epsilon) \geq \overline{\Lambda}(\alpha) - 2\gamma$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. Meanwhile since ϵ can be taken arbitrarily small, we have $\underline{\Lambda}_\omega(\alpha) \geq \overline{\Lambda}(\alpha) - 2\gamma$ for $\mathbb{P}$-a.e. $\omega \in \Omega$. By Lemma 3.10, we have $\underline{\Lambda}(\alpha) \geq \overline{\Lambda}(\alpha) - 2\gamma$, as desired. $\square$

LEMMA 3.12. *Let Λ denote the common functions $\overline{\Lambda}$, $\underline{\Lambda}$. Then Λ is upper semi-continuous and concave on $\Delta_{\mathbb{P}}$.*

Proof. It suffices to show that Λ is upper semi-continuous on $\Delta_{\mathbb{P}}$ and

$$\Lambda\left(\frac{\alpha+\beta}{2}\right) \geq \frac{1}{2}\Lambda(\alpha) + \frac{1}{2}\Lambda(\beta) \quad \text{for any } \alpha,\ \beta \in \Delta_{\mathbb{P}}. \tag{3.24}$$

First we show the upper semi-continuity of Λ. Let $\alpha \in \Delta_{\mathbb{P}}$ and $\gamma > 0$. By Proposition 3.1 and Lemma 3.10, there exist $k \in \mathbb{N}$ and a measurable set $A \subset \Omega$ with $\mathbb{P}(A) > 0$ such that

$$0 \leq \liminf_{n\to\infty} \frac{1}{n} \log f_\omega(\alpha; n, 1/k) < \Lambda(\alpha) + \gamma, \quad \forall\, \omega \in A. \tag{3.25}$$

Now assume $\alpha' \in \Delta_{\mathbb{P}}$ is such that $|\alpha' - \alpha| < 1/(2k)$. Then we have

$$f_\omega(\alpha'; n, 1/(2k)) \leq f_\omega(\alpha; n, 1/k), \qquad \forall\, \omega \in A$$

(since $|S_n\Phi(\omega, y) - n\alpha'| \leq n/(2k)$ implies $|S_n\Phi(\omega, y) - n\alpha| \leq n/k$). This fact, combining with Proposition 3.1 and (3.25), yields

$$0 \leq \liminf_{n\to\infty} \frac{1}{n} \log f_\omega(\alpha'; n, 1/(2k)) < \Lambda(\alpha) + \gamma, \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in A.$$

Therefore we have $\underline{\Lambda}_\omega(\alpha') \leq \Lambda(\alpha) + \gamma$ for $\mathbb{P}$-a.e. $\omega \in A$. Since $\mathbb{P}(A) > 0$, by Lemma 3.10 we have $\Lambda(\alpha') = \underline{\Lambda}(\alpha') \leq \Lambda(\alpha) + \gamma$. This proves the upper semi-continuity of Λ.

To show (3.24), let $\alpha, \beta \in \Delta_{\mathbb{P}}$. We show that for any $\epsilon, \gamma > 0$,

$$\liminf_{n\to\infty} \frac{1}{n} \log f_\omega\left(\frac{\alpha+\beta}{2}; n, 4\epsilon\right) \geq \frac{1}{2}\Lambda(\alpha) + \frac{1}{2}\Lambda(\beta) - 2\gamma \tag{3.26}$$

for $\mathbb{P}$-a.e. $\omega \in \Omega$, from which (3.24) follows. To see (3.26), fix $\epsilon, \gamma > 0$. Without loss of generality, we assume $\frac{1}{2}\Lambda(\alpha) + \frac{1}{2}\Lambda(\beta) - 2\gamma > 0$. Choose $\delta > 0$ such that

$$\delta(\|\Phi\|_\infty + |\alpha| + |\beta|) \leq \epsilon \quad \text{and} \quad (1-\delta)(\Lambda(\alpha) + \Lambda(\beta) - 2\gamma) \geq (\Lambda(\alpha) + \Lambda(\beta) - 4\gamma).$$

Choose a sufficiently large integer ℓ such that $V_\ell(\Phi) < \epsilon\ell$ and the set

$$A = \left\{\omega \in \Omega :\ f_\omega(\alpha; \ell, \epsilon) > e^{\ell(\Lambda(\alpha)-\gamma)},\ f_\omega(\beta; \ell, \epsilon) > e^{\ell(\Lambda(\beta)-\gamma)}\right\}$$

has measure $\mathbb{P}(A) > 1 - \delta$. Denote

$$H := \left\{\omega \in \Omega :\ \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \chi_A(\vartheta^{i\ell}\omega) = \mathbb{P}(A)\right\}.$$

Then $\mathbb{P}(H) = 1$ by Birkhoff ergodic theorem. Fix $\widetilde{\omega} \in H$. Construct a sequence of sets $\Omega_j \subset \Sigma_\ell$ by

$$\Omega_j = \begin{cases} F_{\vartheta^{(j-1)\ell}\widetilde{\omega}}(\alpha; \ell, \epsilon) & \text{if } \vartheta^{(j-1)\ell}\widetilde{\omega} \in A \text{ and } j \text{ is odd,} \\ F_{\vartheta^{(j-1)\ell}\widetilde{\omega}}(\beta; \ell, \epsilon) & \text{if } \vartheta^{(j-1)\ell}\widetilde{\omega} \in A \text{ and } j \text{ is even,} \\ \Sigma_\ell & \text{if } \vartheta^{(j-1)\ell}\widetilde{\omega} \notin A. \end{cases}$$

Let N_0 be an integer such that $\frac{1}{n}\sum_{i=0}^{n-1} \chi_A(\vartheta^{i\ell}\omega) > 1 - \delta$ for $n \geq N_0$. Now fix n such that $n \geq N_0$ and $n\epsilon \geq 2\ell(\|\Phi\|_\infty + |\alpha| + |\beta|)$. Let k be the unique integer such that $\ell k \leq n < \ell(k+1)$. Then a direct estimate (similar to that in the proof of Lemma 3.11) shows that $|S_n\Phi(\widetilde{\omega}, y) - n(\alpha+\beta)/2| \leq 4n\epsilon$ for any $I \in \prod_{j=1}^k \Omega_j$ and $y \in [I]$. It follows that

$$f_{\widetilde{\omega}}((\alpha+\beta)/2; n, 4\epsilon) \geq \#\prod_{j=1}^{k} \Omega_j \geq e^{n((\Lambda(\alpha)+\Lambda(\beta))/2 - 2\gamma)},$$

where the last inequality also follows from an argument similar to that used in the proof of Lemma 3.11. Letting $n \to \infty$, we obtain (3.26) for all $\widetilde{\omega} \in H$. This completes the proof. □

Now we turn to the proof of Proposition 3.9. First we prove a lemma.

LEMMA 3.13. *There is a countable subset Δ_1 of $\Delta_{\mathbb{P}}$ such that for each $\alpha \in \Delta_{\mathbb{P}}$, there exists $\{\alpha_i\}_{i=1}^\infty \subset \Delta_1$ with $\lim_{i\to\infty} \alpha_i = \alpha$ and $\lim_{i\to\infty} \Lambda(\alpha_i) = \Lambda(\alpha)$.*

Proof. The result follows from the upper semi-continuity of Λ on $\Delta_{\mathbb{P}}$. For each $k \in \mathbb{N}$, we can cover $\Delta_{\mathbb{P}}$ by a finite family of closed balls $\{B(z_{k,i}, 1/k)\}_{i=1}^{\ell_k}$ with centers in $\Delta_{\mathbb{P}}$ and radii $1/k$. For each ball $B(z_{k,i}, 1/k)$, by the upper semi-continuity of Λ, we can choose $\alpha_{k,i} \in B(z_{k,i}, 1/k) \cap \Delta_{\mathbb{P}}$ such that $\Lambda(\alpha_{k,i}) = \sup\{\Lambda(z) : \ z \in B(z_{k,i}, 1/k) \cap \Delta_{\mathbb{P}}\}$. Now define

$$\Delta_1 = \{\alpha_{k,i} : \ k \in \mathbb{N}, \ 1 \leq i \leq \ell_k\}.$$

Then Δ_1 satisfies the desired property. To see this, let $\alpha \in \Delta_{\mathbb{P}}$. For each $k \in \mathbb{N}$, pick an integer n_k with $1 \leq n_k \leq \ell_k$ such that $\alpha \in B(z_{k,n_k}, 1/k)$. This implies $|\alpha - \alpha_{k,n_k}| < 2/k$ and hence $\lim_{k\to\infty} \alpha_{k,n_k} = \alpha$. Meanwhile, $\Lambda(\alpha) \leq \Lambda(\alpha_{k,n_k})$. However by the upper semi-continuity, we have $\Lambda(\alpha) \geq \limsup_{k\to\infty} \Lambda(\alpha_{k,n_k})$. This forces $\Lambda(\alpha) = \lim_{k\to\infty} \Lambda(\alpha_{k,n_k})$. □

In our proof of Proposition 3.9, we need to estimate the Hausdorff dimension of a class of Moran-like subsets in symbolic spaces. Let Σ be endowed with a metric d as in (2.1). Let $\{\ell_n\}_{n=1}^\infty$ be a sequence of positive integers. For each $n \in \mathbb{N}$, suppose Υ_n is a non-empty subset of Σ_{ℓ_n}. Denote $\Upsilon = \prod_{n=1}^\infty \Upsilon_n \subset \Sigma$. The following is a special version of a general theorem in [**27**]. The reader is referred to [**21**] for a short proof.

PROPOSITION 3.14. ([**27**])

$$\dim_H \Upsilon \geq \frac{1}{\log m} \liminf_{n\to\infty} \frac{\sum_{i=1}^{n} \log \#\Upsilon_i}{\sum_{j=1}^{n+1} \ell_i},$$

where $\#S$ denotes the cardinality of S.

Proof of Proposition 3.9. By the definition of $\overline{\Lambda}_\omega(\alpha)$, we have $\overline{\dim}_B E_\omega(\alpha) \leq \frac{1}{\log m}\overline{\Lambda}_\omega(\alpha)$ whenever $E_\omega(\alpha) \neq \emptyset$, where $\overline{\dim}_B$ denotes the upper box-counting dimension (see [**19**] for the definition). Hence we have $\dim_H E_\omega(\alpha) \leq \frac{1}{\log m}\overline{\Lambda}_\omega(\alpha)$ whenever $E_\omega(\alpha) \neq \emptyset$. According to Proposition 3.1 and Proposition 3.8, we have for $\mathbb{P}$-a.e. $\omega \in \Omega$, $\dim_H E_\omega(\alpha) \leq \frac{1}{\log m}\Lambda(\alpha)$ for all $\alpha \in \Delta_{\mathbb{P}}$. In the following we show that the reverse inequality also holds.

The proof we give below is similar to the proof of Lemma 3.7. Let Δ_1 be a countable subset of $\Delta_{\mathbb{P}}$ given as in Lemma 3.13. For $\alpha \in \Delta_1$ and $k, j \in \mathbb{N}$, denote

$$A_{\alpha,k,j} := \left\{\omega \in \Omega :\ f_\omega(\alpha;\ n, 1/k) \geq e^{n(\Lambda(\alpha)-1/k)} \text{ for all } n \geq j\right\}.$$

Let H denote the set of all points ω such that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} \chi_{A_{\alpha,k,j}}(\vartheta^{ps+q}\omega) = \mathbb{P}(A_{\alpha,k,j}), \quad \forall\ \alpha \in \Delta_1,\ k, j, p, q \in \mathbb{N}. \tag{3.27}$$

Then $\mathbb{P}(H) = 1$ by Birkhoff ergodic theorem. We will show that $\dim_H E_\omega(\beta) \geq \frac{1}{\log m}\Lambda(\beta)$ for all $\omega \in H$ and $\beta \in \Delta_{\mathbb{P}}$.

Fix $\beta \in \Delta_{\mathbb{P}}$. By Lemma 3.13, there is a sequence $\{\alpha_k\}_{k=1}^{\infty} \subset \Delta_1$ such that $\alpha_k \to \beta$ and $\Lambda(\alpha_k) \to \Lambda(\beta)$, as $k \to \infty$. Let $\epsilon_k = 1/k$ for $k \in \mathbb{N}$. By Proposition 3.8,

$$\mathbb{P}\left(\left\{\omega \in \Omega :\ \liminf_{n\to\infty} \frac{1}{n} \log f_\omega(\alpha_k; n, \epsilon_k) \geq \Lambda(\alpha_k)\right\}\right) = 1.$$

Therefore we can choose a sequence of integers $\{n_k\} \uparrow \infty$ such that for any $k \in \mathbb{N}$, the set

$$G_k := A_{\alpha_k,k,n_k} = \left\{\omega \in \Omega :\ f_\omega(\alpha_k; n, \epsilon_k) \geq e^{n(\Lambda(\alpha_k)-\epsilon_k)} \text{ for } n \geq n_k\right\} \tag{3.28}$$

has measure $\mathbb{P}(G_k) > 1 - 2^{-k}$.

Fix $\widetilde{\omega} \in H$. By (3.27) and (3.28), we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{s=0}^{n-1} \chi_{G_k}(\vartheta^{n_k s+q}\widetilde{\omega}) = \mathbb{P}(G_k) > 1 - 2^{-k}, \qquad \forall\ k, q \in \mathbb{N}. \tag{3.29}$$

We construct inductively a sequence of integers $\{m_k\}_{k=1}^{\infty} \uparrow \infty$ (depending on $\widetilde{\omega}$) as follows. By (3.29), we can choose an integer m_1 large enough such that $m_1 \geq 2^{n_2}$

and

$$\frac{1}{m_1}\sum_{s=0}^{m_1-1}\chi_{G_1}(\vartheta^{n_1 s}\widetilde{\omega}) > 1-2^{-1}, \quad \text{and}$$
$$\frac{1}{\ell}\sum_{s=0}^{\ell-1}\chi_{G_2}(\vartheta^{n_2 s+q}\widetilde{\omega}) \ge 1-2^{-2}, \quad \forall\, \ell \ge \frac{m_1}{n_2},\ 0\le q\le n_2-1.$$

Suppose $m_1,\ldots,m_{k-1}$ have been constructed. By (3.29) again, we choose m_k large enough such that

$$m_k \ge \max\{2^{m_{k-1}}, 2^{n_{k+1}}\},$$
$$\frac{1}{m_k}\sum_{s=0}^{m_k-1}\chi_{G_k}(\vartheta^{n_k s+\sum_{j=1}^{k-1} n_j m_j}\widetilde{\omega}) > 1-2^{-k}, \quad \text{and}$$
$$\frac{1}{\ell}\sum_{s=0}^{\ell-1}\chi_{G_{k+1}}(\vartheta^{n_{k+1}s+q}\widetilde{\omega}) \ge 1-2^{-k}, \quad \forall\, \ell \ge \frac{m_k}{n_{k+1}},\ 0\le q\le n_{k+1}-1.$$

In this way we obtain a sequence $\{m_k\}_{k=1}^{\infty}$. Now for any $k\in\mathbb{N}$ and $1\le j\le m_k$, define $N(k,j)$ the same as in (3.14) and construct $\Upsilon_{k,j}\subset\Sigma_{n_k}$ by

$$\Upsilon_{k,j} = \begin{cases} F_{\vartheta^{N(k,j)}\widetilde{\omega}}(\alpha_k; n_k, \epsilon_k) & \text{if } \vartheta^{N(k,j)}\widetilde{\omega}\in G_k, \\ \Sigma_{n_k} & \text{otherwise.}\end{cases}$$

By the definition of G_k, we have $\#\Upsilon_{k,j}\ge e^{n_k(\Lambda(\alpha_k)-\epsilon_k)}$.

Define $\Upsilon = \prod_{k=1}^{\infty}\prod_{j=1}^{m_k}\Upsilon_{k,j}$. We can show that $E_{\widetilde{\omega}}(\beta)\supset\Upsilon$ by an estimation analogous to that in the proof of Lemma 3.5. Now relabel the sequence

$$\Upsilon_{1,1},\ldots,\Upsilon_{1,m_1},\Upsilon_{2,1},\ldots,\Upsilon_{2,m_2},\ldots$$

as $\{\widetilde{\Upsilon}_n\}_{n=1}^{\infty}$, and relabel the sequence

$$\underbrace{n_1,\ldots,n_1}_{m_1},\underbrace{n_2,\ldots,n_2}_{m_2},\ldots$$

as $\{\ell_n\}_{n=1}^{\infty}$. It is obvious that $\Upsilon=\prod_{n=1}^{\infty}\widetilde{\Upsilon}_n$. Note that $\Upsilon_{k,j}$ is just relabeled as $\widetilde{\Upsilon}_{m_1+\ldots+m_{k-1}+j}$. By Proposition 3.14, we have

$$\dim_H\Upsilon \ge \frac{1}{\log m}\liminf_{n\to\infty}\frac{\sum_{i=1}^{n}\log\#\widetilde{\Upsilon}_n}{\sum_{i=1}^{n+1}\ell_i}.$$

To estimate the lower limit, for a large n we write $n=m_1+m_2+\ldots+m_k+j$ so that $1\le j\le m_{k+1}$. Then we have

$$\frac{\sum_{i=1}^{n}\log\#\widetilde{\Upsilon}_n}{\sum_{i=1}^{n+1}\ell_i} \ge \frac{\sum_{i=1}^{k} m_i n_i(\Lambda(\alpha_i)-\epsilon_i)+jm_{k+1}(\Lambda(\alpha_{k+1})-\epsilon_{k+1})}{m_1n_1+\ldots+m_kn_k+jm_{k+1}n_{k+1}+n_{k+2}}.$$

Since $n_{k+2}=o(m_{k+1})$ and $\Lambda(\alpha_k)\to\Lambda(\beta)$, by taking the lower limit we obtain $\dim_H\Upsilon\ge\frac{1}{\log m}\Lambda(\beta)$. Hence $\dim_H E_\omega(\beta)\ge\dim_H\Upsilon\ge\frac{1}{\log m}\Lambda(\beta)$. This finishes the proof. □

3.3. *A duality principle* To prove Theorem 1.3, we need a duality principle between $P_{\mathbb{P}}(q)$ and $h(\mu|\mathbb{P})$. In this subsection, we will present a more general duality principle, which is based on convex analysis and includes the above duality relation.

Let Y, Z be two locally convex topological vector spaces, and let Y^* and Z^* denote the dual spaces of Y and Z respectively with the weak* topology (cf. [**57**]). Let $f\colon Y \to Z$ be a continuous linear transformation. Assume that D is a compact convex subset of Y. Suppose $t\colon D \to \mathbb{R}$ is a real function such that $\sup_{y\in D} |t(y)| < \infty$ and t is affine on D, i.e., $t(py_1 + (1-p)y_2) = pt(y_1) + (1-p)t(y_2)$ for any $p \in (0,1)$ and any $y_1, y_2 \in D$. Define $w : Z^* \to \mathbb{R}$ by

$$w(z^*) = \sup\{\langle f(y), z^*\rangle + t(y)\colon y \in D\}, \quad \forall\, z^* \in Z^*.$$

Let $g\colon f(D) \to \mathbb{R}$ be defined as

$$g(z) = \sup\{t(y)\colon y \in f^{-1}(z)\}, \quad z \in f(D).$$

PROPOSITION 3.15. *Under the above setting, w is a real-valued convex function on Z^*. Furthermore*

(i) if g is upper semi-continuous at some $z_0 \in f(D)$, then

$$\inf_{z^*\in Z^*}\{w(z^*) - \langle z_0, z^*\rangle\} = g(z_0); \tag{3.30}$$

(ii) in particular, if t is upper semi-continuous on D, then (3.30) holds for all $z_0 \in f(D)$.

Proof. It is routine to verify that w is a real-valued convex function. To prove (i) and (ii), we need some result from convex analysis. Let $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$. Recall that for $u\colon Z \to \overline{\mathbb{R}}$ with $\mathrm{dom}(u) := \{z \in Z\colon u(z) < +\infty\} \neq \emptyset$, the function

$$u^*\colon Z^* \to \overline{\mathbb{R}}, \quad u^*(z^*) = \sup\{\langle z, z^*\rangle - u(z)\colon z \in Z\},$$

is called the *conjugate of* u. Similarly for $v\colon Z^* \to \overline{\mathbb{R}}$ with $\mathrm{dom}(v) \neq \emptyset$, the conjugate of v is defined as

$$v^*\colon Z \to \overline{\mathbb{R}}, \quad v^*(z) = \sup\{\langle z, z^*\rangle - v(z^*)\colon z^* \in Z^*\}.$$

It is well-known (cf. [**58**, Theorem 2.3.4]) that if $u\colon Z \to \overline{\mathbb{R}}$ is convex with non-empty domain $\mathrm{dom}(u)$ and if u is lower semi-continuous at $z_0 \in \mathrm{dom}(u)$, then

$$u^{**}(z_0) = u(z_0). \tag{3.31}$$

Now we return to the proof of (i) and (ii). From the definition of g, it is easy to see that g is a real concave function on $f(D)$. Extend g to be a concave function $\widetilde{g}$ on Z by setting $\widetilde{g} = g$ on $f(D)$ and $\widetilde{g} = -\infty$ on $Z\backslash f(D)$. It is easy to check that for any $z^* \in Z^*$,

$$w(z^*) = \sup\{\langle f(y), z^*\rangle + t(y)\colon y \in D\} = \sup\{\langle z, z^*\rangle + \widetilde{g}(z)\colon z \in Z\} = (-\widetilde{g})^*(z^*).$$

Note that $-\widetilde{g}\colon Z \to \overline{\mathbb{R}}$ is convex with $\mathrm{dom}(-\widetilde{g}) = f(D)$. Assume g is upper semi-continuous at $z_0 \in f(D)$. Then $-\widetilde{g}$ is lower semi-continuous at z_0. Hence by (3.31), $(-\widetilde{g})^{**}(z_0) = -\widetilde{g}(z_0)$. Therefore

$$\inf_{z^* \in Z^*} \{w(z^*) - \langle z_0, z^* \rangle\} = -w^*(z_0) = -(-\widetilde{g})^{**}(z_0) = -(-\widetilde{g})(z_0) = \widetilde{g}(z_0) = g(z_0).$$

This proves (i). To see (ii), it is enough to observe that g is upper semi-continuous on $f(D)$ when t is upper semi-continuous on D. □

3.4. *A variational principle for $\Lambda(\alpha)$ and the proof of Theorem 1.3* In this subsection, we first prove the following variational principle, then we provide a proof of Theorem 1.3.

THEOREM 3.16. *For any $\alpha \in \Delta_{\mathbb{P}}$, we have*

$$\Lambda(\alpha) = \inf\{P_{\mathbb{P}}(q) - \langle \alpha, q \rangle \colon q \in \mathbb{R}^d\} = \sup\{h(\mu|\mathbb{P}) :\ \mu \in \mathcal{G}(\alpha)\},$$

where $\mathcal{G}(\alpha)$ is defined as in (3.3).

The proof of the above theorem is based on some propositions.

PROPOSITION 3.17. *Let $\alpha \in \Delta_{\mathbb{P}}$. Then*

$$\inf\{P_{\mathbb{P}}(q) - \langle \alpha, q \rangle \colon q \in \mathbb{R}^d\} = \sup\{h(\mu|\mathbb{P}) :\ \mu \in \mathcal{G}(\alpha)\}.$$

Proof. It is a direct application of Proposition 3.15. Indeed in the setting of Proposition 3.15, we can take Y to be the dual of $L^1(\Omega, C(\Sigma))$ endowed with the weak* topology, and take $Z = \mathbb{R}^d$. Let D denote $\mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma)$. Then by Proposition 2.1, D is compact convex set of Y. Write $\Phi = (\phi_1, \ldots, \phi_d)$. Define $f\colon Y \to Z$ by $f(y) = (\langle \phi_1, y \rangle, \ldots, \langle \phi_d, y \rangle)$. Let $t\colon D \to \mathbb{R}$ be defined by $t(y) = h(y|\mathbb{P})$. Then f is continuous and linear, and t is affine and upper semi-continuous on D (see §2.3). Applying Propositions 2.3 and 3.15 yields the desired result. □

PROPOSITION 3.18. *Let $\alpha \in \Delta_{\mathbb{P}}$ and $\mu \in \mathcal{G}(\alpha)$. Then $\Lambda(\alpha) \geq h(\mu|\mathbb{P})$.*

Proof. We first assume that $\mu \in \mathcal{G}(\alpha)$ is ergodic. Let $\{\mu_\omega\}_{\omega \in \Omega}$ be the disintegration of μ with respect to $\mathbb{P}$. Let $\xi = \{[1], \ldots, [m]\}$ be the canonical partition of Σ. By Proposition 3.8, (2.4) and Birkhoff ergodic theorem, there exists a measurable set $A \subset \Omega$ with $\mathbb{P}(A) = 1$ such that for any $\omega \in A$, the following properties hold:

$$\begin{aligned} &\underline{\Lambda}_\omega(\alpha) = \overline{\Lambda}_\omega(\alpha) = \Lambda(\alpha); \\ &\lim_{n\to\infty} -\frac{1}{n} \log \mu_\omega(\xi^n(y)) = h(\mu|\mathbb{P}) \quad \text{ for } \mu_\omega\text{-a.e. } y; \\ &\lim_{n\to\infty} \frac{1}{n} S_n\Phi(\omega, y) = \alpha \quad \text{ for } \mu_\omega\text{-a.e. } y. \end{aligned} \tag{3.32}$$

Fix $\omega \in A$ and let $\epsilon > 0$. Then there exist $N \in \mathbb{N}$ and a Borel set $B \subset \Sigma$ with $\mu_\omega(B) \geq 1/2$ such that for all $y \in B$ and $n \geq N$, we have

$$e^{-n(h(\mu|\mathbb{P})+\epsilon)} \leq \mu_\omega(\xi^n(y)) \leq e^{-n(h(\mu|\mathbb{P})-\epsilon)}, \quad |S_n\Phi(\omega, y) - n\alpha| \leq n\epsilon. \tag{3.33}$$

Define $\mathcal{B}_n := \{I \in \Sigma_n \colon [I] \cap B \neq \emptyset\}$ for $n \geq \mathbb{N}$. Then by (3.33), we have $f_\omega(\alpha, n, \epsilon) \geq \#\mathcal{B}_n$ and

$$(\#\mathcal{B}_n) \cdot e^{-n(h(\mu|\mathbb{P})-\epsilon)} \geq \mu_\omega(B) \geq 1/2,$$

which gives $\limsup_{n\to\infty}(1/n)\log f_\omega(\alpha, n, \epsilon) \geq h(\mu|\mathbb{P}) - \epsilon$. Since $\epsilon > 0$ is arbitrary, we have by (3.32) that

$$\Lambda(\alpha) = \lim_{\epsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log f_\omega(\alpha, n, \epsilon) \geq h(\mu|\mathbb{P}).$$

Now assume that $\mu \in \mathcal{G}(\alpha)$ is not ergodic. Let $\gamma > 0$. Since $\Lambda(\cdot)$ is upper semi-continuous on $\Delta_{\mathbb{P}}$, there exists $0 < \epsilon < \gamma/2$ such that

$$\Lambda(\alpha) \geq \Lambda(\alpha') - \frac{\gamma}{2} \quad \text{whenever } \alpha' \in \Lambda \text{ and } \; |\alpha - \alpha'| \leq \epsilon.$$

By Lemma 3.3, there exists a convex combination $\widetilde{\mu} = \Sigma_{i=1}^k p_i\mu_i \in \mathcal{M}_{\mathbb{P}}(\Omega \times \Sigma)$ of some ergodic measures $\mu_1, \ldots, \mu_k$ such that

$$\left|\int \Phi \; d\widetilde{\mu} - \alpha\right| < \epsilon, \quad h(\widetilde{\mu}|\mathbb{P}) \geq h(\mu|\mathbb{P}) - \epsilon.$$

Write $\alpha_i = \int \Phi \; d\mu_i$ for $1 \leq i \leq k$ and $\alpha' = \sum_{i=1}^k p_i\alpha_i$. Then $|\alpha - \alpha'| < \epsilon$ and thus $\Lambda(\alpha) \geq \Lambda(\alpha') - \gamma/2$. By the concavity of Λ (Lemma 3.12) and the previous argument for ergodic elements of $\mathcal{G}(\alpha)$, together with the affine property of $\eta \mapsto h(\eta|\mathbb{P})$, we have

$$\Lambda(\alpha') \quad \geq \quad \sum_{i=1}^k p_i\Lambda(\alpha_i) \geq \sum_{i=1}^k p_i\left(h(\mu_i|\mathbb{P})\right) = h(\widetilde{\mu}|\mathbb{P}) \geq h(\mu|\mathbb{P}) - \epsilon.$$

It follows that $\Lambda(\alpha) \geq h(\mu|\mathbb{P}) - \epsilon - \gamma/2 \geq h(\mu|\mathbb{P}) - \gamma$. Since $\gamma > 0$ is arbitrary, we have $\Lambda(\alpha) \geq h(\mu|\mathbb{P})$, as desired. □

PROPOSITION 3.19. *Let $\alpha \in \Delta_{\mathbb{P}}$. Then $\Lambda(\alpha) \leq \inf\{P_{\mathbb{P}}(q) - \langle\alpha, q\rangle \colon q \in \mathbb{R}^d\}$.*

Proof. Let $\alpha \in \Delta_{\mathbb{P}}$ and $\omega \in \Omega$. Let $q \in \mathbb{R}^d$. By the definition of $P_{\mathbb{P}}(\langle q, \Phi\rangle)(\omega, n)$ (see §2.3), we have for any $n \in \mathbb{N}$ and $\epsilon > 0$,

$$P_{\mathbb{P}}(\langle q, \Phi\rangle)(\omega, n) \geq f_\omega(\alpha; n, \epsilon)\exp(n(\langle\alpha, q\rangle - \epsilon|q|)),$$

where $f_\omega(\alpha; n, \epsilon)$ is defined as in (3.7). It follows that

$$\limsup_{n\to\infty}\frac{1}{n}\log P_{\mathbb{P}}(\langle q, \Phi\rangle)(\omega, n) \geq \overline{\Lambda}_\omega(\alpha) + \langle\alpha, q\rangle.$$

Hence by Proposition 3.8 and (2.2), we have $P_{\mathbb{P}}(q) = P_{\mathbb{P}}(\langle q, \Phi\rangle) \geq \Lambda(\alpha) + \langle\alpha, q\rangle$. □

Proof of Theorem 3.16. It follows from Propositions 3.17, 3.18 and 3.19. □

Proof of Theorem 1.3. Part (i) follows just from Proposition 3.1, and part (ii) follows from Proposition 3.9 and Theorem 3.16. □

4. *The Multifractal analysis for disintegrations of Gibbs measures*

Throughout this section, we let (Σ, σ) be the full shift space over the alphabet $\{1, \ldots, m\}$, and let (X, T) be the full shift space over another alphabet, say, $\{1, \ldots, l\}$. Write $\Sigma_n = \{1, \ldots, m\}^n$ and $X_n = \{1, \ldots, l\}^n$ for $n \in \mathbb{N}$. Let ϕ be a real Hölder continuous function on $X \times \Sigma$ and let $\mu = \mu_\phi$ denote the Gibbs measure associated with ϕ, i.e., μ is the unique $T \times \sigma$-ergodic measure such that one can find constants $c_1 > 0$, $c_2 > 0$ and $P \in \mathbb{R}$ such that

$$c_1 \leq \frac{\mu(I_n(x) \times J_n(y))}{\exp(-Pn + S_n\phi(x, y))} \leq c_2, \quad \forall\, x \in X,\ y \in \Sigma \text{ and } n \in \mathbb{N}, \tag{4.1}$$

where $I_n(x) = \{u = (u_i)_{i=1}^\infty \in X\colon\ u_i = x_i \text{ for } i = 1, \ldots, n\}$, $J_n(y) = \{z = (z_i)_{i=1}^\infty \in \Sigma\colon\ z_i = y_i \text{ for } i = 1, \ldots, n\}$, and $S_n\phi(x, y) = \sum_{i=0}^{n-1} \phi(T^i x, \sigma^i y)$ (see [**13**] for details). The constant P involved in (4.1) is just equal to the classical topological pressure $P(T \times \sigma, \phi)$ of ϕ with respect to $T \times \sigma$, which is defined by

$$P(T \times \sigma, \phi) = \limsup_{n\to\infty} \frac{1}{n} \log \sum_{I \in X_n} \sum_{J \in \Sigma_n} \sup_{x \in [I], y \in [J]} \exp(S_n\phi(x, y)), \tag{4.2}$$

where $[I] = \{u \in X\colon\ u_1 \ldots u_n = I\}$ and $[J] = \{v \in \Sigma\colon\ v_1 \ldots v_n = J\}$ for $I \in X_n$ and $J \in \Sigma_n$.

Let $\pi : X \times \Sigma \to X$ be the projection given by $(x, y) \mapsto x$ and denote $\nu = \mu \circ \pi^{-1}$. In this section, we analyze the multifractal structure of the disintegration $\{\mu_x\}$ of μ with respect to (X, T, ν). Since μ_x is supported on $\{x\} \times \Sigma$ for each $x \in X$, we shall sometimes write $\mu_x(B)$ for $\mu_x(\{x\} \times B)$, whenever $B \subseteq \Sigma$ is Borel measurable.

Denote $\mathcal{C} = \{[I] \times [J] :\ I \in X_n, J \in \Sigma_k, n, k \in \mathbb{N}\}$.

Lemma 4.1. *There exists a Borel measurable set $H \subset X$ with $\nu(H) = 1$ such that for each point $x \in H$,*

(i) $\mu_x(A) = \lim_{n\to\infty} \frac{\mu(A \cap \pi^{-1}(I_n(x)))}{\nu(I_n(x))}$ *for any $A \in \mathcal{C}$;*

(ii) $\lim_{n\to\infty} \frac{1}{n} \log \nu(I_n(x)) = -h_\nu(T)$*, where $h_\nu(T)$ denotes the measure-theoretic entropy of ν with respect to T.*

Proof. Part (ii) just follows from Shannon-McMillan-Breiman Theorem (cf. [**56**, p. 93]), so we only need to prove (i). Since $\mathcal{C}$ is countable, it suffices to prove that for every $A \in \mathcal{C}$, we have $\mu_x(A) = \lim_{n\to\infty} \frac{\mu(A\cap\pi^{-1}(I_n(x)))}{\nu(I_n(x))}$ for ν-a.e. $x \in X$. To see this, fix $A \in \mathcal{C}$ and define $f = \chi_A$. Define $g(x) = \mu_x(A) = \int f\, d\mu_x$ for $x \in X$. Then g is $\mathcal{B}(X)$-measurable and $g(x) = E(f|\pi^{-1}\mathcal{B}(X))(x,y)$ for μ-a.e. (x,y) (see §2.2 and Proposition 2.2). Here $\mathcal{B}(X)$ denotes the Borel σ-algebra on X. Hence for any $B \subset \mathcal{B}(X)$,

$$\mu(A\cap\pi^{-1}(B)) = \int_{\pi^{-1}(B)} f\, d\mu = \int_{\pi^{-1}(B)} E(f|\pi^{-1}\mathcal{B}(X))\, d\mu = \int_B g\, d\nu.$$

In particular, taking $B = I_n(x)$ $(x \in X, n \in \mathbb{N})$ we obtain

$$\frac{\mu(A\cap\pi^{-1}(I_n(x)))}{\nu(I_n(x))} = \frac{1}{\nu(I_n(x))}\int_{I_n(x)} g\, d\nu.$$

Taking the limit on n and applying the differentiation theory of measures (which is valid on Σ) (see, *e.g.*, [**44**, Theorem 2.12]), we have

$$\lim_{n\to\infty} \frac{\mu(A\cap\pi^{-1}(I_n(x)))}{\nu(I_n(x))} = g(x) = \mu_x(A)$$

for ν-a.e. $x \in X$, as desired. □

For any two families of positive numbers $\{a_i\}_{i\in\mathcal{I}}$, $\{b_i\}_{i\in\mathcal{I}}$, we write, for brevity, $a_i \approx b_i$ to mean the existence of a constant $C > 0$ such that $C^{-1}a_i \le b_i \le Ca_i$ for each $i \in \mathcal{I}$.

LEMMA 4.2. *Let H be given as in Lemma 4.1. Then for all $x \in H$,*

(i) $\mu_x(J_n(y)) \approx \frac{\mu(I_n(x)\times J_n(y))}{\nu(I_n(x))} \approx \exp(S_n\phi(x,y) - nP(T\times\sigma,\phi))\frac{1}{\nu(I_n(x))}$ *for any $y \in \Sigma$ (the involved constants in $\approx$ are independent of n, x and y);*

(ii) For $\beta \in \mathbb{R}$,

$$\begin{aligned}&\left\{y\in\Sigma:\ \lim_{n\to\infty}\frac{\log\mu_x(I_n(y))}{\log(m^{-n})} = \beta\right\}\\ =&\left\{y\in\Sigma:\ \lim_{n\to\infty}\frac{1}{n}S_n\phi(x,y) = -\beta\log m + P(T\times\sigma,\phi) - h_\nu(T)\right\}.\end{aligned}$$

(iii) For each $q \in \mathbb{R}$, $\sum_{J\in\Sigma_n}\mu_x([J])^q \approx \exp(nP(T\times\sigma,q\phi) - nqP(T\times\sigma,\phi))\frac{\nu^{(q)}(I_n(x))}{\nu(I_n(x))^q}$, where $\nu^{(q)} = \mu^{(q)}\circ\pi^{-1}$ and $\mu^{(q)}$ denotes the Gibbs measure associated with $q\phi$ (the constant involved in $\approx$ is independent of n and x).

Proof. We remark that some variances of (i) were obtained in [**11, 18, 35**] under some more general settings. Here we provide a simple and self-contained proof. Let $x = (x_i)_{i=1}^\infty \in X$ and $y = (y_i)_{i=1}^\infty \in \Sigma$. Let $n, k \in \mathbb{N}$. According to (4.1), we have

$$\begin{aligned}&\mu([x_1\ldots x_{n+k}]\times[y_1\ldots y_{n+k}])\\ \approx&\mu([x_1\ldots x_n]\times[y_1\ldots y_n])\cdot\mu([x_{n+1}\ldots x_{n+k}]\times[y_{n+1}\ldots y_{n+k}]),\end{aligned}\tag{4.3}$$

here the involved constant in $\approx$ is independent of x, y, n and k. By (4.3), we have

$$\begin{aligned}\nu([x_1 \dots x_{n+k}]) = & \sum_{z_1 \dots z_{n+k} \in \Sigma_{n+k}} \mu([x_1 \dots x_{n+k}] \times [z_1 \dots z_{n+k}]) \\ \approx & \sum_{z_1 \dots z_{n+k} \in \Sigma_{n+k}} \mu([x_1 \dots x_n] \times [z_1 \dots z_n]) \\ & \cdot \mu([x_{n+1} \dots x_{n+k}] \times [z_{n+1} \dots z_{n+k}]) \\ \approx & \nu([x_1 \dots x_n]) \cdot \nu([x_{n+1} \dots x_{n+k}]).\end{aligned} \tag{4.4}$$

Now we turn to the proof of (i). Assume $x \in H$. By Lemma 4.1, we have

$$\mu_x(J_n(y)) = \lim_{k \to \infty} \frac{\mu(I_{n+k}(x) \times J_n(y))}{\nu(I_{n+k}(x))}. \tag{4.5}$$

However, by (4.3) and (4.4) we have

$$\begin{aligned}\mu(I_{n+k}(x) \times J_n(y)) = & \sum_{z_1 \dots z_k \in \Sigma_k} \mu([x_1 \dots x_{n+k}] \times [y_1 \dots y_n z_1 \dots z_k]) \\ \approx & \mu([x_1 \dots x_n] \times [y_1 \dots y_n]) \sum_{z_1 \dots z_k \in \Sigma_k} \mu([x_{n+1} \dots x_{n+k}] \times [z_1 \dots z_k]) \\ \approx & \mu([x_1 \dots x_n] \times [y_1 \dots y_n]) \cdot \nu([x_{n+1} \dots x_{n+k}]) \\ \approx & \mu([x_1 \dots x_n] \times [y_1 \dots y_n]) \cdot \nu([x_1 \dots x_{n+k}])/\nu([x_1 \dots x_n]),\end{aligned}$$

from which we deduce that

$$\frac{\mu(I_{n+k}(x) \times J_n(y))}{\nu(I_{n+k}(x))} \approx \frac{\mu(I_n(x) \times J_n(y))}{\nu(I_n(x))}.$$

This together with (4.5) and (4.1) yields (i). The statement (ii) just follows from (i) and the fact that $\lim_{n \to \infty} \log \nu(I_n(x))/n = -h_\nu(T)$. To see (iii), by (i) we have

$$\begin{aligned}& \sum_{J \in \Sigma_n} \mu_x([J])^q \\ \approx \quad & \nu(I_n(x))^{-q} \sum_{J \in \Sigma_n} \sup_{y \in [J]} \exp\left(q S_n \phi(x, y) - nqP(T \times \sigma, \phi)\right) \qquad (4.6) \\ \approx \quad & \nu(I_n(x))^{-q} \exp(nP(T \times \sigma, q\phi) - nqP(T \times \sigma, \phi)) \\ & \cdot \sum_{J \in \Sigma_n} \sup_{y \in [J]} \exp\left(q S_n \phi(x, y) - nP(T \times \sigma, q\phi)\right) \\ \approx \quad & \nu(I_n(x))^{-q} \exp(nP(T \times \sigma, q\phi) - nqP(T \times \sigma, \phi)) \sum_{J \in \Sigma_n} \mu^{(q)}(I_n(x) \times [J]) \\ \approx \quad & \nu(I_n(x))^{-q} \exp(nP(T \times \sigma, q\phi) - nqP(T \times \sigma, \phi)) \nu^{(q)}(I_n(x)), \qquad (4.7)\end{aligned}$$

as desired. $\square$

LEMMA 4.3. *There is a real-valued concave function τ on $\mathbb{R}$ such that for ν-a.e. $x \in X$,*

$$\tau_x(q) = \tau(q), \quad \forall\, q \in \mathbb{R}.$$

Proof. A standard argument shows that $\tau_x(q)$ is concave about q for each $x \in X$. Let H be given as in Lemma 4.1. By (4.6), $\tau_x(q) \in \mathbb{R}$ if $q \in \mathbb{R}$ and $x \in H$ (for which $\lim_{n\to\infty} \log \nu(I_n(x))/n = -h_\nu(T)$). Hence for each $x \in H$, the function τ_x is concave and continuous on $\mathbb{R}$.

Using an argument similar to the proof of (4.4), we can show that for each $q \in \mathbb{R}$, the measure $\nu^{(q)}$ has the same quasi-Bernoulli property. Hence by Kingman's sub-additive ergodic theorem,

$$\lim_{n\to\infty} \frac{1}{n} \log \nu^{(q)}(I_n(x)) = \lim_{n\to\infty} \int \frac{1}{n} \log \nu^{(q)}(I_n(\widetilde{x}))\, d\nu(\widetilde{x}), \qquad \text{for } \nu\text{-a.e. } x \in X. \tag{4.8}$$

With this, Lemma 4.1(ii) and Lemma 4.2(iii), we conclude that there is a function $\widetilde{\tau} : \mathbb{R} \to \mathbb{R}$ such that for each q, $\tau_x(q) = \widetilde{\tau}(q)$ for ν-a.e. $x \in X$.

Take a countable set Q dense in $\mathbb{R}$. Then there exists a ν-null set $A \subset X$ such that

$$\tau_x(q) = \widetilde{\tau}(q) \qquad \forall\ x \in H \cap (X \backslash A),\ q \in Q. \tag{4.9}$$

Since for each $x \in H$ the function $\tau_x(\cdot)$ is concave and continuous on $\mathbb{R}$, it is uniformly continuous on each relatively compact subset of Q. So is $\widetilde{\tau}$, by (4.9). Therefore $\widetilde{\tau} : Q \to \mathbb{R}$ has a unique continuous extension $\tau : \mathbb{R} \to \mathbb{R}$. By (4.9) again, we have $\tau_x(q) = \tau(q)$ for any $x \in H \cap (X \backslash A)$ and $q \in \mathbb{R}$. Hence $\tau = \widetilde{\tau}$. □

PROPOSITION 4.4. *Let τ be given as in the above lemma. For any $q \in \mathbb{R}$, we have*

$$\tau(q) = -\frac{1}{\log m} \sup \left\{ q \left(h_\nu(T) - P(T \times \sigma, \phi) + \int \phi\, d\widetilde{\mu} \right) + h_{\widetilde{\mu}}(T \times \sigma) - h_\nu(T) \right\},$$

where the supremum is taken over the set of $T \times \sigma$-invariant measures $\widetilde{\mu}$ with $\widetilde{\mu} \circ \pi^{-1} = \nu$.

Proof. By (4.7), (4.8) and Lemma 4.3, we have

$$\tau(q) = -\frac{1}{\log m} \left(q h_\nu(T) - qP(T \times \sigma, \phi) + P(T \times \sigma, q\phi) + \lim_{n\to\infty} \frac{1}{n} \log \nu^{(q)}(I_n(x)) \right) \tag{4.10}$$

for ν-a.e. $x \in X$. However,

$$\nu^{(q)}(I_n(x)) = \sum_{J \in \Sigma_n} \mu^{(q)}(I_n(x) \times [J]) \approx \exp(-nP(T \times \sigma, q\phi)) \cdot \sum_{J \in \Sigma_n} \sup_{y \in [J]} \exp(qS_n\phi(x,y)).$$

It follows that

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log \nu^{(q)}(I_n(x)) &= -P(T \times \sigma, q\phi) + \limsup_{n\to\infty} \frac{1}{n} \log \left(\sum_{J \in \Sigma_n} \sup_{y \in J} \exp(qS_n\phi(x,y)) \right) \\ &=: -P(T \times \sigma, q\phi) + \limsup_{n\to\infty} \log P_\nu(q\phi)(x,n). \end{aligned}$$

Note that $\limsup_{n\to\infty} \log P_\nu(q\phi)(x,n) = P_\nu(q\phi)$ for ν-a.e. x (see (2.2)), and $h_{\widetilde{\mu}}(T\times\sigma) - h_\nu(T) = h(\widetilde{\mu}|\nu)$ by the Abramov-Rohlin formula. This together with (4.10) yields

$$\tau(q) = -\frac{1}{\log m}\left(qh_\nu(T) - qP(T\times\sigma,\phi) + P_\nu(q\phi)\right). \tag{4.11}$$

Combining it with Proposition 2.3, we obtain the desired formula. □

Combining Proposition 4.4 with Proposition 3.15, we obtain the following result.

COROLLARY 4.5. *Write $\beta_{\min} = \lim_{q\to\infty}\tau(q)/q$ and $\beta_{\max} = \lim_{q\to-\infty}\tau(q)/q$. Let $D := \mathcal{M}_\nu(X\times\Sigma)$ be the space of all $T\times\sigma$-invariant measures $\widetilde{\mu}$ on $X\times\Sigma$ satisfying $\widetilde{\mu}\circ\pi^{-1} = \nu$, endowed with the weak* topology. Then*

(i) $\beta_{\min} = \frac{1}{\log m}\left(P(T\times\sigma,\phi) - h_\nu(T) - \max_{\widetilde{\mu}\in D}\int\phi\, d\widetilde{\mu}\right)$ *and* $\beta_{\max} = \frac{1}{\log m}\left(P(T\times\sigma,\phi) - h_\nu(T) - \min_{\widetilde{\mu}\in D}\int\phi\, d\widetilde{\mu}\right)$.

(ii) For any $\beta\in[\beta_{\min},\beta_{\max}]$,

$$\inf_{q\in\mathbb{R}}\{\beta q - \tau(q)\} = \frac{1}{\log m}\sup\left\{h_{\widetilde{\mu}}(T\times\sigma) - h_\nu(T)\right\},$$

where the supremum is taken over the set of $\widetilde{\mu}\in D$ satisfying $\int\phi\, d\widetilde{\mu} = P(T\times\sigma,\phi) - h_\nu(T) - \beta\log m$.

Proof. It is direct to derive (i) from Proposition 4.4, using the boundedness of $h_{\widetilde{\mu}}(T\times\sigma) - h_\nu(T)$. To see (ii), let $\beta\in[\beta_{\min},\beta_{\max}]$ and denote $\alpha = P(T\times\sigma,\phi) - h_\nu(T) - \beta\log m$. Then by (i), we have $\alpha\in\{\int\phi\, d\widetilde{\mu}:\ \widetilde{\mu}\in D\}$. Moreover by (4.11), we have

$$(\log m)\inf_{q\in\mathbb{R}}\{\beta q - \tau(q)\} = \inf_{q\in\mathbb{R}}\{P_\nu(q\phi) - \alpha q\}.$$

However by Theorem 3.16,

$$\inf_{q\in\mathbb{R}}\{P_\nu(q\phi) - \alpha q\} = \sup\{h(\widetilde{\mu}|\nu):\ \widetilde{\mu}\in D \text{ and } \int\phi\, d\widetilde{\mu} = \alpha\}.$$

This completes the proof of (ii). □

Proof of Theorem 1.1. Assertion (i) follows from Lemma 4.3 and (4.11). In the following we prove assertion (ii).

By Lemma 4.2(ii), there exists a Borel set $H\subseteq X$ with $\nu(X) = 1$ such that

$$E_{\mu_x}(\beta)\neq\emptyset \iff E_x(-\beta\log m + P(T\times\sigma,\phi) - h_\nu(T))\neq\emptyset, \qquad \forall\, x\in H,\ \beta\in\mathbb{R}.$$

Let $\Gamma\subseteq X$ be given as in Theorem 1.2. Then

$$E_{\mu_x}(\beta)\neq\emptyset \iff -\beta\log m + P(T\times\sigma,\phi) - h_\nu(T)\in\Delta_{\mathbb{P}}, \qquad \forall\, x\in H\cap\Gamma,\ \beta\in\mathbb{R},$$

where $\Delta_{\mathbb{P}} = \left\{\int \phi d\widetilde{\mu} :\ \widetilde{\mu} \in D\right\}$ and $D = \mathcal{M}_\nu(X \times \Sigma)$. However by Corollary 4.5(i), we have $-\beta \log m + P(T\times\sigma, \phi) - h_\nu(T) \in \Delta_{\mathbb{P}}$ if and only if $\beta \in [\beta_{\min}, \beta_{\max}]$. Hence we obtain the result

$$E_{\mu_x}(\beta) \neq \emptyset \Longleftrightarrow \beta \in [\beta_{\min}, \beta_{\max}] \qquad \forall\, x \in H \cap \Gamma,\ \beta \in \mathbb{R}.$$

By Lemma 4.2(ii), Theorem 1.2 and Corollary 4.5(ii), we have for $x \in H \cap \Gamma$, $\beta \in [\beta_{\min}, \beta_{\max}]$,

$$\dim_H E_{\mu_x}(\beta) = \dim_H E_x(-\beta \log m + P(T \times \sigma, \phi) - h_\nu(T)) = \inf_{q\in\mathbb{R}} \{\beta q - \tau(q)\}.$$

This finishes the proof of Theorem 1.1. $\square$

5. *Geometric realizations and some remarks*

In Theorem 1.1, when $\mu = \mu_\phi$ is a product measure on $X \times \Sigma$ (correspondingly, $\phi(x, y)$ only depends on the first coordinates of (x, y)), the disintegration $\{\mu_x\}$ has a simple form and the corresponding function τ can be determined explicitly. To be more precisely, let $\mathbf{p} = (p_{ij})_{1\le i\le l, 1\le j\le m}$ be a probability vector and let $\mu = \mathbf{p}^{\mathbb{N}}$ be the product measure on $X \times \Sigma$. Denote by $\mathbf{a} = (a_i)_{1\le i\le l}$, where $a_i = \sum_{j=1}^m p_{ij}$. Then $\nu = \mu\circ\pi^{-1}$ is just the product measure $\mathbf{a}^{\mathbb{N}}$ on X. By Lemma 4.1(i), for ν-a.e. $x = (x_i)_{i=1}^\infty \in X$,

$$\mu_x(J_k(y)) = \lim_{n\to\infty} \frac{\mu(I_n(x) \times J_k(y))}{\nu(I_n(x))} = \prod_{i=1}^k p_{x_i y_i}/a_{x_i}$$

for all $y = (y_i)_{i=1}^\infty \in \Sigma$ and $k \in \mathbb{N}$. By the definition of $\tau_x(q)$ and using Birkhoff Ergodic theorem, we obtain the following explicit formula

$$\tau_x(q) = -\frac{1}{\log m} \sum_{i=1}^l \left(a_i \log \left(\sum_{j=1}^m p_{ij}^q \right) - q a_i \log a_i \right) \qquad \text{for } \nu\text{-a.e. } x \in X.$$

We remark that the multifractal formalism also holds for the disintegrations of Gibbs measures on a class of self-affine sets in the plane. Fix numbers $0 < a_{ij}, b_i < 1$ and $0 < c_{ij}, d_i < 1$, for $j = 1, \ldots, m_i$ and $i = 1, \ldots, l$ such that the rectangles $Q_{ij} := [d_i, d_i + b_i] \times [c_{ij}, c_{ij} + a_{ij}]$ are pairwise disjoint subsets of $[0, 1]^2$. Set

$$D = \{(i, j) :\ 1 \le j \le m_i \text{ and } 1 \le i \le l\}.$$

Denote $T_{ij}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} b_i & 0 \\ 0 & a_{ij} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} d_i \\ c_{ij} \end{pmatrix}$, so that T_{ij} maps the unique square onto Q_{ij}. These maps are contractions, so by Hutchinson [**33**] there exists a unique compact set K satisfying

$$K = \bigcup_{(i,j)\in D} T_{ij}(K).$$

The Hausdorff dimension of K was determined by Lalley and Gatzouras [**41**] under an additional assumption $a_{ij} < b_i$ (see [**3**] for an extension). Define $R : D^{\mathbb{N}} \to K$ by

$$R((i_k, j_k)_{k=1}^{\infty}) = \begin{pmatrix} d_{i_1} \\ c_{i_1 j_1} \end{pmatrix} + \sum_{k=1}^{\infty} \begin{pmatrix} b_{i_1} & 0 \\ 0 & a_{i_1 j_1} \end{pmatrix} \cdots \begin{pmatrix} b_{i_k} & 0 \\ 0 & a_{i_k j_k} \end{pmatrix} \begin{pmatrix} d_{i_{k+1}} \\ c_{i_{k+1} j_{k+1}} \end{pmatrix}.$$

The map R is clearly one-to-one. Let μ_ϕ be the Gibbs measure on $D^{\mathbb{N}}$ corresponding to a Hölder continuous function ϕ. Set $\widetilde{\mu} = \mu_\phi \circ R^{-1}$. Then $\widetilde{\mu}$ is a planar measure supported on K. Denote $\widetilde{\nu} = \widetilde{\mu} \circ \pi^{-1}$, where π is the projection defined by $\pi(x, y) = x$. Let $\{\widetilde{\mu}_x\}_{x\in\mathbb{R}} = \{\widetilde{\mu}_{\pi^{-1}(x)}\}_{x\in\mathbb{R}}$ be the disintegration of $\widetilde{\mu}$. Then $\widetilde{\mu}_x$ satisfies the multifractal formalism for $\widetilde{\nu}$-a.e. $x \in \mathbb{R}$. Whenever n_i and a_{ij} are independent of i, j, this result follows directly from Theorem 1.1. In the general case, we may prove the result by taking a suitable modification in our proof of Theorem 1.1. We just omit the details for brevity. For example, let $\mathbf{p} = (p_{ij})_{1\le i\le l, 1\le j\le m_i}$ be a probability vector and let $\mathbf{a} = (a_i)_{1\le i\le l}$, where $a_i = \sum_{j=1}^{m_i} p_{ij}$. Let $\mu_\phi = \mathbf{p}^{\mathbb{N}}$ be the product measure on $D^{\mathbb{N}}$. Then $\widetilde{\mu} = \mu_\phi \circ R^{-1}$ is a self-affine measure on K corresponding to $\mathbf{p}$. In this case, $\tau_x(q)$ satisfies

$$\sum_{i=1}^{l} (a_i \log \sum_{j=1}^{m_i} p_{ij}^q a_{ij}^{-\tau_x(q)} - q a_i \log a_i) = 0 \qquad \text{for } \nu\text{-a.e. } x \in X.$$

We can say something about the structure of the irregular points. Take Theorem 1.3 for example. For $\alpha, \beta \in \Delta_{\mathbb{P}}$, define

$$E_w(\alpha, \beta) = \left\{ \liminf_{n\to\infty} \frac{1}{n} S_n\Phi(w, y) = \alpha,\ \limsup_{n\to\infty} \frac{1}{n} S_n\Phi(w, y) = \beta \right\}.$$

We can show for $\mathbb{P}$ a.e. w,

$$\dim_H E_w(\alpha, \beta) = \min\{\dim E_w(\alpha), \dim E_w(\beta)\},\ \forall\ \alpha,\ \beta \in \Delta_{\mathbb{P}}. \tag{5.1}$$

We give a sketch of the idea for the proof. For $\alpha \in \Delta_{\mathbb{P}}$, define

$$\underline{E}_w(\alpha) = \left\{ y :\ \liminf_{n\to\infty} \frac{1}{n} S_n\Phi(w, y) = \alpha \right\}, \tag{5.2}$$

$$\overline{E}_w(\alpha) = \left\{ y :\ \limsup_{n\to\infty} \frac{1}{n} S_n\Phi(w, y) = \alpha \right\}, \tag{5.3}$$

$$\widetilde{E}_w(\alpha) = \left\{ y :\ \exists\ n_i \uparrow \infty \text{ s.t. } \lim_{i\to\infty} \frac{1}{n_i} S_{n_i}\Phi(w, y) = \alpha \right\}. \tag{5.4}$$

Let Γ be as in Theorem 1.3. By standard Box-counting arguments, using the cylinder covers from $F_w(\alpha; n, \epsilon)$, we obtain

$$\dim_H \widetilde{E}_w(\alpha) \le \Lambda_w(\alpha).$$

Hence we have for $w \in \Gamma$,

$$\dim_H \underline{E}_w(\alpha) = \dim_H \overline{E}_w(\alpha) = \dim_H \widetilde{E}_w(\alpha) = \dim_H E_w(\alpha),$$

from which the " $\leq$ " direction of (5.1) follows immediately. For the other direction, we simply let $\alpha_{2k} = \alpha$, $\alpha_{2k+1} = \beta$ for $k \in \mathbb{N}$ in the proof of Proposition 3.9. Follow the construction there, we can obtain a Moran set $\Upsilon \subset E_w(\alpha, \beta)$ which satisfies

$$\dim_H(\Upsilon) \geq \min \{\dim_H E_w(\alpha),\ \dim_H E_w(\beta)\} .$$

Thus (5.1) holds. Note that from (5.1), we can deduce that for $\mathbb{P}$ a.e. w, the set of divergence points

$$D_w(\Phi) = \left\{ y :\ \liminf_{n\to\infty} \frac{1}{n} S_n\Phi(w, y) \neq \limsup_{n\to\infty} \frac{1}{n} S_n\Phi(w, y) \right\}$$

is either empty or has the same Hausdorff dimension as Σ (cf. [**26, 9**] for the proof in the deterministic case).

We point out that in Theorems 1.2-1.3, Φ may be relaxed to be any uniformly bounded equi-continuous Banach-valued functions. We refer the reader to [**23**] for the corresponding statement and discussions in this aspect in the deterministic case.

Furthermore, in Theorems 1.1-1.3, (Σ, σ) can be relaxed to be a subshift satisfying the specification condition (see [**34**] for the definition). More generally, our method is valid to study the topological entropy (in the sense of Bowen [**12**]) of random level sets corresponding to compact dynamical systems satisfying the specification condition and to set up a random version of the result in [**55**]. In some spirit, our work on the disintegration of measures is related to the multifractal analysis of random statistical self-similar measures (see, e.g., [**1, 4, 28, 46**]) and multiplicative martingale measures (see, e.g., [**5**]). Our approach in this paper may provide some new insights for the possible improvement of the results for those topics.

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