PROJECTIONS OF PLANAR MANDELBROT RANDOM MEASURES

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ABSTRACT. Let μ be a planar Mandelbrot measure and $\pi_*\mu$ its orthogonal projection on one of the principal axes. We study the thermodynamic and geometric properties of $\pi_*\mu$. We first show that $\pi_*\mu$ is exact dimensional, with $\dim(\pi_*\mu) = \min(\dim(\mu), \dim(\nu))$, where ν is the Bernoulli product measure obtained as the expectation of $\pi_*\mu$. We also prove that $\pi_*\mu$ is absolutely continuous with respect to ν if and only if $\dim(\mu) > \dim(\nu)$. Our results provides a new proof of Dekking-Grimmett-Falconer formula for the Hausdorff and box dimension of the topological support of $\pi_*\mu$, as well as a new variational interpretation. We obtain the free energy function $\tau_{\pi_*\mu}$ of $\pi_*\mu$ on a wide subinterval $[0, q_c)$ of $\mathbb{R}_+$. For $q \in [0, 1]$, it is given by a variational formula which sometimes yields phase transitions of order larger than 1. For $q > 1$, it is given by $\min(\tau_\nu, \tau_\mu)$, which can exhibit first order phase transitions. This is in contrast with the analyticity of τ_μ over $[0, q_c)$. Also, we prove the validity of the multifractal formalism for $\pi_*\mu$ at each $\alpha \in (\tau'_{\pi_*\mu}(q_c-), \tau'_{\pi_*\mu}(0+)]$.

CONTENTS

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1. Introduction

Mandelbrot measures are statistically self-similar measures introduced in early seventies by B. Mandelbrot in [41] as a simplified model for energy dissipation in intermittent turbulence. In $\mathbb{R}^2$, such a non-trivial random measure μ is built on $[0,1]^2$ and is characterized by $\mathbb{E}(\mu([0,1]^2)) = 1$ and the equality in law

$$\mu = \sum_{0 \le i,j \le m-1} W_{i,j}\, \mu^{(i,j)} \circ S_{i,j}^{-1}, \tag{1.1}$$

where m is an integer ≥ 2, $S_{i,j}$ are similarity maps on $\mathbb{R}^2$ defined by

$$S_{i,j}(x,y) = \left(\frac{x+i}{m}, \frac{y+j}{m} \right),$$

$W_{i,j}$ are non-negative random variables satisfying

$$\mathbb{E}\left(\sum_{0 \le i,j \le m-1} W_{i,j} \right) = 1$$

and

$$D := -\mathbb{E}\left(\sum_{0 \le i,j \le m-1} W_{i,j} \log_m(W_{i,j}) \right) > 0,$$

$\mu^{(i,j)}$ are independent copies of μ, which are also independent of the weights $W_{i,j}$. Moreover, μ and $(W_{i,j}, \mu^{(i,j)})_{0 \le i,j \le m-1}$ can be constructed on the same probability space so that (1.1) holds not only in law but also almost surely.

The topological support of μ, denoted by K, is a statistically self-similar limit set so that the following equality holds in law:

$$K = \bigcup_{\substack{0 \le i,j \le m-1 \\ W_{i,j} > 0}} S_{i,j}(K_{i,j}),$$

where $K_{i,j}$ are independent copies of K.

The fine geometric properties of μ were initially studied by Mandelbrot himself in [41, 40], as well as by Kahane and Peyrière in [36]. It was established that μ is exact D-dimensional, i.e. the local dimension of μ equals D on a set of full μ-measure. Moreover, a statistical description of the mass distribution of μ at small scales was given by Mandelbrot by using large deviation properties of the branching random walk naturally associated with μ.

On the other hand, the topological and measure theoretic properties of K and the natural branching measure it carries have been studied intensively [42, 51, 12, 18, 23, 31, 19, 26, 9, 39, 44, 54, 20, 46, 52, 53, 27, 50].

Mandelbrot measures, as well as self-similar measures and Gibbs measures, are typical objects illustrating the multifractal formalism, which emerged in the middle of the eighties from turbulence theory [30] and hyperbolic dynamical systems [32, 15], in order to describe geometrically at small scales the distribution of a measure, or the Hölder singularities of a function; this formalism can be viewed as a geometric counterpart of large deviation theory. For measures, it can be defined as follows.

If (X,d) is a locally compact metric space and μ is a positive and finite compactly supported measure, denoting its topological support as $\mathrm{supp}(\mu)$, the L^q-spectrum of μ is a kind of free energy concave function defined by

$$\tau_\mu : q \in \mathbb{R} \mapsto \liminf_{r\to 0+} \frac{\log \sup \left\{ \sum_i \mu(B(x_i,r))^q \right\}}{\log(r)},$$

where the supremum is taken over all the centered packings of $\mathrm{supp}(\mu)$ by closed balls of radius r. When (X,d) possesses the Besicovitch property, i.e. the Bescovitch covering lemma holds in (X,d) (see e.g. [43]), like Euclidean $\mathbb{R}^d$ or any symbolic space endowed with the standard metric, for $\alpha \in \mathbb{R}$, it is always the case that (see e.g. [11, 49, 37])

$$\dim_H E(\mu,\alpha) \le \tau_\mu^*(\alpha) := \inf\{\alpha q - \tau_\mu(q) : q \in \mathbb{R}\},$$

where

$$E(\mu,\alpha) = \left\{ x \in \mathrm{supp}(\mu) : \lim_{r\to 0+} \frac{\log(\mu(B(x,r)))}{\log(r)} = \alpha \right\},$$

here $\dim_H$ stands for the Hausdorff dimension, and we adopt the convention that

$$\dim_H \emptyset = -\infty.$$

We say that *the multifractal formalism holds for μ at α* if $\dim_H E(\mu,\alpha) = \tau_\mu^*(\alpha)$, and we say that *it holds for μ* if this equality holds for all α, i.e. the Hausdorff spectrum $\alpha \mapsto \dim_H E(\mu,\alpha)$ and τ_μ form a Legendre pair. Furthermore, we say that there is *k-th order phase transition* at q for μ if τ_μ has a $(k-1)$-th order derivative but no k-th order derivative at q.

In this paper we will investigate the multifractal structure of the orthogonal projections of a Mandelbrot measure μ on the horizontal and vertical axes, and its relation with that of μ. For this purpose, we recall that under mild assumptions, defining for $q \in \mathbb{R}$

$$T(q) = -\log_m \sum_{0\le i,j\le m-1} \mathbb{E}(\mathbf{1}_{\{W_{i,j}>0\}} W_{i,j}^q),$$

then on the interval $\{q \in \mathbb{R} : T^*(T'(q)) \ge 0\}$, $\tau_\mu = T$ and hence τ_μ is analytic (see Section 2.2).

In our study of projections of μ, we will consider the range $q \ge 0$ for the L^q-spectrum. This restriction is often met in the geometric study of measures obtained via projection schemes, like self-similar measures obtained as projections of Bernoulli products on self-similar sets satisfying the weak separation condition (see e.g. [29] and the references therein) or self-affine measures obtained as projections of Bernoulli products on almost all the attractors associated with a given finite collection of contractive linear maps [25, 7].

For a line ℓ in $\mathbb{R}^2$ passing through the origin, we let π_ℓ denote the orthogonal projection from $\mathbb{R}^2$ to ℓ, and let $\pi_{\ell *}\mu$ denote the push-forward of μ under π_ℓ. For almost every line ℓ, the behavior of the L^q-spectrum $\tau_{\pi_{\ell *}\mu}(q)$ of $\pi_{\ell *}\mu$ is essentially similar to that of the projections of Gibbs measures treated in [6]. In this case, due to Marstrand's projection theorem, one is naturally led to consider the case where $D \leq 1$, for otherwise the projection of μ is absolutely continuous with respect to Lebesgue measure and it is hard to say more about the multifractality in general. Then, since $D \leq 1$, the dimension of the projection is still D, and there are two possible behaviors in terms of the L^q-spectrum. If $\dim_H K \leq 1$ as well, then $\tau_{\pi_{\ell *}\mu} = \tau_\mu$ on the interval $[0, q_2]$, where q_2 is defined by $\tau_\mu(q_2) = 2$ (notice that $q_2 \geq 3$ due to the concavity of τ_μ and the facts that $\tau_\mu(1) = 0$ and $\tau_\mu(0) \geq -\dim_H \mathrm{supp}(\mu) = -1$ in this case). If $\dim_H K > 1$, there is a second order phase transition at the unique $\widetilde{q} \in [0, 1]$ at which $\tau_\mu^*(\tau_\mu'(\widetilde{q})) = 1$; more precisely, the L^q-spectrum $\tau_{\pi_{\ell *}\mu}$ is analytic over $(0, \widetilde{q})$ and $(\widetilde{q}, q_2)$ but not twice differentiable at $\widetilde{q}$; specifically, it is linear on $[0, \widetilde{q}]$ and equals τ_μ on $[\widetilde{q}, q_2]$. Also, the multifractal formalism is valid at any $\alpha \in \tau'_{\pi_{\ell *}\mu}([0, q_2])$. It is worth mentioning that the preservation of the L^q-spectrum over $[1, q_2]$ is a fact valid for any measure (see [34, 3]).

The situation is significantly different with the principal axes. To begin with, it is worth noticing that for a Gibbs measure associated with a Hölder potential on the unit square, e.g. for the self-similar measures obtained when the weights $W_{i,j}$ are constant, its projection on any of the main directions is still a Gibbs measure of this kind [13], so no special new phenomenon appears related to its multifractal nature. Things turn out to be more interesting with (random) Mandelbrot measures.

Let π denote the orthogonal projection on the first principal axis. It is known (Dekking and Grimmett [18], Falconer [23]), that $\dim_H \pi(K)$ in general differs from the typical value obtained by Marstrand's projection theorem when one projects on almost every line. Instead of being equal to $\min(\dim_H K, 1)$, $\dim_H \pi(K)$ is given by the following variational formula:

$$\dim_H \pi(K) = \inf_{0 \leq h \leq 1} \log_m \sum_{i=0}^{m-1} \mathbb{E}(N_i)^h, \tag{1.2}$$

where $N_i = \#\{0 \leq j \leq m-1 : W_{i,j} > 0\}$. Moreover, this dimension equals the box counting dimension of $\pi(K)$. It turns out that understanding the geometric structure of the projection $\pi_*\mu$ of the Mandelbrot measure μ heavily relies on its expectation, which is the Bernoulli product measure ν associated with the probability vector $\Big(p_i = \sum_{j=0}^{m-1} \mathbb{E}(W_{i,j})\Big)_{0 \leq i \leq m-1}$, for which it is known that

$$\tau_\nu(q) = -\log_m \sum_{\substack{0 \leq i \leq m-1 \\ p_i > 0}} p_i^q.$$

In this paper, we show (see Theorems 3.1, 3.3 and 10.2) that when $\mu \neq 0$, $\pi_*\mu$ is exact dimensional with $\dim(\pi_*\mu) = \dim(\mu) = D$ if and only if $\dim(\mu) \leq \dim(\nu)$, in which case $\pi_*\mu$ is singular with respect to ν, while if $\dim(\mu) > \dim(\nu)$ then $\pi_*\mu$ is absolutely continuous with respect to ν. Exact dimensionality and "dimension conservation properties" of projections of Mandelbrot measures on *all* the lines have already been established in [27];

however, the result of [27] is not quantitative, whilst for the principal axes we provide the precise values for the dimensions, which differ from those given by Marstrand's theorem for almost every line when ν is not the Lebesgue measure. Also, as a consequence of Theorem 3.3 we get a new variational interpretation of Dekking-Grimmett-Falconer formula for $\dim_H \pi(K)$ (see Corollary 3.5 and Theorem 10.2).

Regarding the multifractal analysis (see Theorems 3.7 and 10.3), for $q \geq 1$ we prove that

$$\tau_{\pi_*\mu}(q) = \min(\tau_\mu(q), \tau_\nu(q))$$

on a non-trivial interval $[1, \widetilde{q}_c)$. This fact is a source of first order phase transitions when the graphs of τ_ν and T cross each other transversally. For $0 < q \leq 1$, we prove that $\tau_{\pi_*\nu}$ is given by the following variational formula:

$$\tau_{\pi_*\mu}(q) = -\inf\left\{ \log_m \sum_{i=0}^{m-1} \left(\sum_{j=0}^{m-1} \mathbb{E}(W_{i,j}^s) \right)^{q/s} : q \leq s \leq 1 \right\},$$

which converges to the value of $\dim_H \pi(K)$ given by (1.2) as q tends to 0. The function $\tau_{\pi_*\mu}$ is differentiable over $[0,1]$. It coincides with $\tau_\mu(q)$ when the infimum is attained at $s = q$ and $\tau_\nu(q)$ when it is attained at $s = 1$. Otherwise, the infimum is attained at a unique $s(q) \in (q, 1)$, and this property holds on a neighborhood of q over which by definition of $s(q)$ we have $\tau_{\pi_*\mu}(q) > \max(\tau_\mu(q), \tau_\nu(q))$; see figures in Section 4. These possible changes of analytic expressions lead to phase transitions of orders greater than or equal to 2. Also, each transversal crossing of the graphs of τ_ν and T in the domain $(1, \widetilde{q}_c)$ gives rise to a first order phase transition.

We also verify the validity of the multifractal formalism over $(\tau'_{\pi_*\mu}(\widetilde{q}_c-), \tau'_{\pi*\mu}(0+)]$. When applied to the so-called branching measure on K, our result yields a partial multifractal classification of the box-counting dimension of the fibers $\pi^{-1}(\{x\})$, $x \in \pi(K)$ (see Corollary 11.1).

Let us finally mention that Mandelbrot martingales in various Bernoulli random environments play an important role in our study.

The paper is organized as follows. We will mainly work with Mandelbrot measures on the symbolic space $\{0, \ldots, m-1\}^{\mathbb{N}} \times \{0, \ldots, m-1\}^{\mathbb{N}}$, for this offers a simpler framework to expose ideas and techniques. The transfer of the results from the symbolic space to the Euclidean plane is explained in Section 10. In Section 2 we recall basic facts from multifractal formalism, as well as the formal definition of Mandelbrot measures and a precise known result for their multifractal analysis on the symbolic space. In Section 3 we present in complete rigor our main results in this symbolic context, while Section 4 contains comments and examples related to phase transitions. Section 5 provides the proof of our results related to the dimension of the projected measures, as well as the new variational interpretation of the Hausdorff dimension of their topological support. Sections 6 to 8 provide the proof of Theorem 3.7 about the multifractal analysis of the projection. Specifically, Section 6 deals with the differentiability property of the function identified to be the L^q-spectrum of $\pi_*\mu$, Section 7 exhibits the sharp lower bound for the L^q-spectrum, and Section 8 deals both with the sharp upper bound for the L^q-spectrum and the Hausdorff spectrum. Sections 5, 7 and 8 use moments estimates developed in Section 9 for quantities related to Mandelbrot martingales in Bernoulli environments, as well as other basic results gathered in the Appendix.

2. Preliminaries on multifractal formalism and Mandelbrot measures on symbolic spaces

Throughout this paper, we use $\mathbb{N}$ to denote the set of natural numbers, i.e. $\mathbb{N} = \{1, 2, \ldots\}$. Let us first restate the multifractal formalism in this context.

2.1. **Multifractal formalism on symbolic spaces.**

Let $m \geq 2$ be an integer. For $n \geq 0$ let $\Sigma_n = \{0, \ldots, m-1\}^n$. By convention, Σ_0 consists of the empty word ϵ. Then define $\Sigma_* = \bigcup_{n\geq 0} \Sigma_n$, $(\Sigma\times\Sigma)_* = \bigcup_{n\geq 0}(\Sigma_n \times \Sigma_n)$, and $\Sigma = \{0, \ldots, m-1\}^{\mathbb{N}}$. The sets Σ_* and $(\Sigma\times\Sigma)_*$ act in the standard way by concatenation on $\Sigma_*\cup\Sigma$ and $(\Sigma\times\Sigma)_*\cup(\Sigma\times\Sigma)$ respectively. We denote by σ the standard left shift operation on $\Sigma_*\cup(\Sigma\times\Sigma)$. The length of a word $w \in \Sigma_*$, i.e. its number of letters, is denoted as $|w|$.

For $x = x_1 \cdots x_p \cdots \in \Sigma$, set $x_{|n} = x_1 \cdots x_n$ if $n \geq 1$ and ϵ if $n = 0$. For $u \in \Sigma_*$, set $[u] = \{x \in \Sigma : x_{||u|} = u\}$.

The set Σ is endowed with the standard metric distance

$$d(x, x') = m^{-\sup\left\{n:\, x_{|n}=x'_{|n}\right\}},$$

and $\Sigma\times\Sigma$ is endowed with the distance $d((x, y), (x', y')) = \max(d(x, x'), d(y, y'))$.

Given a positive and finite Borel measure ρ on Σ or $\Sigma\times\Sigma$, its topological support, i.e. the smallest closed set carrying the whole mass of μ is denoted as $\mathrm{supp}(\rho)$, and its lower and upper local dimensions at $x \in \mathrm{supp}(\rho)$ are defined as

$$\underline{\dim}_{\mathrm{loc}}(\rho, x) = \liminf_{n\to\infty} \frac{\log(\rho([x_{|n}]))}{(-n\log(m))} \quad \text{and} \quad \overline{\dim}_{\mathrm{loc}}(\rho, x) = \limsup_{n\to\infty} \frac{\log(\rho([x_{|n}]))}{(-n\log(m))}$$

respectively. Let

$$\begin{aligned}
&\underline{\dim}_H(\rho) = \inf\{\dim_H E :\ \rho(E) > 0 \text{ and } E \text{ is a Borel set}\} \text{ and}\\
&\overline{\dim}_P(\rho) = \inf\{\dim_P E : \rho(E) = \|\rho\| \text{ and } E \text{ is a Borel set}\},
\end{aligned}$$

where $\dim_P E$ stands for the packing dimension of E (see e.g. [43]) and $\|\rho\|$ stands for the total mass of ρ .

It is well known that (see e.g. [16, 17])

$$\begin{aligned}\underline{\dim}_H(\rho) &= \sup\{s: \ \underline{\dim}_{\rm loc}(\rho,x)\ge s \text{ for } \rho\text{-almost every } x\} \text{ and}\\ \overline{\dim}_P(\rho) &= \inf\{s: \ \overline{\dim}_{\rm loc}(\rho,x)\le s \text{ for } \rho\text{–almost every } x\};\end{aligned}$$

when these two dimensions coincide, we say that ρ is exact dimensional and writes $\dim(\rho)$ for the common value.

The L^q-spectrum of ρ is the mapping $\tau_\rho: \ \mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ given by

$$\tau_\rho(q)=\liminf_{n\to\infty}\frac{-1}{n}\log_m\sum_{w\in S_n}\mathbf{1}_{\{\rho([w])>0\}}\rho([w])^q \quad (q\in\mathbb{R}),$$

where S_n stands for Σ_n or $\Sigma_n\times\Sigma_n$. It is well known that (cf. [47])

$$\tau_\rho'(1+)\le\underline{\dim}_H(\rho)\le\overline{\dim}_P(\rho)\le\tau_\rho'(1-).$$

For all $\alpha\in\mathbb{R}$, set

$$\begin{aligned}\underline{E}(\rho,\alpha)&=\{x\in\operatorname{supp}(\rho):\underline{\dim}_{\rm loc}(\rho,x)=\alpha\},\\ \overline{E}(\rho,\alpha)&=\{x\in\operatorname{supp}(\mu):\overline{\dim}_{\rm loc}(\rho,x)=\alpha\}\end{aligned}$$

and

$$E(\rho,\alpha)=\underline{E}(\rho,\alpha)\cap\overline{E}(\rho,\alpha).$$

Then it is always the case that (see e.g. [37, 49])

$$\dim_H E(\rho,\alpha)\le\max(\dim_H\underline{E}(\rho,\alpha),\dim_H\overline{E}(\rho,\alpha))\le\tau_\rho^*(\alpha),$$

where the Legendre transform of $f:\mathbb{R}\to\mathbb{R}\cup\{-\infty\}$ is defined as

$$f^*:\alpha\in\mathbb{R}\mapsto\inf_{q\in\mathbb{R}}(\alpha q-f(q)),$$

and a negative dimension means that the set is empty. We say that the multifractal formalism holds at α if

$$\dim_H E(\rho,\alpha)=\tau_\rho^*(\alpha).$$

It is well-known (see e.g. [37, 49]) that for $\alpha\le\tau_\rho'(0+)$,

$$\dim_H\{x\in\operatorname{supp}(\rho):\underline{\dim}_{\rm loc}(\rho,x)\le\alpha\}\le\inf_{q\ge0}\{\alpha q-\tau_\rho(q)\}=\tau_\rho^*(\alpha). \tag{2.1}$$

2.2. **Multifractal analysis of the Mandelbrot measures on $\Sigma\times\Sigma$.** Now let us formally define the Mandelbrot measures on $\Sigma\times\Sigma$. We consider a non-negative random vector

$$W=(W_{i,j})_{(i,j)\in\Sigma_1\times\Sigma_1}$$

whose entries are integrable. For $q\in\mathbb{R}$ we define

$$T(q)=T_W(q)=-\log_m\sum_{(i,j)\in\Sigma_1\times\Sigma_1}\mathbb{E}(\mathbf{1}_{\{W_{i,j}>0\}}W_{i,j}^q). \tag{2.2}$$

Let $N=\sum_{(i,j)\in\Sigma_1\times\Sigma_1}\mathbf{1}_{\{W_{i,j}>0\}}$, and assume that $\mathbb{P}(N\in\{0,1\})<1$.

To build a Mandelbrot measure on $\Sigma\times\Sigma$ we assume that $T(1)=0$ and consider a sequence $(W(u,v))_{(u,v)\in\bigcup_{n\geq 0}\Sigma_n\times\Sigma_n}$ of independent copies of W, defined on a probability space $(\Omega,\mathcal{A},\mathbb{P})$.

Let η be the uniform measure on $\Sigma\times\Sigma$, i.e. $\eta([u,v])=m^{-2n}$ for each cylinder $[u,v]:=[u]\times[v]$ of generation n. For each $n\geq 1$ let $\mu_n=\mu_{W,n}$ be the measure on $\Sigma\times\Sigma$ whose density with respect to η is constant over each cylinder $[u,v]:=[u]\times[v]$ of generation n and given by $m^{2n}Q(u,v)$, where

$$Q(u,v)=\prod_{j=1}^{n}W_{u_j,v_j}(u_{|j-1},v_{|j-1}).$$

Denote the total mass of μ_n as Y_n, i.e.

$$Y_n=\sum_{|u|=|v|=n}Q(u,v).$$

By construction the sequence $(Y_n)_{n\geq 1}$ is a non-negative martingale of expectation 1 with respect to the filtration $(\sigma(W(u,v):|u|=|v|\leq n-1))_{n\geq 1}$, thus it converges to a limit, which we denote by Y.

Let $T_n=\{(u,v)\in\Sigma_n\times\Sigma_n : Q(u,v)>0\}$. The sequence $(T_n)_{n\geq 1}$ represents the generations of a Galton-Watson process with offspring distribution given by that of N. We have

$$K_n:=\operatorname{supp}(\mu_n)=\bigcup_{(u,v)\in T_n}[u]\times[v].$$

For $n\geq k\geq 1$ and $(u,v)\in\Sigma_k\times\Sigma_k$, the statistical self-similarity of the construction yields $\mu_n([u]\times[v])=Q(u,v)Y_{n-k}(u,v)$, with $(Y_{n-k}(u,v))_{(u,v)\in\Sigma_k\times\Sigma_k}$ a family of independent copies of Y_{n-k}, also independent of $\sigma(W(u,v):|u|=|v|\leq k-1)$.

Consequently, with probability 1, there exists a family $(Y(u,v))_{(u,v)\in\Sigma_k\times\Sigma_k,k\geq 1)}$ of copies of Y such that for each $k\geq 1$ and $(u,v)\in\Sigma_k\times\Sigma_k$,

$$\lim_{n\to\infty}\mu_n([u]\times[v])=Q(u,v)Y(u,v). \tag{2.3}$$

Moreover, the random variables $Y(u,v)$, $(u,v)\in\Sigma_k\times\Sigma_k$, are independent, and generate a σ-field independent of $\sigma(W(u,v):|u|=|v|\leq k-1)$. By construction, this means that μ_n weakly converges to a measure μ defined by

$$\mu([u]\times[v])=Q(u,v)Y(u,v).$$

Moreover, μ is positive (i.e. $Y>0$) with positive probability if and only if $T'(1-)>0$; and this is also equivalent to the uniform integrability of $(Y_n)_{n\geq 1}$, that is $\mathbb{E}(Y)=1$ ([36, 22]). From now on we assume that this condition (i.e., $T'(1-)>0$) holds; in this case, it is known (cf. [36, 35]) that the measure μ, if non-degenerate, is exact dimensional and

$$\dim(\mu)=T'(1-)\text{ almost surely on }\{\mu\neq 0\}.$$

Also, the events $\{\mu \neq 0\}$ and $\{K := \bigcap_{n\geq 1} K_n \neq \emptyset\}$ coincide up to a set of probability 0 over which $K = \operatorname{supp}(\mu)$ (see Proposition A.1 for a proof). In addition, the inequality $T'(1-) > 0$ and the concavity of T imply that $T(0) = -\log_m(\mathbb{E}(N)) < 0$, i.e. $\mathbb{E}(N) > 1$.

We have the following result regarding the multifractal analysis of μ (see also [33, 24, 48, 45, 5] for slightly less sharp versions). Recall that f^* stands for the Legendre transform of f.

Theorem 2.1 ([1])**.** *Suppose that T is finite on a neighborhood of 0 and $N \geq 2$ conditional on $\{N \neq 0\}$. Define $f(\alpha) = T^*(\alpha)$ if $T^*(\alpha) \geq 0$ and $f(\alpha) = -\infty$ otherwise. With probability 1, conditional on $\{\mu \neq 0\}$, $\tau_\mu = f^*$ and the multifractal formalism holds at all α in the domain of $\tau_\mu^* = f$. In particular, $\tau_\mu(q) = T(q)$ at each $q \in \mathbb{R}$ such that $T^*(T'(q)) \geq 0$.*

Since we mainly want to focus on new phenomena associated with $\pi_*\mu$, to avoid too many technicalities we discard the case when

$$\sup\{q \geq 1 : T(q) > -\infty\} = \sup\{q \geq 1 : T^*(T'(q)) > 0\},$$

for which τ_μ itself exhibits a first order phase transition on $(1, \infty)$ [1].

Thus, when we study the validity of the multifractal formalism for $\pi_*\mu$, our assumptions will be:

(2.4)

- $\mathbb{P}(N \in \{0, 1\}) < 1,\ T'(1-) > 0$;
- T is finite on a neighborhood of 0;
- either $\exists\ q_c > 1$ such that $T^*(T'(q_c^-)) = 0$
  or $T^*(T'(q)) > 0$ for all $q \geq 0$, in which case we set $q_c = \infty$.

We drop the assumption that $N \geq 2$ when $N \neq 0$ because this does not affect the validity of Theorem 2.1 for the local dimensions α associated with non-negative q by Legendre duality, and for our study of $\pi_*\mu$ we will only focus on the case $q \geq 0$. The assumption that T is finite on a neighborhood of 0 implies that $\mathbb{E}(Y^{-h}) < \infty$ for some $h > 0$ (see [38, Theorem 2.4]).

3. Main results for projections of Mandelbrot measures on the symbolic space

Throughout this section we assume that $\mathbb{P}(N \in \{0, 1\}) < 1$ and $T'(1-) > 0$. We are interested in the geometric properties of the measure $\pi_*\mu$, where π stands for the canonical projection onto the first factor of $\Sigma \times \Sigma$.

For $0 \le i, j \le m-1$ set

$$p_i = \sum_{j=0}^{m-1} \mathbb{E}(W_{i,j}) \text{ and } V_{i,j} = \begin{cases} W_{i,j}/p_i & \text{if } p_i > 0, \\ 1/m & \text{otherwise.} \end{cases} \tag{3.1}$$

(In fact those i for which $p_i = 0$ will play no role in our study.) Then write $V_i := (V_{i,j})_{j \in \Sigma_1}$ and define

$$T_i(q) = T_{V_i}(q) = -\log_m \sum_{j \in \Sigma_1} \mathbb{E}(\mathbf{1}_{\{V_{i,j}>0\}} V_{i,j}^q), \qquad q \in \mathbb{R}. \tag{3.2}$$

Notice that $T_i(1) = 0$ for all $0 \le i \le m-1$.

Let ν stand for the Bernoulli product measure on Σ associated with the probability vector $(p_0, \dots, p_{m-1})$, that is

$$\nu([x_1 \dots x_n]) = p_{x_1} \dots p_{x_n}$$

for $n \ge 1$ and $x_1, \dots, x_n \in \{0, 1, \dots, m-1\}$.

By construction

$$m^{-T(q)} = \sum_{i,j} \mathbb{E}(\mathbf{1}_{\{W_{i,j}>0\}} W_{i,j}^q) = \sum_{i,j} \mathbf{1}_{\{p_i>0\}} p_i^q \mathbb{E}(V_{i,j}^q) = \sum_i \mathbf{1}_{\{p_i>0\}} p_i^q m^{-T_i(q)}. \tag{3.3}$$

Consequently,

$$T'(1-) = \sum_i p_i (T_i'(1-) - \log_m(p_i)) = \Big(\sum_i p_i T_i'(1-)\Big) + \dim(\nu), \tag{3.4}$$

where we recall that

$$\dim(\nu) = -\sum_{i=0}^{m-1} p_i \log(p_i)/\log(m).$$

Notice that $\nu = \mathbb{E}(\pi_* \mu)$, and recall that a direct calculation yields

$$\tau_\nu(q) = -\log_m \sum_{i=0}^{m-1} p_i^q \quad (q \in \mathbb{R}). \tag{3.5}$$

For $q \in \mathbb{R}$, we denote by ν_q the Bernoulli product measure on Σ associated with the probability vector $\big(p_0^q m^{\tau_\nu(q)}, \dots, p_{m-1}^q m^{\tau_\nu(q)}\big)$.

Below we discard two trivial situations.

We first discard the case when $p_i = 1$ for some $0 \le i \le m-1$, which means that the measure μ is supported on a deterministic vertical line hence is a Mandelbrot measure on a line, for which the multifractal nature is analogue to that of a 1-dimensional Mandelbrot measure. For $0 \le i \le m-1$, we set

$$N_i = \#\{0 \le j \le m-1 : W_{i,j} > 0\}.$$

We also discard the case when $N_i = 1$ almost surely for all $0 \le i \le m-1$, which implies that $\pi_* \mu$ is a Mandelbrot measure on a line as well.

3.1. **Absolute continuity and dimension.** This section gathers our results on the absolute continuity/singularity of $\pi_*\mu$ with respect to $\nu = \mathbb{E}(\pi_*\mu)$, and on the dimension of $\pi_*\mu$ and its associated conditional measures in the natural disintegration of μ along $\pi_*\mu$-almost every fiber $\{x\} \times \Sigma$. The result on $\dim(\pi_*\mu)$ also yields a new variational principle for $\dim \pi(K)$.

Theorem 3.1. *With probability 1, conditional on $\{\mu \neq 0\}$:*

(1) *If $\dim(\mu) > \dim(\nu)$, then*
   (i) *$\pi_*\mu$ is absolutely continuous with respect to ν.*
   (ii) *Suppose that T is finite in a neighborhood of 1. Then the density of $\pi_*\nu$ with respect to ν is in $L^s(\nu)$ for all s in the following non-empty set*
$$\left\{ s \in (1,2] : \ T(s) > 0 \ \text{and} \ \sum_{i=0}^{m-1} p_i m^{-T_i(s)} < 1 \right\}.$$
(2) *If $\dim(\mu) \leq \dim(\nu)$, then $\pi_*\mu$ and ν are mutually singular.*

Remark 3.2. Sufficient conditions for $\pi_*\mu$ to be equivalent to ν can be found in [8].

Theorem 3.3. *With probability 1, conditional on $\{\mu \neq 0\}$:*

(1) *If $\dim(\mu) > \dim(\nu)$ then $\pi_*\mu$ is exact dimensional with dimension $\dim(\nu)$; while if $\dim(\mu) \leq \dim(\nu)$ and T is finite in a neighborhood of 1, then $\pi_*\mu$ is exact dimensional with dimension $\dim(\mu)$.*
(2) *Suppose that T is finite in a neighborhood of 1. For $\pi_*\mu$-almost every x, the conditional measure μ^x is exact dimensional with dimension $\dim(\mu) - \dim(\pi_*\mu) = \dim(\mu) - \dim(\nu) = \sum_{i=0}^{m-1} p_i T_i'(1)$ if $\dim(\mu) > \dim(\nu)$, and dimension 0 if $\dim(\mu) \leq \dim(\nu)$.*

Remark 3.4. Recall that under the assumption that T is finite in a neighborhood of 1, in [27] Falconer and Jin have already proven that with probability 1, conditional on $\{\mu \neq 0\}$, for $\pi_*\mu$-almost every x, $\dim(\mu^x) = \dim(\mu) - \dim(\pi_*\mu)$ without specifying the value of $\dim(\pi_*\mu)$, hence of $\dim(\mu^x)$.

When $\dim(\mu) > \dim(\nu)$, a direct proof of the equality $\dim(\mu^x) = \sum_{i=0}^{m-1} p_i T_i'(1^-)$ without the additional assumption on the finiteness of T near 1 can be found in [8].

The previous statement makes it possible to derive the dimension formula of $\pi(K)$ by using an adapted Mandelbrot measure, whilst in [23] Falconer builds statistically self-similar subsets of $\pi(K)$ of Hausdorff dimension smaller than but arbitrarily close to the value given by (1.2). The new point is the variational principle invoking Mandelbrot measures in (3.7) and the related uniqueness property.

Corollary 3.5 (Dekking-Grimmett-Falconer formula revisited)**.** *Let*

$$\varphi : h \geq 0 \mapsto \log\left(\sum_{i=0}^{m-1} \mathbb{E}(N_i)^h\right) / \log(m). \tag{3.6}$$

With probability 1, conditional on $\{K \neq \emptyset\}$,

$$\begin{aligned}\dim_H \pi(K) &= \dim_B(\pi(K)) \\ &= \inf_{0\leq h\leq 1} \varphi(h) \\ &= \max\{H(\pi_*\mu') : \mu' \text{ is a Mandelbrot measure supported on } K\},\end{aligned} \tag{3.7}$$

where $H(\pi_*\mu') = \min(\dim(\mu'), \dim \mathbb{E}(\pi_*\mu'))$*. Moreover, the maximum in* (3.7) *is attained at a unique point if and only if* $\varphi'(0) \leq 0$*, i.e.* $\sum_{i=0}^{m-1} \log(\mathbb{E}(N_i)) \leq 0$*. Also,* $\dim(\pi_*\mu') = H(\pi_*\mu')$ *at some point* μ' *where the maximum is attained.*

Remark 3.6. It is readily to check that $\dim_H \pi(K) = \dim_H K$ if and only if the inequality $\sum_{i=0}^{m-1} \mathbb{E}(N_i) \log \mathbb{E}(N_i) \leq 0$ holds, i.e. in (3.7) the infimum is attained at $h = 1$.

3.2. **Validity of the multifractal formalism.** Assume (2.4) (in which q_c is well-defined) and set

$$\widetilde{q}_c = \begin{cases} q_c & \text{if } q_c < \infty \text{ and } \tau_\nu(q_c) \geq T(q_c) \\ \inf\{q > q_c : \tau_\nu(q) \geq T(q)\} & \text{otherwise} \end{cases}, \tag{3.8}$$

with the convention that $\inf \emptyset = q_c$. Let

(3.9)

$$\tau : q \mapsto \begin{cases} -\inf\left\{\log_m \sum_{i=0}^{m-1} \mathbb{E}(N_i)^h : 0 \leq h \leq 1\right\} & \text{if } q = 0, \\ -\inf\left\{\log_m \sum_{i=0}^{m-1} p_i^q m^{-qT_i(s)/s} : q \leq s \leq 1\right\} & \text{if } 0 < q \leq 1, \\ \min(\tau_\nu(q), T(q)) & \text{if } 1 < q < \widetilde{q}_c \text{ or } q = \widetilde{q}_c < \infty. \end{cases}$$

Theorem 3.7. *The function* τ *is differentiable everywhere except at the possible points in* $(1, \widetilde{q}_c)$ *at which the graphs of* T *and* τ_ν *cross each other transversally. Moreover,*

(1) *with probability 1, conditional on* $\{\mu \neq 0\}$*, for all* $q \in [0, \widetilde{q}_c)$*,*

$$\tau(q) = \lim_{n\to\infty} \frac{-1}{n} \log_m \sum_{|u|=n} \mathbf{1}_{\{\pi_*\mu([u])>0\}} \pi_*\mu([u])^q. \tag{3.10}$$

In particular $\tau_{\pi_*\mu}(q) = \tau(q)$*. Also, if* $\widetilde{q}_c = q_c < \infty$*, then* $\tau_{\pi_*\mu}(q) = qT'(q_c-)$ *for* $q > q_c$*.*

(2) *If* $\alpha \in (\tau'(\widetilde{q}_c-), \tau'(0+)]$*, with probability 1, conditional on* $\{\mu \neq 0\}$*, the multifractal formalism holds at* α*.*

Remark 3.8. Notice that when $q_c < \infty$, the equality $\tau_\nu(q_c) = T(q_c)$ cannot hold if $\tau_\nu'(q_c) \leq T'(q_c-)$, for this would imply that $\tau_\nu^*(\tau_\nu'(q_c)) \leq T^*(T'(q_c-)) = 0$, while $\tau_\nu^* \circ \tau_\nu'$ is always positive over $\mathbb{R}$.

Remark 3.9. The same conclusion as in (3.10) and Theorem 3.7(2) holds if we replace $\pi_*\mu([x_{|n}])$ by $\pi_*\mu_n([x_{|n}])$ in the definition of the level sets $E(\pi_*\mu,\alpha)$.

Remark 3.10. (1) The identities $\tau = T = \tau_\nu$ over $[0,q_c)$ hold if and only if for all $i\in\{0,\dots,m-1\}$ with $p_i>0$, $\mathbb{E}(N_i)=1$ and $W_{i,j}\in\{0,p_i\}$ almost surely for all $0\le j\le m-1$, i.e. T_i is identically equal to 0 (see Section 6.1).

(2) On the other hand, a sufficient condition to have $\tau=\tau_\nu$ over $\mathbb{R}_+$ and $\tau_\nu>T$ over $(0,1)$ and $\tau_\nu<T$ over $(1,\infty)$ is $(\mathcal{P})$: there exists a partition $\{I_1,\dots,I_L\}$ of $\{0\le i\le m-1 : p_i>0\}$ such that (a) p_i does not depend on $i\in I_k$ and $\prod_{i\in I_k}\mathbb{E}(N_i)=1$ for each $1\le k\le L$; (b) there exists $1\le k\le L$ such that $\#I_k\ge 2$ and $\mathbb{E}(N_i)\neq 1$ for at least two values of $i\in I_k$; (c) for all $i\in\{0,\dots,m-1\}$ with $p_i>0$, $W_{i,j}\in\{0,p_i/\mathbb{E}(N_i)\}$ almost surely for all $0\le j\le m-1$. See the proof of Lemma 8.6.

Remark 3.11. In all the examples we have examined numerically and for which we do not have $\tau_\nu=T$, the functions τ_ν and T coincide at three points at most. We do not know whether this is a general fact.

Remark 3.12. We think (and know that it is true on some intervals) that the validity of the multifractal formalism for $\pi_*\mu$ holds almost surely for all $\alpha\in(\tau'(\widetilde{q}_c-)),\tau'(0+)]$. However, we dediced to limit the technicalities as much as possible, the most important facts being the new behaviors associated with the projection. In particular, the proof of the validity of the multifractal formalism will show that the possible phase transitions separate the domain of possible exponents α into intervals over which the computation of the Hausdorff dimension of the sets $E(\pi_*\mu,\alpha)$ uses different arguments, this being in contrast with what happens for the measure μ itself, indeed we can use one uniform approach to deal with all the level sets $E(\mu,\alpha)$ (see [1]).

Remark 3.13 (Similar result for critical Mandelbrot measures)**.** When $T'(1-)=0$, under mild assumptions there exists a substitute to the degenerate Mandelbrot measure μ, namely a critical Mandelbrot measure $\widetilde{\mu}$, which satisfies the same statistical self-similarity (1.1) with the set K as its support, but $\mathbb{E}(\|\mu\|)=\infty$; the multifractal analysis of this measure is considered in [5]. Defining q_c like when $T'(1-)>0$, we have $q_c=1$. Furthermore, defining $\widetilde{q}_c=1$ and ν as for μ, the conclusions of Theorem 3.7 holds for $\pi_*\widetilde{\mu}$.

4. Phase transition. Remarks and examples

This section gathers a series of remarks and examples related to phase transitions associated with $\pi_*\mu$.

Remark 4.1. Let S denote the set of non-analytic points of τ in $(0,\widetilde{q}_c)$. Then S is discrete and possibly empty. Moreover, the cardinality of $S\cap(0,1]$ is not less than the number of times that the graphs of T and τ_ν cross each other transversally over $(0,1)$. These properties will be established in Section 5.

Now we give some examples to illustrate Theorem 3.7.

Example 4.2 (Lognormal canonical cascades)**.** Let us consider the standard lognormal canonical cascade, for which the weights $W_{i,j}$ are independent and $W_{i,j} \sim m^{-2}\exp(\beta N - \beta^2/2)$, where $\beta \geq 0$ and $N \sim \mathcal{N}(0,1)$. We have

$$T(q) = 2(q-1) - \frac{\beta^2}{2\log(m)} q(q-1).$$

A necessary and sufficient condition for μ to be almost surely positive is $T'(1) = 2 - \frac{\beta^2}{2\log(m)} > 0$, i.e. $\beta \in [0, 2\sqrt{\log m})$.

Fix $\beta \in (0, 2\sqrt{\log m})$ (we discard the case $\beta = 0$ which corresponds to μ being the restriction of the Lebesgue measure to $[0,1]^2$). Then, the dimension of μ equals $2 - \frac{\beta^2}{2\log(m)}$, and the measure ν is simply the Lebesgue measure restricted to $[0,1]$, so $\tau_\nu(q) = q-1$. Also, due to Theorem 3.1, the measure $\pi_*\mu$ is almost surely equivalent to Lebesgue measure if and only if $T'(1) > 1$, i.e. $\beta \in [0, \sqrt{2\log(m)})$.

We also have $T'(q)q - T(q) = 2 - \frac{\beta^2 q^2}{2\log(m)}$, so $q_c = 2\sqrt{\log(m)}/\beta$. Moreover, $T(q) = \tau_\nu(q)$ if and only if $q = 1$ or $q = q_0 := 2\log(m)/\beta^2$.

Thus, if $\beta \in [0, \sqrt{\log(m)}]$, we have $q_c \leq \widetilde{q}_c = q_0$; if $\beta \in (\sqrt{\log(m)}, \sqrt{2\log(m)})$, $q_c = \widetilde{q}_c$ and τ_ν and T cross once transversally at $q_0 \in (1, q_c)$, and do not cross over $[0,1)$; if $\beta = \sqrt{2\log(m)}$ then τ_ν and T cross at $1 = q_0$ only and $q_c = \widetilde{q}_c$; if $\beta \in (\sqrt{2\log(m)}, 2\sqrt{\log(m)})$, then T and τ_ν cross once transversally at $q_0 \in (0,1)$ and do not cross over $(1,\infty)$.

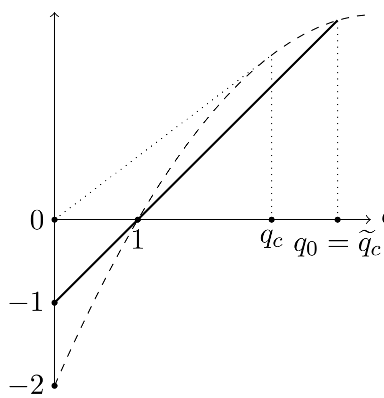
The previous observations and the definition of τ yield, with probability 1:

- if $\beta \in (0, \sqrt{\log m}]$, then $\tau(q) = q-1$ over $[0, q_0 = \widetilde{q}_c]$ (and $q_0 > 1$).
- If $\beta \in (\sqrt{\log(m)}, \sqrt{2\log(m)})$, $\tau(q) = q-1$ over $[0, q_0]$, $\tau(q) = T(q)$ over $[q_0, q_c = \widetilde{q}_c]$, and $q_0 \in (1, q_c)$.

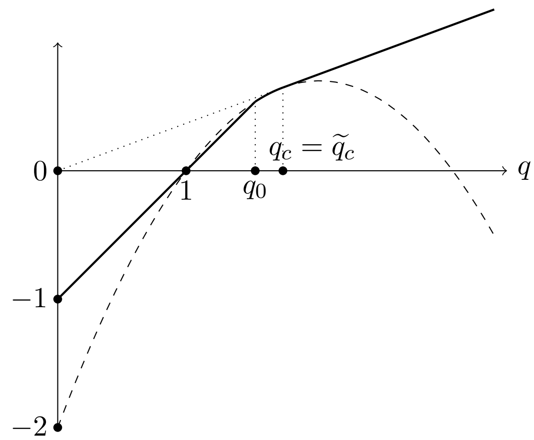
  In this case $\pi_*\mu$ provides new examples of statistically self-similar measures absolutely continuous with respect to the Lebesgue measure over $[0,1]$, with a non-trivial Hausdorff spectrum and a first order phase transition, here at q_0 (see also [28] for deterministic examples for which, however, the Hausdorff spectrum is not described at the phase transition).
- If $\beta = \sqrt{2\log(m)}$, $\tau(q) = q-1$ over $[0, 1 = q_0]$ and $\tau(q) = T(q)$ over $[1, q_c = \widetilde{q}_c]$.
- If $\beta \in (\sqrt{2\log(m)}, 2\sqrt{\log(m)})$ then $q_0 < 1$, and a calculation using the definition of τ over $[0,1]$ shows that $\tau(q) = -1 + T'(\sqrt{q_0})q$ over $[0, \sqrt{q_0}]$ and $\tau(q) = T(q)$ over $[\sqrt{q_0}, q_c = \widetilde{q}_c]$.

In the last two cases, for which $\dim(\mu) \le 1$, our result provides, for the special directions of projection considered in this paper, the same information as that given by [6] for almost every direction, and recalled in Section 1.

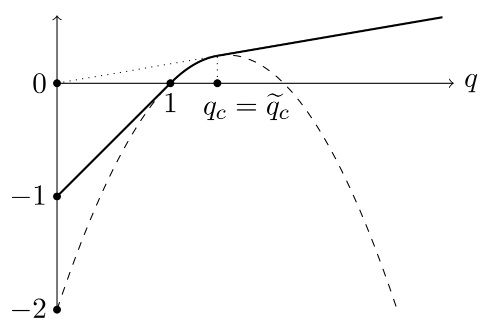
Illustrations are provided by Figure 1.



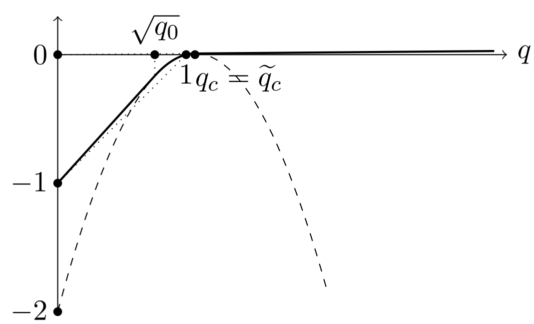
(A) $\beta \in (0, \sqrt{\log m}]$.



(B) $\beta \in (\sqrt{\log m}, \sqrt{2\log m})$. First order phase transition at q_0 and second order phase transition at q_c.



(C) $\beta = \sqrt{2\log m}$. Second order phase transitions at $q_0 = 1$ and $q_c = \sqrt{2}$.



(D) $\beta \in (\sqrt{2\log m}, 2\sqrt{\log m})$. Second order phase transitions at $\sqrt{q_0}$ and q_c.

FIGURE 1. The thick curve represents $\tau_{\pi_*\mu}$ over $[0, \widetilde{q}_c]$ in case (A) and $[0, \infty]$ in the other cases, while the dashed curve represents T.

Below we construct a concrete example so that $q_c = \infty$ and the function $\tau_{\pi_*\mu}$ has a non-differentiable point in $(1, \infty)$ (i.e. first order phase transition), and a non-C^∞ smooth point in $(0, 1)$ (i.e. phase transition of order ≥ 2). It is illustrated in Figure 2.

Example 4.3. Let $(p_0, \ldots, p_{m-1})$ be a positive probability vector different from the vector $(m^{-1}, \ldots, m^{-1})$. We have $p_{\max} = \max\{p_i : 0 \le i \le m-1\} > m^{-1}$. We assume that $p_0 = p_{\max} > \sqrt{p_1} = \ldots = \sqrt{p_{m-1}}$. Fix β in the interval $(m, mp_{\max}^{-1})$ and $\lambda \in (0, 1)$. Let $(V_{i,j})_{1 \le i \le m-1, 0 \le j \le m-1}$ be a family of random variables which take value β/m with probability $\lambda\beta^{-1}$ and $c_{m,\beta,\lambda} = \frac{\beta(1-\lambda)}{m(\beta-\lambda)}$ with probability $1-\lambda\beta^{-1}$. Let

$$V_{0,0} \in \left(\max_{1\le i\le m-1} \frac{p_i}{p_{\max}^2},\ 1 \right),$$

$V_{0,1} = 1 - V_{0,0}$, and $V_{0,j} = 0$ if $j \ge 2$; also suppose that $V_{0,1} < V_{0,0}$. Set $W_{i,j} = p_i V_{i,j}$ for all $0 \le i, j \le m-1$ and define the functions T_i and T as previously.

For all $1 \le i \le m-1$ and $0 \le j \le m-1$, by construction, $W_{i,j} \le p_i\beta/m < W_{0,0} < p_0 < 1$, and also $W_{0,1} < W_{0,0} < 1$. Consequently, $T'(1) > 0$. Also, for all $0 \le i \le m-1$, $\mathbb{E}\left(\sum_{j=0}^{m-1} V_{i,j}\right) = 1$ and for $1 \le i \le m-1$,

$$-\log(m) T_i'(1) = \mathbb{E}\left(\sum_{j=0}^{m-1} V_{i,j}\log(V_{i,j})\right) = \lambda\log\left(\frac{\beta}{m}\right) + m(1-\lambda\beta^{-1})c_{m,\beta,\lambda}\log(c_{m,\beta,\lambda})$$

and $-\log(m)T_0'(1) = V_{0,0}\log(V_{0,0}) + V_{0,1}\log(V_{0,1})$. Thus, if we take λ close enough to 1 and $V_{0,0}$ close enough to 1, then $\sum_{i=0}^{m-1} p_i T_i'(1) < 0$, so that $0 < T'(1) < \tau_\nu'(1)$, and $T < \tau$ near $1+$. Now let us make $T(q)$ explicit:

$$T(q) = -\log_m\left((p_0V_{0,0})^q + (p_0V_{0,1})^q + \sum_{i=1}^{m-1} m\lambda\beta^{-1}\left(p_i\frac{\beta}{m}\right)^q + m(1-\lambda\beta^{-1})(p_i c_{m,\beta,\lambda})^q \right).$$

Hence $T(q) = -q\log_m(p_0V_{0,0}) + o(1)$ as $q \to \infty$, with $-\log_m(p_0V_{0,0}) > -\log_m(p_0) > 0$ since $V_{0,0} < 1$. This shows that $T^* \circ T'$ does not vanish over $\mathbb{R}_+$ and $q_c = \infty$. Moreover $\tau_\nu(q) = -q\log_m(p_{\max}) + o(1)$ as $q \to \infty$, so $\tau_\nu(q) < T(q)$ near ∞. It follows that there is a first order phase transition over $(1,\infty)$.

Now let us look at the situation over $[0,1]$. Clearly $-\log_m(m(m-1)+2) = T(0) < \tau_\nu(0) = -1$, and $T'(1) < \tau_\nu'(1)$ which implies that $\tau_\nu < T$ near $1-$. Thus, the graphs of τ_ν and T cross each other on $[0,1]$, and we know from Theorem 3.7 that there is at least one phase transition of order at least 2. Let us be a little bit more precise. Set

$$G(q,s) = \sum_{i=0}^{m-1} p_i^q m^{-T_i(s)q/s}.$$

Then

$$\frac{\partial G}{\partial s}(q,s) = -s^{-2} q\log(m) \sum_{i=0}^{m-1} p_i^q m^{-T_i(s)q/s} T_i^*(T_i'(s)).$$

By construction, $\frac{\partial G}{\partial s}(1,1) = -\log(m)\sum_{i=0}^{m-1} p_i T_i'(1) > 0$.

Thus, by continuity of $\frac{\partial G}{\partial s}(q,s)$, for q near $1-$, $\frac{\partial G}{\partial s}(q,s) > 0$ for all $s \in [q,1]$, which implies that τ is attained at $s = q$ and $\tau(q) = T(q)$. Also $T_i^*(T_i'(0)) = -T_i(0) = 1 > 0$ if $i \ge 1$, $T_0^*(T_0'(0)) = -T_0(0) = \log_m(2) > 0$ and by construction $\sum_{i=0}^{m-1} T_i'(1) < 0$. Consequently, for all q near $0+$, $\frac{\partial G}{\partial s}(q,q) < 0$ and $\frac{\partial G}{\partial s}(q,1) > 0$, which implies that $\tau(q)$ is attained at some $s \in (q,1)$ and $\tau(q) > \max(\tau_\nu(q), T(q))$.

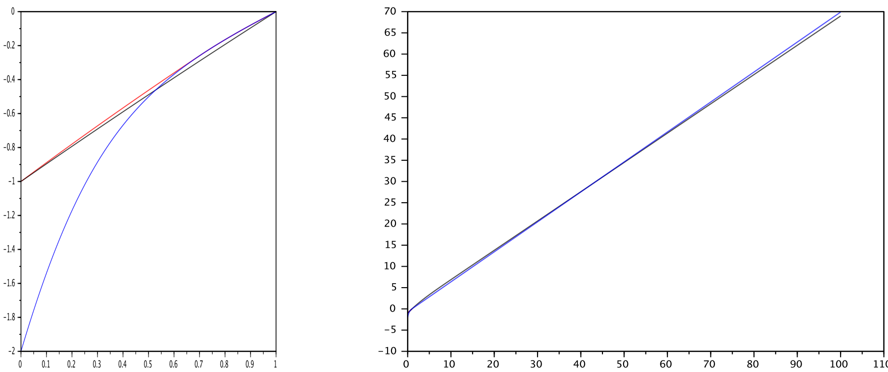


FIGURE 2. Illustration of Example 4.3 with $m = 2$, $p_0 = .62$, $\beta = 3.22$, $\lambda = .99$ and $V_{00} = .99$ (blue curve: T; black curve: τ_ν; red curve: τ). $q_c = \widetilde{q}_c = \infty$. One phase transition of order ≥ 2 at some $q_0 \in (0,1)$ and one first order phase transition at some $q_0' \in (1,\infty)$. $\tau_{\pi_*\mu} = \tau > \max(\tau_\nu, T)$ over $[0, q_0)$, $\tau_{\pi_*\mu} = T = \tau_\mu$ over $[q_0', q_0]$, and $\tau_{\pi_*\mu} = \tau_\nu$ over $[q_0, \infty)$.

Example 4.4. This example exhibits two phase transitions over $[0,1]$ and no first order phase transition over $(1, q_c)$, with $q_c < \infty$ and $\tau_{\pi_*\mu} = \tau_\mu$ over $(1,\infty)$. Take $m = 2$, $p_0 \in (0,1)$, and N_0 and N_1 two random integers taking values in $\{0,1,2\}$ and with positive expectation. Then for $0 \leq i,j \leq 1$ define $V_{i,j} = (\mathbb{E}(N_i))^{-1}\mathbf{1}_{\{j \leq N_i - 1\}}$. This yields $T_i(q) = (q-1)\log \mathbb{E}(N_i)$, hence $T_i^*(T_i'(s)) = -T_i(0) = T_i'(1) = \log \mathbb{E}(N_i)$ for all $s \geq 0$. Also,

$$T(q) = -\log_2\left(p_0^q \mathbb{E}(N_0)^{1-q} + (1-p_0)^q \mathbb{E}(N_1)^{1-q}\right).$$

We require $\mathbb{E}(N_0) < 1$,

$$\mathbb{E}(N_0)\mathbb{E}(N_1) > 1, \tag{4.1}$$

$$\mathbb{E}(N_0)^{p_0}\mathbb{E}(N_1)^{1-p_0} < 1, \tag{4.2}$$

$$\left(\frac{\mathbb{E}(N_0)}{p_0}\right)^{p_0}\left(\frac{\mathbb{E}(N_1)}{1-p_0}\right)^{1-p_0} > 1. \tag{4.3}$$

Properties (4.2) and (4.3) yield $\tau_\nu'(1) > T'(1) > 0$. Also (4.1) implies $\mathbb{E}(N_1) + \mathbb{E}(N_2) > 2$ hence $T(0) < -1 = \tau_\nu(0)$. The graphs of τ_ν and T cross each other on $[0,1]$. Let G be defined as in the previous example. Property (4.1) yields $\frac{\partial G}{\partial s}(q,s) < 0$ for all $s \in [q,1]$ if q is close enough to 0, hence $\tau(q)$ is attained at $q = 1$; $\tau(q) = \tau_\nu(q)$. Moreover, (4.2) implies that $\frac{\partial G}{\partial s}(q,s) > 0$ for all $s \in [q,1]$ if q is close to 1, hence $\tau(q)$ is attained at $s = q$: $\tau(q) = T(q)$. Then our study of τ in Section 6 implies that $\tau(q) > \max(\tau_\nu(q), T(q))$ on a non-trivial interval, i.e. τ is given by a third analytic expression.

It is also possible to choose the parameters so that $T'(1) < -\log_2(p_0) = \tau_\nu'(+\infty)$ and $p_0 > \mathbb{E}(N_0)$, hence $\tau_\nu > T$ over $(1,\infty)$ and $q_c < \infty$, which implies that $\tau_{\pi_*\mu}(q) = \tau_\mu(q)$ over $[1,\infty)$. A concrete choice is $p_0 = .8$, $\mathbb{E}(N_0) = .6$ and $\mathbb{E}(N_1) = 1.8$. See Figures 3.

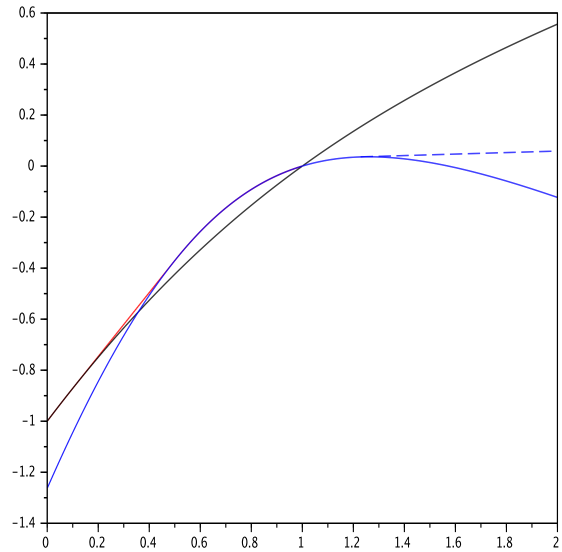


FIGURE 3. Illustration of Example 4.4 with $p_0 = .8$, $\mathbb{E}(N_0) = .6$ and $\mathbb{E}(N_1) = 1.8$ (blue curve: T; dashed blue curve: τ_μ; black curve: τ_ν; red curve: τ). $q_c = \widetilde{q}_c \simeq 1.229 < \infty$. Phase transitions of orders ≥ 2 at some $q_0 < q_0'$ in $(0,1)$, no first order phase transition over $(1, q_c)$, and one second order phase transition at q_c. $\tau_{\pi_*\mu} = \tau_\nu$ over $[0, q_0]$, $\tau_{\pi_*\mu} = \tau > \max(\tau_\nu, T)$ over (q_0, q_0'), $\tau_{\pi_*\mu} = T = \tau_\mu$ over $[q_0', q_c]$, and $\tau_{\pi_*\mu}(q) = \tau_\mu(q) = T'(q_c)q$ over $[q_c, \infty)$.

Example 4.5 (Previous example continued)**.** We can use the same model as in Example 4.4 to get other different behaviors by modifying the values of the parameters p_0, $\mathbb{E}(N_0)$ and $\mathbb{E}(N_1)$. See Figures 4 to 6.

5. Proofs of Theorem 3.1, Theorem 3.3(1), and Corollary 3.5

We first introduce the following new notation and definitions.

For each $u \in \Sigma_*$, set

$$\pi_*\mu([u]) = \sum_{v\in\Sigma_{|u|}} \mu([u,v]) = \sum_{v\in\Sigma_{|u|}} Q(u,v)Y(u,v) = \nu(u)X(u), \tag{5.1}$$

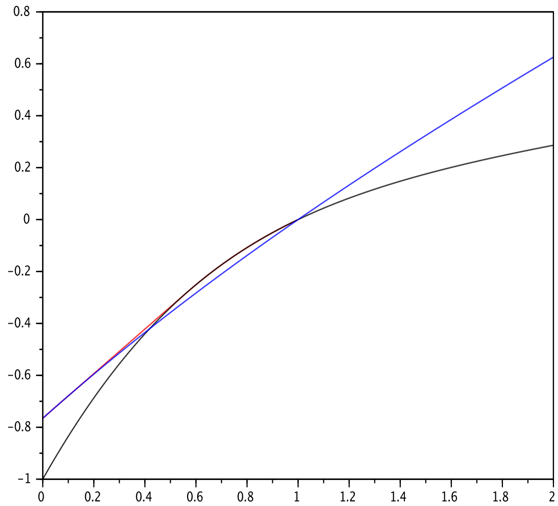


FIGURE 4. Same model as in Example 4.4 with $p_0 = .1$, $\mathbb{E}(N_0) = .4$ and $\mathbb{E}(N_1) = 1.3$ (blue curve: T; black curve: τ_ν; red curve: τ). $q_c = \widetilde{q}_c = \infty$. Phase transitions of orders ≥ 2 at some $q_0 < q_0'$ in $(0,1)$, no first order phase transition over $(1,\infty)$. $\tau_{\pi_*\mu} = T = \tau_\mu$ over $[0, q_0]$, $\tau_{\pi_*\mu} = \tau > \max(\tau_\nu, T)$ over (q_0, q_0'), and $\tau_{\pi_*\mu} = \tau_\nu$ over $[q_0', \infty)$.

where

$$X(u) = \sum_{v\in\Sigma_{|u|}} Y(u,v) \prod_{j=1}^{|u|} V_{u_j,v_j}(u_{|j-1}, v_{|j-1}). \tag{5.2}$$

Define also

$$\widetilde{X}(u) = \sum_{v\in\Sigma_{|u|}} \prod_{j=1}^{|u|} V_{u_j,v_j}(u_{|j-1}, v_{|j-1}),$$

and for all $x \in \Sigma$, and $n \geq 0$, set

$$X_n(x) = X(x_{|n}) \quad \text{and} \quad \widetilde{X}_n(x) = \widetilde{X}(x_{|n}). \tag{5.3}$$

Keep in mind that all variables defined above depend implicitly on ω.

Now, let us start by presenting two results that will be used in this section. They are proved in Section 9 as parts of Proposition 9.8 and Corollary 9.9 respectively.

Proposition 5.1. *Let $q \in (1,2]$ such that $T(q) > 0$. Let η be the Bernoulli product measure on Σ generated by a probability vector $(p_0', \dots, p_{m-1}')$. Set $A := \max\{1, \sum_{i=0}^{m-1} p_i' m^{-T_i(q)}\}$. Then there exists a constant C_q depending on W and q such that*

$$A^n \leq \mathbb{E}_{\mathbb{P}\otimes\eta}(X_n^q) \leq C_q A^n, \quad \forall n \in \mathbb{N},$$

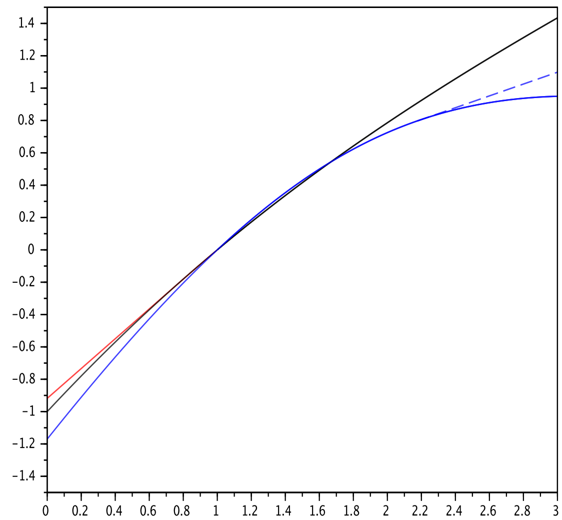


FIGURE 5. Same model as in Example 4.4 with $p_0 = .3$, $\mathbb{E}(N_0) = .25$ and $\mathbb{E}(N_1) = 2$ (blue curve: T; dashed blue curve: τ_μ; black curve: τ_ν; red curve: τ). $q_c = \widetilde{q}_c \simeq 2.176 < \infty$. One phase transition of order ≥ 2 at some $q_0 \in (0,1)$. One first order phase transition at some $q_0' \in (1, q_c)$, and one second order phase transition at q_c. $\tau_{\pi_*\mu} = \tau > \max(\tau_\nu, T)$ over $[0, q_0)$ (in particular $-\tau(0) = \dim_H \pi(K) < \min(-\tau_\nu(0), -T(0))$), $\tau_{\pi_*\mu} = \tau_\nu$ over $[q_0, q_0']$ and $\tau_{\pi_*\mu} = T = \tau_\mu$ over $[q_0, q_c]$, and $\tau_{\pi_*\mu}(q) = \tau_\mu(q) = T'(q_c)q$ over $[q_c, \infty)$.

where X_n is defined as in (5.3).

Corollary 5.2. *Let $q \in (1,2]$ such that $T(q) > 0$. Then there exists a constant C_q depending on W and q such that for all $n \in \mathbb{N}$:*

$$m^{-n\min\{\tau_\nu(q),T(q)\}} \leq \mathbb{E}\Big(\sum_{u\in\Sigma_n} \pi_*\mu([u])^q\Big) \leq C_q m^{-n\min\{\tau_\nu(q),T(q)\}}.$$

5.1. **Proof of Theorem 3.1: absolute continuity.**

Proof of Theorem 3.1(1). (i) Since (Σ, d) satisfies the Besicovitch covering property, almost surely $\pi_*\mu_\omega(\mathrm{d}x) = f(\omega,x)\,\nu(\mathrm{d}x) + \rho_\omega(\mathrm{d}x)$, where ρ_ω is a Borel measure singular with respect to ν and

$$f(\omega,x) = \lim_{n\to\infty}\left(X_n(\omega,x) = \frac{\pi_*\mu_\omega([x_{|n}])}{\nu(x_{|n})}\right), \tag{5.4}$$

ν-almost everywhere. Thus, if $\mathbb{E}_{\mathbb{P}\otimes\nu}(f) = \mathbb{E}(\|\pi_*\mu\|) = 1$, then $\rho_\omega = 0$ almost surely, i.e. $\pi_*\mu$ is almost surely absolutely continuous with respect to ν.

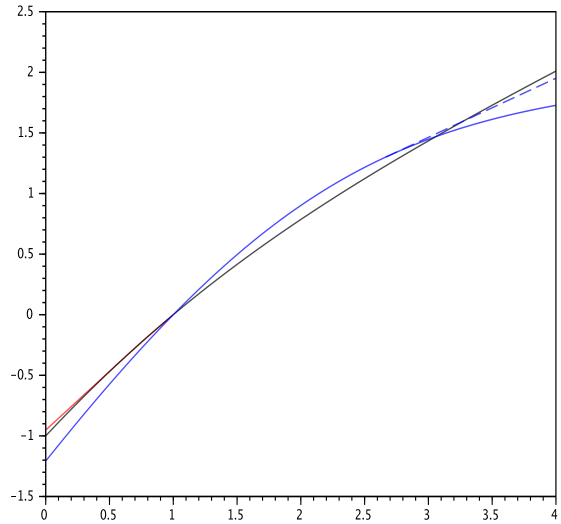


FIGURE 6. Same model as in Example 4.4 with $p_0 = .3$, $\mathbb{E}(N_0) = .3$ and $\mathbb{E}(N_1) = 2$ (blue curve: T; dashed blue curve: τ_μ; black curve: τ_ν; red curve: τ). $q_c \simeq 2.665 < \infty$ and $\widetilde{q}_c \simeq 3.059$. One phase transition of order ≥ 2 at some $q_0 \in (0,1)$. No first order phase transition over $[1, \widetilde{q}_c)$. $\tau_{\pi_*\mu} = \tau > \max(\tau_\nu, T)$ over $[0, q_0)$ (in particular $-\tau(0) = \dim_H \pi(K) < \min(-\tau_\nu(0), -T(0)))$ and $\tau_{\pi_*\mu} = \tau_\nu$ over $[q_0, \widetilde{q}_c]$.

We know by the construction of μ that $\mathbb{E}_{\mathbb{P}\otimes\nu}(X_n) = \mathbb{E}(\|\mu_n\|) = 1$ for all $n \geq 1$. This implies that for all $\lambda \in (0,1)$ the sequence $(X_n^\lambda)_{n\geq 1}$ is uniformly integrable with respect to $\mathbb{P}\otimes\nu$, hence by (5.4), $\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}\otimes\nu}(X_n^\lambda) = \mathbb{E}_{\mathbb{P}\otimes\nu}(f^\lambda)$.

Next we claim that under $\mathbb{P}\otimes\nu$, X_n converges in law to a random variable $\widetilde{X}$. We postpone its proof to the next paragraph. Since for any given $\lambda \in (0,1)$ the sequence $(X_n^\lambda)_{n\geq 1}$ is uniformly integrable, it follows that $\lim_{n\to\infty} \mathbb{E}_{\mathbb{P}\otimes\nu}(X_n^\lambda) = \mathbb{E}_{\mathbb{P}\otimes\nu}(\widetilde{X}^\lambda)$. Hence $\mathbb{E}_{\mathbb{P}\otimes\nu}(f^\lambda) = \mathbb{E}_{\mathbb{P}\otimes\nu}(\widetilde{X}^\lambda)$ for all $\lambda \in (0,1)$. Furthermore, if $\mathbb{E}_{\mathbb{P}\otimes\nu}(\widetilde{X}) = 1$, letting λ tend to 1 we get $\mathbb{E}_{\mathbb{P}\otimes\nu}(f) = 1$.

We now prove that X_n converges in law to a random variable $\widetilde{X}$ such that $\mathbb{E}_{\mathbb{P}\otimes\nu}(\widetilde{X}) = 1$.

By the definition of $X_n(x)$, for any $t > 0$,

$$\mathbb{E}_{\mathbb{P}\otimes\nu}(e^{-tX_n}) = \mathbb{E}_{\mathbb{P}\otimes\nu}\Big(\prod_{v\in\Sigma_n} \phi_Y\Big(t \prod_{j=1}^n V_{x_j,v_j}(x_{|j-1}, v_{|j-1})\Big)\Big),$$

where ϕ_Y stands for the Laplace transform of Y, i.e. $\phi_Y(t) = \mathbb{E}(e^{-tY})$.

Let us show that

$$M_n(x) := \max_{v\in\Sigma_n} \prod_{j=1}^n V_{x_j,v_j}(x_{|j-1}, v_{|j-1})$$

converges in law to 0 under $\mathbb{P}\otimes\nu$, as n tends to ∞. For $x\in\Sigma$, let $\mathbb{Q}_x$ be the probability measure on

$$(\Omega\times\Sigma, \sigma(V_{x_n}(x_{|n-1},v) : n\geq 1,\, v\in\Sigma_{n-1})\otimes\mathcal{B}(\Sigma))$$

whose restriction to

$$\sigma(V_{x_j}(x_{|n-1},v) : 1\leq j\leq n,\, v\in\Sigma_{j-1})\otimes\sigma([v] : v\in\Sigma_n)$$

is determined by

$$\mathbb{Q}_{x,n}(A\times[v]) = \mathbb{E}\big(\mathbf{1}_A(\omega)\prod_{j=1}^n V_{x_j,v_j}(x_{|j-1},v_{|j-1})\big)$$

for $A\in\sigma(V_{x_j}(v) : 1\leq j\leq n,\, v\in\Sigma_{j-1})$ and $v\in\Sigma_n$. This yields a new skew product measure $\rho(\mathrm{d}\omega,\mathrm{d}x,\mathrm{d}y)=\nu(\mathrm{d}x)\mathbb{Q}_x(\mathrm{d}\omega,\mathrm{d}y)$ on $\Omega\times\Sigma^2$. A direct computation shows that the random variables $(\omega,x,y)\mapsto V_{x_j,y_j}(x_{|j-1},y_{|j-1})$ are i.i.d. with respect to ρ, and their logarithms are of expectation

$$\sum_{i=0}^{m-1}p_i\sum_{j=0}^{m-1}\mathbb{E}(V_{i,j}\log V_{i,j}) = -\log(m)\sum_{i=0}^{m-1}p_iT_i'(1-)<0.$$

By the strong law of large numbers, for ρ-almost every (ω,x,y),

$$\lim_{n\to\infty}\prod_{j=1}^n V_{x_j,y_j}(x_{|j-1},y_{|j-1})=0.$$

Now fix $\epsilon>0$. We have

$$\begin{aligned}
\mathbb{P}\otimes\nu(M_n(x)\geq\epsilon) &\leq \mathbb{E}_{\mathbb{P}\otimes\nu}\Big(\sum_{v\in\Sigma_n}\mathbf{1}_{\{\prod_{j=1}^n V_{x_j,v_j}(x_{|j-1},v_{|j-1})\geq\epsilon\}}\Big)\\
&\leq \mathbb{E}_{\mathbb{P}\otimes\nu}\Big(\sum_{v\in\Sigma_n}\epsilon^{-1}\mathbf{1}_{\{\prod_{j=1}^n V_{x_j,v_j}(x_{|j-1},v_{|j-1})\geq\epsilon\}}\prod_{j=1}^n V_{x_j,v_j}(x_{|j-1},v_{|j-1})\Big)\\
&= \epsilon^{-1}\rho\Big(\Big\{\prod_{j=1}^n V_{x_j,y_j}(x_{|j-1},y_{|j-1})\geq\epsilon\Big\}\Big),
\end{aligned}$$

and the right hand side converges to 0.

Consequently, since $\mathbb{E}(Y)=1$, it follows that $\phi_Y(u)=e^{-u+o(u)}$ near $0+$, so for each $t>0$ we can write

$$\begin{aligned}
\text{(5.5)}\quad \mathbb{E}_{\mathbb{P}\otimes\nu}(e^{-tX_n}) &= \mathbb{E}_{\mathbb{P}\otimes\nu}\Big(\mathbf{1}_{\{M_n(x)<\epsilon\}}e^{-t\widetilde{X}_n(1+O(\epsilon))}\Big)+\mathbb{E}_{\mathbb{P}\otimes\nu}\Big(\mathbf{1}_{\{M_n(x)\geq\epsilon\}}e^{-tX_n}\Big)\\
&= \mathbb{E}_{\mathbb{P}\otimes\nu}\Big(e^{-t\widetilde{X}_n(1+O(\epsilon))}\Big)+R_n,
\end{aligned}$$

where $|R_n|\leq 2\mathbb{P}\otimes\nu(M_n(x)\geq\epsilon)$ and $\widetilde{X}_n(1+O(\epsilon))\geq 0$. On the other hand, the information gathered in Appendix B applied with $\eta=\nu$ and $U_i=V_i$ shows that $(\widetilde{X}_n(x,\cdot))_{n\geq1}$ is a Mandelbrot martingale in the random environment defined by ν, and $\widetilde{X}_n$ converges $\mathbb{P}\otimes\nu$-almost surely to a limit $\widetilde{X}$. We then deduce from the bounded convergence theorem and the fact that $\mathbb{P}\otimes\nu(M_n(x)\geq\epsilon)$ tends to 0 as $n\to\infty$ that $\mathbb{E}_{\mathbb{P}\otimes\nu}(e^{-tX_n})$ converges to

$\mathbb{E}_{\mathbb{P}\otimes\nu}(e^{-t\widetilde{X}})$. Moreover, the condition $\dim(\mu)-\dim(\nu)=\sum_{i=0}^{m-1}p_iT_i'(1-)>0$ is sufficient for $(\widetilde{X}_n)_{n\geq 1}$ to be uniformly integrable (Theorem B.1), hence $\mathbb{E}_{\mathbb{P}\otimes\nu}(\widetilde{X})=1$.

(ii) Since $T'(1-)>0$, the assumption that T is finite on a neighborhood of 1 implies that $T(s)>0$, hence $\mathbb{E}(Y^s)<\infty$ on a right neighborhood of 1 (see [36] or [22]). Moreover, the assumption $\dim(\mu)>\dim(\nu)$ is equivalent to $\sum_{i=0}^{m-1}p_iT_i'(1)>0$, hence $\sum_{i=0}^{m-1}p_im^{-T_i(s)}<1$ on a right neighborhood of 1. Also, if $s\in(1,2]$ and both $\mathbb{E}(Y^s)<\infty$ and $\sum_{i=0}^{m-1}p_im^{-T_i(s)}<1$, then $\sup_{n\geq 1}\mathbb{E}_{\mathbb{P}\otimes\nu}(X_n(x)^s)<\infty$ by Proposition 5.1. For any such $s>1$, using (5.1) we get

$$\int_\Sigma\Big(\frac{\pi_*\mu([x_{|n}])}{\nu([x_{|n}])}\Big)^{s-1}\pi_*\mu(\mathrm{d}x)=\sum_{u\in\Sigma_n}\mathbf{1}_{\{\nu([u])>0\}}\Big(\frac{\pi_*\mu([u])}{\nu([u])}\Big)^{s-1}\pi_*\mu([u])=\sum_{u\in\Sigma_n}\nu([u])X(u)^s.$$

Thus

$$\sup_{n\geq 1}\mathbb{E}\left(\int_\Sigma\Big(\frac{\pi_*\mu([x_{|n}])}{\nu([x_{|n}])}\Big)^{s-1}\pi_*\mu(\mathrm{d}x)\right)=\sup_{n\geq 1}\mathbb{E}_{\mathbb{P}\otimes\nu}(X_n(x)^s)<\infty.$$

Consequently, by the Fatou lemma,

$$\begin{aligned}\mathbb{E}\left(\liminf_{n\to\infty}\int_\Sigma\Big(\frac{\pi_*\mu([x_{|n}])}{\nu([x_{|n}])}\Big)^{s-1}\pi_*\mu(\mathrm{d}x)\right)&\leq\liminf_{n\to\infty}\mathbb{E}\left(\int_\Sigma\Big(\frac{\pi_*\mu([x_{|n}])}{\nu([x_{|n}])}\Big)^{s-1}\pi_*\mu(\mathrm{d}x)\right)\\&<\infty,\end{aligned}$$

from which we get

$$\liminf_{n\to\infty}\int_\Sigma\Big(\frac{\pi_*\mu([x_{|n}])}{\nu([x_{|n}])}\Big)^{s-1}\pi_*\mu(\mathrm{d}x)<\infty\quad\text{a.s.}$$

Due to [43, Theorem 2.12(3)], this implies both the absolute continuity of $\pi_*\mu$ with respect to ν and the desired result about the density of $\pi_*\mu$ with respect to ν. ☐

Proof of Theorem 3.1(2). If $\dim(\mu)<\dim(\nu)$, there is nothing to prove since $\overline{\dim}_P(\pi_*\mu)\leq\dim(\mu)$.

Suppose now that $\dim(\mu)=T'(1)=\dim(\nu)$. This time, under $\mathbb{P}\otimes\nu$, the martingale $\widetilde{X}_n(\omega,x)$ converges to 0 almost surely since $\sum_{i=0}^{m-1}p_iT_i'(1-)=0$ (see Theorem B.1 again). This implies that $M_n(x)=\max_{v\in\Sigma_n}\prod_{j=1}^nV_{x_j,v_j}(x_{|j-1},v_{|j-1})$ converges to 0 almost surely under $\mathbb{P}\otimes\nu$. Using (5.5) this time yields the convergence in law to 0 for X_n, and $\mathbb{E}_{\mathbb{P}\otimes\nu}(f^\lambda)=0$ for all $\lambda\in(0,1)$. Consequently, $f=0$ with $\mathbb{P}\otimes\nu$ probability 1, which is equivalent to the fact that $\pi_*\mu$ and ν are almost surely mutually singular. ☐

5.2. **Proof of Theorem 3.3(1): dimension.** When $\dim(\mu)>\dim(\nu)$, since by Theorem 3.1(1)(i) $\pi_*\mu$ is absolutely continuous with respect to ν, we already know that if $\mu\neq 0$, then $\dim(\pi_*\mu)=\dim(\nu)$. However, under the assumption that T is finite in a neighborhood of 1, we give an alternative proof which works regardless of the respective positions of $\dim(\mu)$ and $\dim(\nu)$, and independently of absolute continuity considerations.

We will use Corollary 5.2 and the following elementary lemma.

Lemma 5.3. *Let ρ be a positive and finite Borel measure on Σ. Let $D \geq 0$. If for all $\epsilon > 0$ there exists $q > 1$ such that $\sum_{n\geq 1} m^{n(q-1)(D-\epsilon)} \sum_{|u|=n} \rho([u])^q < \infty$, then $\underline{\dim}_{\rm loc}(\rho, x) \geq D$ for ρ-almost every x. Also, if for all $\epsilon > 0$ there exists $q \in (0,1)$ such that $\sum_{n\geq 1} m^{n(q-1)(D+\epsilon)} \sum_{|u|=n} \rho([u])^q < \infty$, then $\overline{\dim}_{\rm loc}(\rho, x) \leq D$ for ρ-almost every x.*

Proof. Fix $\epsilon > 0$. For all $q > 1$ and $n \geq 1$, applying Markov's inequality yields

$$\begin{aligned} \rho\Big(\Big\{x \in \Sigma :\ \frac{\log(\rho([x_{|n}]))}{-n\log(m)} \leq D - \epsilon\Big\}\Big) &= \rho\Big(\Big\{x \in \Sigma :\ \rho([x_{|n}])^{q-1} \geq m^{-n(q-1)(D-\epsilon)}\Big\}\Big) \\ &\leq m^{n(q-1)(D-\epsilon)} \int_\Sigma \rho([x_{|n}])^{q-1}\, \rho(\mathrm{d}x) \\ &= m^{n(q-1)(D-\epsilon)} \sum_{|u|=n} \rho([u])^q. \end{aligned}$$

Consequently, if $\sum_{n\geq 1} m^{n(q-1)(D-\epsilon)} \sum_{|u|=n} \rho([u])^q < \infty$, by the Borel-Cantelli lemma we get $\underline{\dim}_{\rm loc}(\rho, x) \geq D - \epsilon$ for ρ-almost every x.

The upper local dimension of ρ is dealt with similarly. $\square$

Recall that $\dim(\nu) = \tau_\nu'(1)$ and that almost surely, conditional on $\{\mu \neq 0\}$, $\dim(\mu) = T'(1)$. We deduce from Corollary 5.2 that for $q > 1$ close enough to 1, there exists a constant C_q such that for all $n \geq 1$,

$$\mathbb{E}\Big(\sum_{u\in\Sigma_n} \pi_*\mu([u])^q\Big) \leq C_q \cdot \begin{cases} m^{-n(q-1)\dim(\nu)+o(q-1)} & \text{if } T'(1) > \tau_\nu'(1) \\ m^{-n(q-1)T'(1)+o(q-1)} & \text{if } T'(1) \leq \tau_\nu'(1) \end{cases}$$

as $q \to 1+$. Fix $\epsilon > 0$. Take q close enough to 1 so that the previous upper bound holds with $|o(q-1)| \leq \epsilon(q-1)/4$. By Lemma C.1 we conclude that, with probability 1, conditional on $\{\mu \neq 0\}$, for n large enough,

$$m^{n(q-1)(D-\epsilon)} \sum_{|u|=n} \pi_*\mu([u])^q \leq m^{-n\epsilon(q-1)/2},$$

with $D = \tau_\nu'(1)$ if $T'(1) > \tau_\nu'(1)$, and $D = T'(1)$ otherwise. Then Lemma 5.3 yields the expected lower bound for $\underline{\dim}_{\rm loc}(\pi_*\mu, x)$, $\pi_*\mu$-almost everywhere.

To control $\overline{\dim}_{\rm loc}(\pi_*\mu, x)$, $\pi_*\mu$-almost everywhere, we only need to deal with the case $T'(1) > \tau_\nu'(1)$. Indeed, for $\pi_*\mu$-almost every x, we obviously have $\overline{\dim}_{\rm loc}(\pi_*\mu, x) \leq \dim(\mu)$.

Now assume $T'(1) > \tau_\nu'(1)$. Let $q \in (0,1)$. We have

$$\begin{aligned} \mathbb{E}\Big(\sum_{u\in\Sigma_n} \pi_*\mu([u])^q\Big) &= \sum_{u\in\Sigma_n} \nu([u])^q \mathbb{E}(X(u)^q) \\ &\leq \sum_{u\in\Sigma_n} \nu([u])^q \mathbb{E}(X(u))^q = \sum_{u\in\Sigma_n} \nu([u])^q = m^{-n\tau_\nu(q)}. \end{aligned}$$

This is enough to conclude that $\overline{\dim}_{\rm loc}(\pi_*\mu, x) \leq \tau_\nu'(1)$ for $\pi_*\mu$-almost every x by using again Lemmas C.1 and 5.3.

Putting together the previous arguments we conclude that with probability 1, conditional on $\{\mu \neq 0\}$, $\pi_*\mu$ is exact dimensional with $\dim(\pi_*\mu) = \dim(\nu)$ if $T'(1) > \tau'_\nu(1)$ and $\dim(\pi_*\mu) = \dim(\mu)$ if $T'(1) \leq \tau'_\nu(1)$.

5.3. **Proof of Corollary 3.5: variational principle.**

Let $\varphi: \ h \in \mathbb{R}_+ \mapsto \log_m \sum_{i=0}^{m-1} \mathbb{E}(N_i)^h$.

We begin with the proof of (3.7). The upper bound for the box dimension of $\pi(K)$ can be obtained as a consequence of our approach to the multifractal analysis, or by using Falconer's argument in [23] (see also [18]). To see it, notice that at a given generation n of the construction, $\pi(K_n)$ is covered by at most $\sum_{|u|=n} (\#\{v \in \Sigma_n : Q(u,v) > 0\})^h$, for all $0 \leq h \leq 1$, which yields that the expectation of this number is at most $(\inf_{0 \leq h \leq 1} m^{\varphi(h)})^n$. Applying Lemma C.1, we obtain that $\overline{\dim}_B(\pi(K)) \leq \inf_{0\leq h \leq 1} \varphi(h)$. Thus it remains to derive a lower bound for the Hausdorff dimension of $\pi(K)$.

Let h_0 be a point at which $\inf_{0 \leq h \leq 1} \varphi(h)$ is attained. Due to the convexity and the analyticity of φ, such a point is not unique if and only if $\mathbb{E}(N_i) = 1$ when $\mathbb{E}(N_i) > 0$. Let us consider the Mandelbrot measure associated with the following weights:

$$W'_{i,j} = p'_i V'_{i,j} \quad \text{with } V'_{i,j} = \begin{cases} \dfrac{\mathbf{1}_{\{W_{i,j} > 0\}}}{\mathbb{E}(N_i)} & \text{if } \mathbb{E}(N_i) > 0 \\ 0 & \text{otherwise} \end{cases},$$

where

$$p' = (p'_i)_{0 \leq i \leq m-1} = \left(\frac{\mathbb{E}(N_i)^{h_0}}{\sum_{k=0}^{m-1} \mathbb{E}(N_k)^{h_0}} \right)_{0 \leq i \leq m-1}.$$

Let μ' be the associated Mandelbrot measure and ν' the Bernoulli product associated with p'. Notice that for this measure the function $T = T_{W'}$ is everywhere finite, so that Theorem 3.3(1) applies to μ' in any case: $\dim(\pi_*\mu') = H(\pi_*\mu')$. Also,

$$\dim(\nu') = -\frac{h_0 \sum_{i=0}^{m-1} \mathbb{E}(N_i)^{h_0} \log_m(\mathbb{E}(N_i))}{\sum_{i=0}^{m-1} \mathbb{E}(N_i)^{h_0}} + \log_m \left(\sum_{i=0}^{m-1} \mathbb{E}(N_i)^{h_0} \right) = \varphi(h_0) - h_0 \varphi'(h_0)$$

and

$$\sum_{i=0}^{m-1} p'_i T'_{V'_i}(1) = \frac{\sum_{i=0}^{m-1} \mathbb{E}(N_i)^{h_0} \log_m(\mathbb{E}(N_i))}{\sum_{i=0}^{m-1} \mathbb{E}(N_i)^{h_0}} = \varphi'(h_0). \tag{5.6}$$

Next we show that $\dim_H \pi(K) \geq H(\pi_*\mu') \geq \varphi(h_0)$, by considering the scenarios $h_0 = 1$, $h_0 \in (0,1)$ and $h_0 = 0$, separately. First suppose that $h_0 = 1$. Then μ' is the so-called branching measure on K, and we see that

$$\dim(\nu') + \sum_{i=0}^{m-1} p'_i T'_{V'_i}(1) = \varphi(1) = \log(\mathbb{E}(N))/\log(m) > 0,$$

hence μ' is non-degenerate with positive probability (a fact that can also be directly seen from $T'_{W'}(1)$). Moreover, since on $[0,1]$ φ takes its minimum at $h = 1$, by smoothness of

φ we must have $\varphi'(1) \leq 0$, consequently $\sum_{i=0}^{m-1} p_i' T'_{V_i'}(1) \leq 0$, and thus by Theorem 3.3, $\dim(\pi_*\mu') = \dim(\mu')$ and $\dim_H \pi(K) = \dim_H(K) = \varphi(1)$ when $K \neq \emptyset$.

Next suppose that $0 < h_0 < 1$. We have $\varphi'(h_0) = 0$, hence $\sum_{i=0}^{m-1} p_i' T'_{V_i'}(1) = 0$ and thus by Theorem 3.3,

$$\dim(\pi_*\mu') = \dim(\mu') = \dim(\nu') = \varphi(h_0),$$

yielding $\dim_H \pi(K) \geq \varphi(h_0)$ when $K \neq \emptyset$.

Finally suppose that $h_0 = 0$. Then $\varphi'(0) \geq 0$, so $\sum_{i=0}^{m-1} p_i' T'_{V_i'}(1) \geq 0$ and thus by Theorem 3.3, $\dim(\pi_*\mu') = \dim(\nu') = \varphi(0) \leq \dim(\mu')$ when $\mu' \neq 0$, and consequently, $\dim_H \pi(K) \geq \varphi(0)$ when $K \neq \emptyset$.

So far we have proved (3.7). Below we discuss the uniqueness problem regarding the last variational relation in (3.7).

Notice that the Mandelbrot measure μ' considered above has a dimension equal to $\dim_H K$ if and only if $T'_{W'}(1) = -T_{W'}(0) = \log_m(\mathbb{E}(N))$, that is $T_{W'}$ is linear. In this case, if $h_0 = 1$, μ' is the branching measure. If $h_0 < 1$, since

$$T_{W'}(q) = q \log_m \Big(\sum_{i=0}^{m-1} \mathbb{E}(N_i)^{h_0} \Big) - \log_m \sum_{i=0}^{m-1} \mathbf{1}_{\{p_i > 0\}} \mathbb{E}(N_i)^{1+q(h_0-1)}$$

and the second derivative of $T_{W'}$ vanishes, we get that $\mathbb{E}(N_i) = 1$ for each i such that $p_i > 0$. Once again μ' is the branching measure.

For the uniqueness problem, the case when $\dim_H K = \dim_H \pi(K)$ is clear from the above discussion, since the same argument in fact shows that a Mandelbrot measure supported on K whose dimension equals that of K must be the branching measure. Thus we can suppose that $\dim_H K > \dim_H \pi(K)$.

Suppose that the maximum in (3.7) is attained at a Mandelbrot measure μ'' defined simultaneously with μ' and supported on K conditional on non-vanishing. Then it is easily seen that μ'' is generated by a random vector W'' such that $W''_{i,j} > 0$ only if $W_{i,j} > 0$, and we can associate with W'' the probability vector $(p_i'' = \sum_{j=0}^{m-1} \mathbb{E}(W''_{i,j}))_{0 \leq i \leq m-1}$ and the vectors $V_i'' = (W''_{i,j}/p_i'')_{0 \leq j \leq m-1}$ if $p_i'' > 0$ and 0 otherwise. Moreover, $p_i' > 0$ implies $p_i'' > 0$ for otherwise the formula $\inf_{0 \leq h \leq 1} \log_m \sum_{i=0}^{m-1} \mathbb{E}(N_i)^h$ for the Hausdorff dimension of $\pi(K)$ would give a strictly smaller dimension. Recall that

$$H(\pi_*\mu'') = \min \Big(\dim(\nu''),\ \dim(\nu'') + \sum_{i=0}^{m-1} p_i'' T'_{V_i''}(1-) \Big). \tag{5.7}$$

Now, let us observe that $\sum_{i=0}^{m-1} p_i'' T'_{V_i''}(1-)$ is always smaller than or equal to

$$\sum_{i=0}^{m-1} p_i'' T'_{V_i'}(1-) = \sum_{i=0}^{m-1} p_i'' \log_m \mathbb{E}(N_i).$$

This is due to the fact that $T_{V_i''}$ is concave, equal to 0 at 1, and

$$T_{V_i''}(0) = -\log_m \mathbb{E}(\sum_{j=0}^{m-1} \mathbf{1}_{\{W''_{i,j}>0\}}) \geq -\log_m \mathbb{E}(N_i), \tag{5.8}$$

implying that $T'_{V_i''}(1-) \leq -T_{V_i''}(0) \leq \log_m \mathbb{E}(N_i)$.

Consequently, in order to optimize $\dim(\nu'') + \sum_{i=0}^{m-1} p_i'' T'_{V_i''}(1-)$, μ'' must satisfy the condition that $T'_{V_i''}(1-) = T'_{V_i'}(1-) = \log_m \mathbb{E}(N_i)$. On the other hand, by concavity of $T_{V_i''}$ on $[0,1]$, $T_{V_i''}(0) \leq -T'_{V_i''}(1-)$. Finally, since by (5.8), $T_{V_i''}(0) \geq -\log_m \mathbb{E}(N_i) = -T'_{V_i''}(1-)$, we get $T_{V_i''}(0) = -\log_m \mathbb{E}(N_i) = -T'_{V_i''}(1-)$, hence $T_{V_i''}$ is linear on $[0,1]$. This means that like for V_i', the coordinates of the vector V_i'' equal either 0 or $1/\mathbb{E}(N_i)$. Since, moreover, $W''_{i,j} = 0$ as soon as $W'_{i,j} = 0$, we get $V_i'' = V_i'$ almost surely. On the other hand, a simple study using Lagrange multipliers shows that $\dim(\nu'') + \sum_{i=0}^{m-1} p_i'' \log_m \mathbb{E}(N_i)$ is optimal for $p'' = p'$, the maximum being unique. In other words, the maximum over μ'' of $\dim(\nu'') + \sum_{i=0}^{m-1} p_i'' T'_{V_i''}(1-)$ is reached uniquely at μ'.

Now, suppose first that $\varphi'(0) \leq 0$, i.e. the infimum of φ over $[0,1]$ is reached at a unique $h_0 \in (0,1]$, or at $h_0 = 0$ with $\varphi'(0) = 0$. In both cases, we have $\varphi'(h_0) \leq 0$, and our study of μ' (cf. (5.6)) shows that $\sum_{i=0}^{m-1} p_i' T'_{V_i'}(1^-) = \sum_{i=0}^{m-1} p_i' \log_m \mathbb{E}(N_i) = \varphi'(h_0) \leq 0$, showing that $H(\pi_*\mu') = \dim(\mu')$. Consequently, by the arguments in the last paragraph, for any Mandelbrot measure μ'' supported on K,

$$\dim(\nu'') + \sum_{i=0}^{m-1} p_i'' T'_{V_i''}(1-) \leq \dim(\nu') + \sum_{i=0}^{m-1} p_i' T'_{V_i'}(1-) = \dim(\mu') = H(\pi_*\mu'),$$

where the first equality holds if and only if $\mu'' = \mu'$. Then, the relation (5.7) yields μ' as the unique Mandelbrot measure such that $H(\pi_*\mu')$ is maximal.

Next suppose that $\varphi'(0) > 0$. Fix $\lambda > 1$ and U_λ a random variable independent of V' and taking value $\lambda > 1$ with probability λ^{-1} and 0 with probability $1 - \lambda^{-1}$. Take $p'' = p'$ and replace V' by $V'' = (V_0'', V_1'', V_2'', \dots, V_{m-1}'')$ with $V_i'' = U_\lambda \cdot V_i'$. This yields a Mandelbrot measure μ'' different from μ', with the same expectation ν' and $\sum_{i=0}^{m-1} p_i'' T'_{V_i''}(1) = \sum_{i=0}^{m-1} p_i' T'_{V_i'}(1) - \log_m(\lambda) > 0$ if λ is close enough to 1. Consequently, $H(\pi_*\mu'') = \dim(\nu') = H(\pi_*\mu')$, and there is no uniqueness in this case.

6. Proof of Theorem 3.7: Differentiability properties of the function τ

6.1. Differentiability over $(0,1-]$.

Notice that the differentiability of τ over $(0,1-]$ automatically holds if $\tau \equiv T$ over $(0,1]$, and that this holds in particular if T_i is linear and $\mathbb{E}(N_i) = 1$ for all $0 \leq i \leq m-1$ such that $\mathbb{E}(N_i) > 0$, i.e. $T_i \equiv 0$ so that $T = \tau_\nu = \tau$ (it is shown below that this is also a necessary condition, which is equivalent to having $\mathbb{E}(N_i) = 1$ and $V_{i,j} = \mathbf{1}_{\{W_{i,j}>0\}}$ for all

$0 \le j \le m-1$ almost surely). Moreover, still in this case, since we have excluded the case that $N_i = 1$ for all $0 \le i \le m-1$ such that $\mathbb{E}(N_i) > 0$, by Theorem 3.1(2), $\pi_*\mu$ and ν are mutually singular, and thus $\pi_*\mu \neq \nu$ almost surely.

Now suppose that $\tau \not\equiv T$ over $(0, 1]$. For $0 < q \le s \le 1$ set

$$G(q,s) = \sum_{i=0}^{m-1} p_i^q m^{-qT_i(s)/s}$$

and

$$g(q,s) = s^2(-q\log(m))^{-1}\frac{\partial G}{\partial s}(q,s) = \sum_{i=1}^{m-1} p_i^q m^{-qT_i(s)/s} T_i^*(T_i'(s)).$$

Let $q \in (0, 1]$. To begin with suppose that the infimum defining $\tau(q)$, i.e. the infimum of $G(q, \cdot)$, is reached at $s \in (q, 1)$ (hence $q < 1$). We claim that s is unique and for all q' in an open neighborhood of q there exists a unique $s(q') \in (q', 1)$ such that $\tau(q') = -\log_m G(q, s(q'))$. To show this claim, notice that at any $s_0 \in (q, 1)$ at which the infimum defining $\tau(q)$ is reached, $g(q, s_0) = 0$. Moreover, for all $s \in [q, 1]$,

$$\frac{\partial g}{\partial s}(q,s) = \sum_{i=1}^{m-1} p_i^q m^{-qT_i(s)/s}(-q\log(m)s^{-2}(T_i^*(T_i'(s)))^2 + sT_i''(s)) \le 0.$$

Suppose that $T_i''(s) = 0$ for some i. Then

$$\left(\mathbb{E}\sum_{j=0}^{m-1} V_{i,j}^s(\log(V_{i,j}))^2\right)\left(\mathbb{E}\sum_{j=0}^{m-1} V_{i,j}^s\right) = \left(\mathbb{E}\sum_{j=0}^{m-1} V_{i,j}^s(\log(V_{i,j}))\right)^2.$$

It follows that by the Cauchy-Schwarz inequality, there exists a constant c such that almost surely either $V_{i,j} = 0$ or $V_{i,j} = c$, hence $c = 1/\mathbb{E}(N_i)$. In this case, $T_i^*(T_i'(s)) = \log(\mathbb{E}(N_i))$. Consequently, for $\frac{\partial g}{\partial s}(q,s)$ to be equal to 0 we need to have $\mathbb{E}(N_i) = 1$ and $V_{i,j} = \mathbf{1}_{\{W_{i,j}>0\}}$ for all $0 \le i, j \le m-1$ such that $p_i > 0$, a situation that we have discarded by assuming that $\tau \neq T$ (notice that this property is equivalent to requiring that $T_i \equiv 0$ for all $0 \le i \le m-1$ such that $p_i > 0$). Thus $\frac{\partial g}{\partial s}(q,s) < 0$, hence $g(q, s)$ can vanish only at one point of $(q, 1)$, that we denote by $s(q)$. Then, because $\frac{\partial g}{\partial s}(q, s(q)) < 0$, the implicit function theorem implies our claim, as well as the analyticity of $s(\cdot)$ and τ on any maximal interval of points q such that $s(q) \in (q, 1)$. In addition, $s'(q) = -\frac{\frac{\partial g}{\partial q}(q,s(q))}{\frac{\partial g}{\partial s}(q,s(q))}$. We also notice that the study of $s \mapsto g(q, s)$ shows that $s \mapsto \frac{\partial G}{\partial s}(q,s)$ is negative on the left hand side of $s(q)$ and positive on the right hand side, so the infimum of $G(q, \cdot)$ over $[q, 1]$ can be reached neither at q nor at 1.

Now suppose that the infimum of $G(q, \cdot)$ is reached at $s_0 \in \{q, 1\}$. Suppose that this infimum is reached at another point of $[q, 1]$ as well (this can hold only if $q < 1$). Then, let $s_1 \in (q, 1)$ at which $G(q, \cdot)$ reaches a local maximum, hence $g(q, \cdot)$ vanishes. Our previous analysis of the sign of $g(q, \cdot)$, which is the opposite of the sign of $\frac{\partial G}{\partial s}(q, \cdot)$, shows that

$\frac{\partial G}{\partial s}(q,\cdot)$ is negative on the left of s_1, which is a contradiction. Thus the infimum of $G(q,\cdot)$ at s_0 is strict. We again denote this point s_0 by $s(q)$.

We notice that the argument in the above paragraph also shows that if q is a point of $(0,1)$ at which τ_ν and T coincide, i.e. $G(q,1)=G(q,q)$, then $\tau(q)$ cannot be attained at q or 1. This entails the fact that $\tau=T=\tau_\nu$ only if $T_i\equiv 0$ when $p_i>0$.

Next we prove that both τ and $s(\cdot)$ are continuous over $(0,1]$. Suppose that $q\in(0,1]$. Let $(q_n)_{n\geq 1}$ be a sequence of points in $(0,1]$ such that $q_n\to q$. Without loss of generality, we can assume that $s(q_n)$ converges as well, to a number, say s_q, which necessarily belongs to $[q,1]$ since $s(q_n)\in[q_n,1]$. It follows by continuity of G that $G(q_n,s(q_n))\to G(q,s_q)$. Suppose that $s_q\neq s(q)$. Then, $G(q,s(q))<G(q,s_q)$, hence there exist $n_0>1$ and $\epsilon>0$ such that for all $n\geq n_0$, for all $s\in[q_n,1]$,

$$G(q_n,s)\geq G(q_n,s(q_n))>G(q,s(q))+\epsilon.$$

However, there exists a sequence $(s_n)_{n\geq 1}$ such that $s_n\in[q_n,1]$ for all $n\geq n_0$ and $(q_n,s_n)\to(q,s(q))$. By continuity of G over $[0,1]$, $G(q_n,s_n)\to G(q,s(q))$, but $G(q_n,s_n)>G(q,s(q))+\epsilon$, which gives a contradiction. Consequently, we obtained the desired continuity property of $s(\cdot)$, and that of $\tau=-\log_m G(\cdot,s(\cdot))$.

Let us denote by $\mathcal{I}$ the set of the connected components of $\{q\in(0,1):s(q)\in(q,1)\}$.

Let $E=(0,1]\setminus\bigcup_{I\in\mathcal{I}}I$. Let $q_0\in E$. If q_0 is an interior point of E, then by continuity of s, we must have either $s(q)=q$ or $s(q)=1$ on the maximal interval I_{q_0} containing q_0 and contained in E; as a consequence, both $s(\cdot)$ and τ are analytic on the interior of I_{q_0}.

Suppose that $q_0\in\partial E$ and $q_0<1$. Notice that since q_0 is an accumulating point of $\bigcup_{I\in\mathcal{I}}I$, by continuity of $\frac{\partial G}{\partial s}$ and $s(\cdot)$, either $\frac{\partial G}{\partial s}(q_0,q_0)=0$ if $s(q_0)=q_0$ or $\frac{\partial G}{\partial s}(q_0,1)=0$ if $s(q_0)=1$.

Up to symmetry between the left and the right hand sides of q_0, there are essentially three situations. There exists $\eta>0$ such that either $s(q)=q$ over $[q_0-\eta,q_0)$ and $s(q)\in(q,1)$ over $(q_0,q_0+\eta]$, $s(q)=1$ over $[q_0-\eta,q_0)$ and $s(q)\in(q,1)$ over $(q_0,q_0+\eta]$, or $s(q)\in(q,1)$ both over $[q_0-\eta,q_0)$ and $(q_0,q_0+\eta]$. It means that q_0 cannot be an accumulating point of boundary points of E. Indeed, suppose that on the contrary q_0 is such a point. Then $s(q_0)\in\{q_0,1\}$. First assume that $s(q_0)=q_0$. By the remark in the last paragraph, $\frac{\partial G}{\partial s}(q,q)$ should have infinitely many zeros accumulating at q_0, which would imply that $\frac{\partial G}{\partial s}(q,q)=0$ for all $q\in(0,1)$ by analyticity of G; but this does not hold, for otherwise $\tau=T$, a case that we discarded. Indeed if $\tau\neq T$, there exists $q_0\in(0,1)$ such that $s(q_0)\in(q_0,1)$. Then our previous study of $g(q_0,\cdot)$ shows that $\frac{\partial G}{\partial s}(q_0,q_0)=-(\log)g(q_0,q_0)/q_0<0$ since $g(q_0,\cdot)$ is strictly decreasing and $g(q_0,s(q_0))=0$. Next assume $s(q_0)=1$. Again by the remark in the last paragraph we should have $\frac{\partial G}{\partial s}(q,1)=0$ and thus $g(q,1)=0$ for all $q\in(0,1)$, and it follows that $g(q,s(q))>0$ whenever $s(q)\neq 1$, leading to a contradiction.

Finally suppose that $q_0 = 1$. The same approach as above shows that there exists $\eta > 0$ such that either $s(q) = 1$ or $s(q) \in (q, 1)$ over $[1 - \eta, 1)$. Also, we notice that 0 cannot be an accumulating point of ∂E since we assumed that the T_i are finite and analytic in a neighborhood of 0.

We summarize the above in the following proposition.

Proposition 6.1. *The functions τ and $s(\cdot)$ are continuous over $(0, 1]$. There exists a set S, finite or empty, such that for each connected component I of $(0, 1] \setminus S$, the functions τ and $s(\cdot)$ restricted to I are analytic, and I is a maximal interval over which either $s(q) = q$, $s(q) \in (q, 1)$ or $s(q) = 1$.*

It remains to prove the differentiability of τ at each $q \in S$. Let $q_0 \in S$. If $q_0 = 1$, then there exists $\eta > 0$ such that $s(q) \in (q, 1)$ over $[1 - \eta, 1)$. The formula

$$\tau'(q) = -\frac{\frac{\partial G}{\partial q}(q, s(q))}{\log(m) G(q, s(q))} \tag{6.1}$$

implies that $\tau'(q)$ has a limit at $1-$, hence by the mean value theorem τ is left differentiable at 1.

Suppose that $q_0 < 1$. If $s(q) \in (q, 1)$ for all q in $[q_0 - \eta, q_0 + \eta] \setminus \{q_0\}$ for some $\eta > 0$, then formula (6.1) and the continuity of $s(\cdot)$ combined with the mean value theorem yield the fact that τ is C^1 at q_0. If $s(q) = q$ on $[q_0 - \eta, q_0)$ and $s(q) \in (q, 1)$ on $(q_0, q_0 + \eta]$, we first notice that $s(q)/q$ tends to 1 as $q \to q_0+$ by continuity of $s(\cdot)$. It is then almost direct to see that $\tau'(q)$ given by (6.1) converges to $T'(q_0)$ as $q \to q_0+$. Indeed,

$$\frac{\partial G}{\partial q}(q, s(q)) = \sum_{i=0}^{m-1} p_i^q m^{-T_i(s(q))q/s(q)}(\log(p_i) - \log(m) T_i(s(q))/s(q)) \tag{6.2}$$

$$= \sum_{i=0}^{m-1} p_i^q m^{-T_i(s(q))q/s(q)}(\log(p_i) - \log(m) T_i'(s(q)), \tag{6.3}$$

due to the equality $\frac{\partial G}{\partial s}(q, s(q)) = 0$. Then, letting q tend to q_0+ and using the fact that $s(q)/q$ tends to 1, we get $\lim_{q \to q_0+} \tau'(q) = T'(q_0)$. On the other hand, $\tau = T$ over $[q_0 - \eta, q_0)$, hence τ is C^1 at q_0. The other cases can be treated similarly.

6.2. **Concavity of τ over** $[0, 1]$**.** We will show later that the differentiability of τ over $(0, 1]$ combined with other arguments yields the equality of τ with the L^q-spectrum of $\pi_* \mu$ over this interval, conditional on $\{\mu \neq 0\}$. Consequently, τ is concave on $[0, 1]$ and automatically differentiable at the right hand side of 0 as soon as it is right continuous at 0.

6.3. **Continuity and differentiability at 0.** Due to the previous discussion, it is enough to prove the continuity at 0. However, we will examine the value of $\tau'(0+)$. We distinguish two cases.

To begin with, suppose that $h(q) = q/s(q)$ does not tend to 0 as q tends to 0. It follows that $s(q)$ tends to 0. Suppose that for some sequence $(q_n)_{n\geq 0}$ tending to 0, $h(q_n) \to h_* \in (0,1]$. The study achieved above gives $g(q_n, s(q_n)) = 0$ if $q_n < s(q_n) < 1$ and $g(q_n, s(q_n)) \leq 0$ if $s(q_n) = q_n$. This implies that

$$\sum_{i=0}^{m-1} \mathbb{E}(N_i)^{h_*} \log_m(\mathbb{E}(N_i)) = \lim_{n\to+\infty} g(q_n, s(q_n))$$

vanishes if $h_* < 1$ and is non-positive if $h_* = 1$. By convexity of the mapping $h \in [0,1] \mapsto \log_m \sum_{i=0}^{m-1} \mathbb{E}(N_i)^h$, we conclude that in any case,

$$-\log_m \sum_{i=0}^{m-1} \mathbb{E}(N_i)^{h_*} = -\inf_{0\leq h\leq 1} \log_m \sum_{i=0}^{m-1} \mathbb{E}(N_i)^h,$$

i.e. h_* is the point at which the minimum in (1.2) is attained. Moreover $\lim_{n\to\infty} \tau(q_n) = -\log_m \sum_{i=0}^{m-1} \mathbb{E}(N_i)^{h_*} = \tau(0)$. It follows that τ is right continuous at 0.

Now suppose that $h(q) = q/s(q)$ tends to 0 as q tends to 0. We have $q < s(q) \leq 1$ for q small enough. From this it follows that $g(q, s(q)) \geq 0$. Consequently, since $\sum_{i=0}^{m-1} \mathbf{1}_{\{p_i>0\}} \log(\mathbb{E}(N_i)) = \lim_{q\to 0+} g(q, s(q))$ (because $h(q)$ tends to 0), this number is non-negative. This implies that $\log_m \sum_{i=0}^{m-1} \mathbf{1}_{\{p_i>0\}} = \inf_{0\leq h\leq 1} \log_m \sum_{i=0}^{m-1} \mathbb{E}(N_i)^h$. On the other hand $\lim_{q\to 0+} \tau(q) = -\log_m \sum_{i=0}^{m-1} \mathbf{1}_{\{p_i>0\}}$, hence τ is right continuous at 0, and $\tau(0) = \tau_\nu(0)$. In this case we set $h_* = 0$.

In all the cases, we set

$$p'_i = \left(\frac{\mathbb{E}(N_i)^{h_*}}{\sum_{i'=0}^{m-1} \mathbb{E}(N_{i'})^{h_*}} \right)_{0\leq i\leq m-1}, \tag{6.4}$$

with the convention $0^0 = 0$, and we denote by ν' the associated Bernoulli product.

6.4. **The value of $\tau'(0+)$.** Now we use Proposition 6.1 to determine the value of $\tau'(0+)$ and examine more precisely the behavior of $s(q)$ at $0+$. This will be used to prove the validity of the multifractal formalism for $\pi_*\mu$ at $\tau'(0+)$. Our observation is the following:

Proposition 6.2. *Let p'_i be defined as in (6.4). One of the three following situations occurs:*

(i) $\tau = T$ *near* $0+$ *and* $\tau'(0+) = T'(0)$.
(ii) $\tau = \tau_\nu$ *near* $0+$ *and* $\tau'(0+) = \tau'_\nu(0)$. *Moreover,* $\sum_{i=1}^{m-1} p'_i T_i^*(T'_i(1)) \geq 0$.
(iii) $\tau > \max(T, \tau_\nu)$ *near* $0+$, *and there exists* $s_0 \in [0,1]$ *such that*

$$\tau'(0+) = -\sum_{i=0}^{m-1} p'_i(\log_m(p_i) - T'_i(s_0)).$$

Moreover, $\sum_{i=0}^{m-1} p'_i T_i^*(T'_i(s_0)) = 0$.

Proof. We treat the three cases considered in the statement separately.

Case 1: $\tau = T$ near $0+$. In this case, $h_* = 1$ and $\tau'(0+) = T'(0)$.

Case 2: $\tau = \tau_\nu$ near $0+$. We have $h_* = 0$ and $\tau'(0+) = \tau_\nu'(0)$. Moreover, for all $q > 0$ close enough to 0, $s(q) = 1$ which implies that $g(q, s(q)) = g(q, 1) = \sum_{i=1}^{m-1} p_i^q T_i^*(T_i'(1)) \geq 0$. Consequently, letting q tend to 0 we get $\sum_{i=1}^{m-1} p_i' T_i^*(T_i'(1)) \geq 0$.

Case 3: $\tau > \max(T, \tau_\nu)$ near $0+$.

Assume at first that $h_* \in (0, 1]$. Letting q tend to $0+$ in the equality $g(q, s(q)) = 0$ we obtain $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(0)) = -\sum_{i=0}^{m-1} p_i' T_i(0) = 0$. Then, applying (6.1) and (6.3) at q close enough to $0+$ and letting q tend to 0 we obtain $\tau'(0+) = -\sum_{i=0}^{m-1} p_i'(\log_m(p_i) - T_i'(0))$; we then set $s_0 = 0$.

Next assume that $h_* = 0$. From the discussion of the continuity of τ at 0 we deduce that $\tau(0) = \tau_\nu(0)$. Next, consider a sequence $(q_n)_{n\geq 1}$ converging to $0+$ such that $s(q_n)$ (which belongs to $(q_n, 1)$) tends to $s_0 \in [0, 1]$. From the equality $g(q_n, s(q_n)) = 0$ we deduce that $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s_0)) = 0$ by letting n tend to ∞. Moreover, using (6.1) and (6.3) with q_n and letting n tend to ∞ yields $\tau'(0+) = -\sum_{i=0}^{m-1} p_i'(\log_m(p_i) - T_i'(s_0))$. $\square$

6.5. **Differentiability at 1.** Due to (6.1), if $q < s(q) < 1$ in a left neighborhood of 1, by (6.1), $\tau'(1-) = T'(1)$. This, together with the facts that $\tau \geq \max(\tau_\nu, T)$ over $[0, 1]$ and $\tau(1) = T(1) = \tau_\nu(1)$ implies that $\tau_\nu'(1) \geq T'(1)$. Then, if the last inequality is strict, $\tau_\nu > T$ hence $\tau = T$ on a right neighborhood of 1, which yields the differentiability of τ at 1. If $\tau_\nu'(1) = T'(1)$, then $\min(\tau_\nu, T)$ must have a derivative equal to $T'(1)$ on the right of 1, and we get the desired conclusion as well.

If $s(q) = 1$ in a left neighborhood 1, then there we have $\tau = \tau_\nu \geq T$, and $\tau'(1-) = \tau_\nu'(1)$. Then a similar argument as in the previous case (with the roles of τ_ν and T exchanged) yields the existence of $\tau'(1)$.

The case $s(q) = q$ in a left neighborhood 1 is treated similarly.

In conclusion, we get

$$\tau'(1) = \begin{cases} T'(1) & \text{if } T'(1) \leq \tau_\nu'(1) \\ \tau'(1) = \tau_\nu'(1) & \text{otherwise} \end{cases}. \tag{6.5}$$

6.6. **Differentiability and concavity over $(1, q_c)$.** Recall that q_c is defined in (2.4). The definition of τ clearly implies its concavity and differentiability at points at which the graphs of τ_ν and T do not cross transversally. Due to the analyticity of τ_ν and T, there are at most finitely many such points in a given bounded interval.

7. Proof of Theorem 3.7: Lower bound for the L^q-spectrum

Recall that $T(q)$, $\tau_\nu(q)$ and $\tau(q)$ are defined/given in (2.2), (3.5) and (3.9), respectively.

Proposition 7.1. *With probability 1, conditional on $\{\pi_*\mu \neq 0\}$,*

(1) *for all $q \geq 1$ the following properties hold:*
  (i) $\tau_{\pi_*\mu}(q) \leq \tau_\mu(q)$*;*
  (ii) *if $T(q) > 0$, then $\tau_{\pi_*\mu}(q) \geq \min(\tau_\nu(q), T(q))$;*
  (iii) *if $T^*(T'(q)) \geq 0$ then $T(q) > 0$. If, in addition, $\min(\tau_\nu(q), T(q)) = T(q)$, then $\tau_{\pi_*\mu}(q) = T(q)$.*

(2) *For all $0 < q \leq 1$, $\tau_{\pi_*\mu}(q) \geq \tau(q)$.*

Since, as a L^q-spectrum, the function $\tau_{\pi_*\mu}$ is continuous over $(0,\infty)$ and τ, τ_ν and T are continuous, to prove the above proposition we only need to get the desired inequalities for each $q > 0$.

Proof of Proposition 7.1. (1) (i) The fact that $\tau_{\pi_*\mu}(q) \leq \tau_\mu(q)$ for $q \geq 1$ is general and comes from the super-additivity of $x \mapsto x^q$ over $\mathbb{R}_+$ applied to $\Big(\pi_*\mu([u]) = \sum_{v\in\Sigma_n} \mu([u,v])\Big)^q$.

(ii) The almost sure inequality $\tau_{\pi_*\mu}(q) \geq \min(\tau_\nu(q), T(q))$ for a given $q \geq 1$ such that $T(q) > 0$ is a direct consequence of Corollary 9.10.

(iii) Let $q \geq 1$ be such that $T^*(T'(q)) \geq 0$ and suppose that $T(q) \leq 0$. Recall that T is concave, so its derivative is non-increasing. Also, $T(1) = 0$ and $T'(1) > 0$. This implies that T' is negative at some point of $(1, q)$, otherwise T could not take non-positive values over $(1, q]$. Since T' is non-increasing, it follows that T has a unique zero q_0 over $(1, q]$ at which $T'(q_0) < 0$. This implies that $T^*(T'(q_0)) = q_0 T'(q_0) < 0$. Since $T^*(T')$ is non-increasing on $\mathbb{R}_+$ (its derivative is $q \mapsto qT''(q)$), we get $T^*(T'(q)) \leq T^*(T'(q_0)) < 0$, which is a contradiction. So $T(q) > 0$.

Now recall that by Theorem 2.1, $T(q) = \tau_\mu(q)$ as soon as $T^*(T'(q)) \geq 0$. Thus, if $\min(\tau_\nu(q), T(q)) = T(q)$, the equality $\tau_{\pi_*\mu}(q) = T$ comes from (i) and (ii).

(2) Recall that $\pi_*\mu([u]) = \nu([u])X(u)$ for $u \in \Sigma_n$, where

$$X(u) = \sum_{v\in\Sigma_n} \frac{\mu([u,v])}{\nu([u])} = \sum_{v\in\Sigma_n} Y(u,v) \prod_{k=1}^{n} V_{u_k,v_k}(u_{|k-1}, v_{|k-1}),$$

as defined in (5.2). For $0 < q \leq 1$ and $q \leq s \leq 1$, using Jensen's inequality, for each $n \geq 1$ we get

$$\begin{aligned}\mathbb{E}\Big(\sum_{u\in\Sigma_n} \nu([u])^q X(u)^q\Big) &= \mathbb{E}\Big(\sum_{u\in\Sigma_n} \nu([u])^q X(u)^{s\cdot q/s}\Big) \\ &\leq \sum_{u\in\Sigma_n} \nu([u])^q \mathbb{E}(X(u)^s)^{q/s}.\end{aligned}$$

Then, using the fact that $\mathbb{E}(Y^s) \leq \mathbb{E}(Y)^s = 1$, and the branching property, we obtain

$$\begin{aligned}
\sum_{u\in\Sigma_n} \nu([u])^q \mathbb{E}(X(u)^s)^{q/s} &= \sum_{u\in\Sigma_n} \nu([u])^q \mathbb{E}\Big(\Big(\sum_{v\in\Sigma_n} \frac{\mu([u,v])}{\nu([u])}\Big)^s\Big)^{q/s} \\
&\leq \sum_{u\in\Sigma_n} \nu([u])^q \mathbb{E}\Big(\sum_{v\in\Sigma_n} \Big(\frac{\mu([u,v])}{\nu([u])}\Big)^s\Big)^{q/s} \\
&= \sum_{u\in\Sigma_n} \nu([u])^q \mathbb{E}\Big(\sum_{v\in\Sigma_n} Y(u,v)^s \prod_{k=1}^n V_{u_k,v_k}(u_{|k-1}, v_{|k-1})^s\Big)^{q/s} \\
&= \mathbb{E}(Y^s)^{q/s} \sum_{u\in\Sigma_n} \prod_{k=1}^n p_{u_k}^q m^{-T_{u_k}(s)q/s} \leq \Big(\sum_{i=0}^{m-1} p_i^q m^{-qT_i(s)/s}\Big)^n,
\end{aligned}$$

where p_i and T_i are defined as in (3.1)-(3.2). Since this holds for all $s \in [q, 1]$, for each $n \geq 1$ we obtain

$$\mathbb{E}\Big(\sum_{u\in\Sigma_n} \nu([u])^q X(u)^q\Big) \leq \Big(\inf_{q\leq s\leq 1} \sum_{i=0}^{m-1} p_i^q m^{-qT_i(s)/s}\Big)^n.$$

Consequently, Lemma C.1 yields that $\tau_{\pi_*\mu}(q) \geq \tau(q)$ almost surely. $\square$

8. Proof of Theorem 3.7: Upper bound for the L^q-spectrum and validity of the multifractal formalism

Recall that $\widetilde{q}_c$ and τ are defined in (3.8), (3.9). Proposition 7.1 and the fact that both $\tau(0)$ and $\tau_{\pi_*\mu}(0)$ equal the box counting dimension of $\pi(K)$ yield the following lemma.

Lemma 8.1. *With probability 1, conditional on $\{\mu \neq 0\}$, $\tau_{\pi_*\mu} \geq \tau$ over $[0, \widetilde{q}_c)$.*

Consequently, due to the general inequality $\dim E(\pi_*\mu, \alpha) \leq \tau^*_{\pi_*\mu}(\alpha)$, valid for all α, to prove the validity of the multifractal formalism at any $\alpha \in [\tau'(q^+), \tau'(q^-)]$ for some $q \in (0, \widetilde{q}_c)$ or at $\alpha = \tau'(0+)$ almost surely, as well as the almost sure equality of $\tau_{\pi_*\mu} = \tau$ over $[0, \widetilde{q}_c)$, it is enough to show that, for each $q \in [0, \widetilde{q}_c)$, with probability 1, conditional on $\{\mu \neq 0\}$, $\dim E(\pi_*\mu, \alpha) \geq \tau^*(\alpha)$ for $\alpha \in [\tau'(q^+), \tau'(q^-)]$ if $q > 0$ and $\alpha = \tau'(0+)$ if $q = 0$.

Indeed, once this is done, we automatically have that almost surely, conditional on $\{\mu \neq 0\}$, $\tau^*(\alpha) = \alpha q - \tau(q) \leq \dim E(\pi_*\mu, \alpha) \leq \tau^*_{\pi_*\mu}(\alpha) \leq \alpha q - \tau_{\pi_*\mu}(q) \leq \alpha q - \tau(q)$. Moreover, the information $\dim E(\pi_*\mu, \alpha) \geq \tau^*(\alpha)$ for $\alpha = \tau'(q)$, where q describes a dense countable subset of values of q, is enough to get the equality $\tau = \tau_{\pi_*\mu}$ over $[0, \widetilde{q}_c)$. Also, the fact $\tau_{\pi_*\mu}(q) = qT'(q_c-)$ for $q \geq q_c$ when $\widetilde{q}_c = q_c < \infty$ follows from Proposition 10.4.

Then, to get (3.10) for $q \in (0, \widetilde{q}_c)$, we notice that if $\alpha \in \{\tau'(q^+), \tau'(q^-)\}$, for any $\epsilon > 0$ and large enough n, $\#\{u \in \Sigma_n : \pi_*\mu([u]) \geq m^{-n(\alpha+\epsilon)}\} \geq m^{n(\tau^*(\alpha)-\epsilon)}$, for otherwise a simple covering argument would give $\dim E(\pi_*\mu,\alpha)<\tau^*(\alpha)$. This implies

$$\sum_{|u|=n}\mathbf{1}_{\{\pi_*\mu([u])>0\}}\pi_*\mu([u])^q\ge m^{n(\tau^*(\alpha)-\epsilon)}m^{-nq(\alpha+\epsilon)}\ge m^{-n(\tau(q)+(q+1)\epsilon)}.$$

Since ϵ is arbitrary, this yields $\limsup_{n\to\infty}-\frac{1}{n}\log_m\sum_{|u|=n}\mathbf{1}_{\{\pi_*\mu([u])>0\}}\pi_*\mu([u])^q\le\tau(q)$. Moreover, we already know (by Proposition 7.1) that

$$\tau_{\pi_*\mu}(q)=\liminf_{n\to\infty}-\frac{1}{n}\log_m\sum_{|u|=n}\mathbf{1}_{\{\pi_*\mu([u])>0\}}\pi_*\mu([u])^q\ge\tau(q).$$

This yields (3.10) and the equality $\tau_{\pi_*\mu}(q)=\tau(q)$ for $q>0$. The case when $q=0$ just comes from the fact that $\dim_H K=\dim_B K$.

In the remaining part of this section we will prove the desired inequality (that is, with probability 1, conditional on $\{\mu\neq 0\}$, $\dim E(\pi_*\mu,\alpha)\ge\tau^*(\alpha)$) by distinguishing the following 4 cases:

Case (I): $\alpha=T'(q)$ and $\tau(q)=T(q)$ with $q\in(0,q_c)\setminus\{1\}$.
Case (II): $\alpha=\tau'(q)$ with $\tau(q)\neq T(q)$ and $q\in(0,\widetilde{q}_c)\setminus\{1\}$, or $\alpha\in\{\tau'(q^+),\tau'(q^-)\}$ with $q\in(1,q_c)$ and a first order phase transition occurs at q.
Case (III): $\alpha\in(\tau'(q^+),\tau'(q^-))$ with $q\in(1,q_c)$ and a first order phase transition occurs at q.
Case (IV): $\alpha=\tau'(0+)$.

Remark 8.2. To follow the different cases distinguished above, it is useful to have the following properties in mind.

(1) If $\alpha=\tau'(1)$, our study of the exact dimensionality of $\pi_*\mu$ and (6.5) show that $\dim E(\pi_*\mu,\alpha)=\alpha=\tau^*(\alpha)$ almost surely conditional on $\{\mu\neq 0\}$.
(2) The study of the differentiability of τ achieved in Section 6 shows that if $q\in(0,1)$ then either $\tau(q)=T(q)$ and $\tau'(q)=T'(q)$ or $\tau(q)=\tau_\nu(q)$ and $\tau'(q)=\tau_\nu'(q)$.
(3) Simple considerations about the concave function $\min(\tau_\nu,T)$ show that at $q\in(1,q_c)$, if $\tau(q)=T(q)<\tau_\nu(q)$ then $\tau'(q)=T'(q)$, if $q\in(1,\widetilde{q}_c)$ and $\tau(q)=\tau_\nu(q)<T(q)$ then $\tau'(q)=\tau_\nu'(q)$, and if $q\in(1,q_c)$ and $\tau(q)=T(q)=\tau_\nu(q)$, then $\{\tau'(q^+),\tau'(q^-)\}=\{T'(q),\tau_\nu'(q)\}$.

8.1. **Case (I).** To begin with we recall some known facts about the multifractal analysis of μ.

For $q\ge 0$, let μ_q be the Mandelbrot measure built with the random vectors

$$W_q(u,v)=(m^{T(q)}W_{i,j}(u,v)^q)_{0\le i,j\le m-1},\quad (u,v)\in\bigcup_{n\ge 0}\Sigma_n\times\Sigma_n.$$

According to the study in [5], with probability 1, conditional on $\{\mu\neq 0\}$, all the Mandelbrot measures μ_q, $q\in[0,q_c)$, are defined simultaneously, moreover, $\dim(\mu_q)=T^*(T'(q))>0$ and $E(\mu,T'(q))$ is of full μ_q-measure.

Proposition 8.3. *Fix $q \in (0, q_c) \setminus \{1\}$ such that $\tau(q) = T(q)$. With probability 1, conditional on $\{\mu \neq 0\}$, $\dim(\pi_*\mu_q) = T^*(T'(q))$.*

The following corollary is our main goal.

Corollary 8.4. *Fix $q \in (0, q_c)\setminus\{1\}$ such that $\tau(q) = T(q)$. With probability 1, conditional on $\{\mu \neq 0\}$, $\dim E(\pi_*\mu, T'(q)) \geq T^*(T'(q)) = \tau^*(T'(q))$.*

We start with the proof of the corollary, assuming Proposition 8.3.

Proof of Corollary 8.4. Suppose $\mu \neq 0$. We first show that $\tau'_{\pi_*\mu}(0+) \geq \tau'(0+) \geq T'(q)$. To see this, observe at first that $\tau_{\pi_*\mu}(0) = \tau(0)$ since $-\tau'_{\pi_*\mu}(0)$ is the upper box dimension of $\pi(K)$ and by (1.2), $-\tau(0) = \dim_B \pi(K)$. Since, moreover, $\tau_{\pi_*\mu} \geq \tau$ over $(0, 1]$ by Proposition 7.1(2), we get the first inequality that $\tau'_{\pi_*\mu}(0+) \geq \tau'(0+)$. Now if $\tau'(0+) < T'(q)$, the equality $\tau(q) = T(q)$ yields

$$T^*(T'(q)) = qT'(q) - T(q) > q\tau'(0+) - \tau(q) \geq \tau^*(\tau'(0+)) = -\tau(0) = \dim_B(\pi(K)).$$

However, by Proposition 8.3, $\dim(\pi_*\mu_q) = T^*(T'(q))$, so $\dim(\pi_*\mu_q) > \dim_B \pi(K)$, which is impossible since $\pi_*\mu_q$ is supported on $\pi(K)$. Thus $\tau'(0) \geq T'(q)$.

There is a subset F_q of $\mathrm{supp}(\mu)$ of full μ_q-measure such that for all $t \in F_q$, $\dim_{\mathrm{loc}}(\mu_q, t) = \dim_{\mathrm{loc}}(\pi_*\mu_q, \pi(t)) = T^*(T'(q))$ (by Proposition 8.3 and the fact that $\dim(\mu_q) = T^*(T'(q))$) and $\dim_{\mathrm{loc}}(\mu, t) = T'(q)$ (by the multifractal analysis of μ [5]). This implies that for all $t \in F_q$, $\underline{\dim}_{\mathrm{loc}}(\pi_*\mu, \pi(t)) \leq \overline{\dim}_{\mathrm{loc}}(\pi_*\mu, \pi(t)) \leq \overline{\dim}_{\mathrm{loc}}(\mu, t) = T'(q)$. On the other hand, since $T'(q) \leq \tau'_{\pi_*\mu}(0+)$, it follows that for all $\alpha' < T'(q)$, by (2.1),

$$\begin{aligned} \dim\{x \in \mathrm{supp}(\pi_*\mu) : \underline{\dim}_{\mathrm{loc}}(\pi_*\mu, x) \leq \alpha'\} &\leq \tau^*_{\pi_*\mu}(\alpha') \leq \alpha' q - \tau_{\pi_*\mu}(q) \\ &< T'(q)q - \tau_{\pi_*\mu}(q) \\ &\leq T'(q)q - \tau(q) = T^*(T'(q)). \end{aligned}$$

Consequently, since the family $(\underline{E}(\pi_*\mu, \alpha'))_{\alpha'<T'(q)}$ is non-decreasing and $\dim(\pi_*\mu_q) = T^*(T'(q))$, we get $\pi_*\mu_q\big(\bigcup_{\alpha'<T'(q)} \underline{E}(\pi_*\mu, \alpha')\big) = 0$. Now, set $\widetilde{F}_q = \pi(F_q)\setminus\bigcup_{\alpha'<T'(q)} \underline{E}(\pi_*\mu, \alpha')$. By construction, $\widetilde{F}_q \subset E(\pi_*\mu, T'(q))$ and $\pi_*\mu_q(\widetilde{F}_q) = 1$. Finally

$$\dim E(\pi_*\mu, T'(q)) \geq \dim \widetilde{F}_q \geq \dim \mu_q = T^*(T'(q)).$$

Moreover, by Remark 8.2, if $q \leq 1$ then $\tau'(q) = T'(q)$, and if $q > 1$, then $T'(q) \in \{\tau'(q^+), \tau'(q^-)\}$, so $T^*(T'(q)) = \tau^*(T'(q))$. This completes the proof of the corollary. □

Proposition 8.3 is a consequence of Theorem 3.3 and the following lemma.

Lemma 8.5. *If $q \in (0, q_c)\setminus\{1\}$ and $\tau(q) = T(q)$, then, conditional on $\{\mu_q \neq 0\}$, $\dim(\mu_q) = T^*(T'(q)) \leq \dim(\mathbb{E}(\pi_*\mu_q))$.*

Proof. We first show that for any $q \in (0, q_c)$, almost surely, conditional on $\{\mu_q \neq 0\}$,

$$\dim(\mu_q) - \dim(\mathbb{E}(\pi_*\mu_q)) = \sum_{i=0}^{m-1} p_i^q m^{T(q)-T_i(q)} T_i^*(T_i'(q)). \tag{8.1}$$

To see this, notice that $\mathbb{E}(\pi_*\mu_q)$ is a Bernoulli product measure on Σ generated by the probability vector $(p_0', \ldots, p_{m-1}')$ with

$$p_i' := \sum_{j=0}^{m-1} m^{T(q)} \mathbb{E}(W_{i,j}^q) = p_i^q m^{T(q)-T_i(q)}.$$

A simple computation yields that

(8.2)

$$\begin{aligned}\dim(\mathbb{E}(\pi_*\mu_q)) &= -\frac{1}{\log m} \sum_{i=0}^{m-1} p_i' \log p_i' \\ &= -T(q) + \left(\sum_{i=0}^{m-1} p_i^q m^{T(q)-T_i(q)} T_i(q) \right) - \frac{q}{\log m} \left(\sum_{i=0}^{m-1} p_i^q (\log p_i) m^{T(q)-T_i(q)} \right).\end{aligned}$$

In the meantime, since $\sum_{i=0} p_i^q m^{T(q)-T_i(q)} = 1$, differentiating with respect to q yields

$$T'(q) = \left(\sum_{i=0}^{m-1} p_i^q m^{T(q)-T_i(q)} T_i'(q) \right) - \frac{1}{\log m} \left(\sum_{i=0}^{m-1} p_i^q (\log p_i) m^{T(q)-T_i(q)} \right). \tag{8.3}$$

Since $\dim(\mu_q) = T^*(T'(q)) = T'(q)q - T(q)$ almost surely, by (8.2) and (8.3) we obtain (8.1).

Next we show that if $\tau(q) = T(q)$ for some $q \in (0, q_c) \setminus \{1\}$, then

$$\sum_{i=0}^{m-1} p_i^q m^{T(q)-T_i(q)} T_i^*(T_i'(q)) \leq 0.$$

We consider the cases $q \in (1, q_c)$ and $0 < q < 1$ separately. First suppose $q \in (1, q_c)$. Recall that $\pi_*\mu([u]) = \nu([u])X(u)$ for $u \in \Sigma_n$, where

$$X(u) = \sum_{v \in \Sigma_n} \frac{\mu([u,v])}{\nu([u])} = \sum_{v \in \Sigma_n} Y(u,v) \prod_{k=1}^{n} V_{u_k, v_k}(u_{|k-1}, v_{|k-1}).$$

For $1 \leq s \leq q$ and $n \geq 1$, using Jensen's inequality and calculations similar to those displayed in the proof of Proposition 7.1(2) (with reversed inequalities), as well as the fact that $\mathbb{E}(Y^s)^{1/s} \geq \mathbb{E}(Y) = 1$, we can get that

$$\mathbb{E}\Big(\sum_{u \in \Sigma_n} \pi_*\mu([u])^q \Big) \geq \Big(\sum_{i=0}^{m-1} p_i^q m^{-qT_i(s)/s} \Big)^n.$$

Consequently, due to Corollary 9.9,

$$-\tau(q) = \max(-\tau_\nu(q), -T(q)) \geq \sup_{1 \leq s \leq q} \log_m \sum_{i=0}^{m-1} p_i^q m^{-qT_i(s)/s}.$$

Since $\tau(q) = T(q)$, this implies that the supremum is reached at $s = q$. Differentiating with respect to s at $s = q$ then yields $\sum_{i=0}^{m-1} p_i^q m^{-T_i(q)} T_i^*(T_i'(q)) \leq 0$, hence $\sum_{i=0}^{m-1} p_i^q m^{T(q)-T_i(q)} T_i^*(T_i'(q)) \leq 0$.

Finally, suppose that $0 < q < 1$. By the definition of τ, the condition $\tau(q) = T(q)$ also implies that the following infimum

$$\inf_{q \leq s \leq 1} \log_m \sum_{i=0}^{m-1} p_i^q m^{-qT_i(s)/s}$$

is attained at q. Hence differentiating with respect to s at $s = q$ yields

$$\sum_{i=0}^{m-1} p_i^q m^{-T_i(q)} T_i^*(T_i'(q)) \leq 0.$$

This completes the proof of the lemma. □

8.2. **Case (II).**

In this section, we suppose that we do not have $\tau \equiv \tau_\nu \equiv T$ over $[0, \widetilde{q}_c)$, i.e. we are not in the case where for each $0 \leq i \leq m-1$ such that $p_i > 0$ the function T_i is equal to 0.

We will use the notation of Section 6. Recall that for $q \in (0, 1]$, $s(q)$ is the unique $s \in [q, 1]$ at which $\sum_{i=1}^{m-1} p_i^q m^{-q(T_i(s)/s)}$ gets minimized on $[q, 1]$. For $q > 1$, we define $s(q) = 1$ if $\tau(q) = \tau_\nu(q)$ holds. Also we recall Remark 8.2.

For $q \in (0, \widetilde{q}_c)$ such that $s(q)$ is defined, for $0 \leq i \leq m-1$ set

$$p_i' = p_{q,i}' = m^{\tau(q)} p_i^q m^{-qT_i(s(q))/s(q)}.$$

Also let $\nu' = \nu_q'$ be the Bernoulli measure associated with $p' = (p_0', \dots, p_{m-1}')$.

For $s > 0$ and $0 \leq i, j \leq m-1$, set

$$V_{s,i,j}' = \mathbf{1}_{\{V_{i,j} > 0\}} V_{i,j}^s m^{T_i(s)}, \tag{8.4}$$

so that for $q' \geq 0$

$$T_{V_{s,i}'}(q') := -\log_m \sum_{j=0}^{m-1} \mathbb{E}({V_{s,i,j}'}^{q'}) = T_i(q's) - q'T_i(s).$$

Set $W_s' = (W_{s,i,j}' = p_i' V_{s,i,j}')_{0 \leq i,j \leq m-1}$. We have

$$T_{W_s'}(q') = \sum_{i=0}^{m-1} (p_i')^{q'} m^{-T_{V_{s,i}'}(q')}.$$

For all $(u, v) \in \bigcup_{n \geq 1} \Sigma_n \times \Sigma_n$, let $W_{s,i,j}'(u, v) = \left(p_i' \mathbf{1}_{\{V_{i,j}(u,v) > 0\}} V_{i,j}(u, v)^s m^{T_i(s)} \right)_{0 \leq i,j \leq m-1}$. This family of random weights generates a Mandelbrot mesure $\mu_{W_s'}$ simultaneously with μ_W.

We start with a first lemma.

Lemma 8.6. (1) *If $q \in (0, 1)$ and $s(q) \in (0, 1)$, then $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) > 0$ for all $s \in (0, s(q))$.*

(2) *If $q \in (0,\widetilde{q}_c)\setminus\{1\}$ and $s(q)=1$, then either $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) = 0$ for all $s \in [0,1]$, or $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) > 0$ for all $s \in (0,1)$ according to whether T_i is affine (and equal to $q \mapsto (q-1)\log_m(\mathbb{E}(N_i))$ for each i such that $p_i > 0$ and $\sum_{i=0}^{m-1} p_i' T_i'(1) = 0$, or not.*

Moreover, either the set $\widetilde{S}$ of those $q \in (0,\widetilde{q}_c)$ for which $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) = 0$ for all $s \in [0,1]$ is discrete or it is equal to $(0,\widetilde{q}_c)$). The later case holds if and only if property $(\mathcal{P})$ of Remark 3.10(2) holds. In particular, T is finite over $\mathbb{R}_+$, $\widetilde{q}_c = \infty$, $\tau = \tau_\nu > T$ over $(0,1)$ and $\tau = \tau_\nu < T$ over $(1,\infty)$.

Proof. (1) Suppose $q \in (0,1)$ and $s(q) \in (q,1)$. The study of the differentiability of τ achieved in Section 6.1 yields $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s(q))) = m^{\tau(q)} g(q,s(q)) = 0$ and since $\frac{\partial g}{\partial s}(q,s(q)) < 0$, we have $g(q,s) = m^{-\tau(q)} \sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) > 0$ for all $s \in (0,s(q))$.

(2) First suppose that $q \in (0,1)$ and $s(q) = 1$. That means that $\tau(q) = \tau_\nu(q)$. Here again, we can use the study of τ to get that $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(1)) = \sum_{i=0}^{m-1} p_i' T_i'(1) = m^{\tau_\nu(q)} g(q,1) \geq 0$. Now, notice that the derivative of $s \mapsto \sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s))$ is $s \mapsto \sum_{i=0}^{m-1} p_i' s T_i''(s)$. If one of the T_i is not affine, then by an argument given in the study of the differentiability of τ we can prove that T_i'' is strictly negative so $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) > 0$ for all $s \in (0,1)$. Otherwise, the function $\sum_{i=0}^{m-1} p_i' T_i^* \circ T_i'$ is identically equal to 0 over its domain by analyticity.

Suppose now that $q \in (1,\widetilde{q}_c)$ and $s(q) = 1$. The condition $\tau(q) = \tau_\nu(q) \leq T(q)$ implies that $\sum_{i=0}^{m-1} p_i' m^{-T_i(q)} = \sum_{i=0}^{m-1} p_i^q m^{\tau_\nu(q) - T_i(q)} \leq 1$. Since, moreover, $\sum_{i=0}^{m-1} p_i' m^{-T_i(1)} = 1$, by convexity of $q \mapsto \sum_{i=0}^{m-1} p_i' m^{-T_i(q)}$, we must have $\sum_{i=0}^{m-1} p_i' T_i'(1) \geq 0$. Then, the same arguments as in previous paragraph yield the same conclusion.

For each q such that $s(q) = 1$ and $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) = 0$ for all $s \in [0,1]$, the functions T_i are linear and $p_i' = p_i^q m^{\tau_\nu(q)}$, so $\sum_{i=0}^{m-1} p_i^q T_i'(1) = m^{-\tau_\nu(q)} \sum_{i=0}^{m-1} p_i' T_i^*(T_i'(1)) = 0$ and $\sum_{i=0}^{m-1} p_i^q \log(\mathbb{E}(N_i)) = m^{-\tau_\nu(q)} \sum_{i=0}^{m-1} p_i' T_i^*(T_i'(0)) = 0$. If the set of such points q has an accumulation point, then by analyticity, we must have $\sum_{i=0}^{m-1} p_i^q \log(\mathbb{E}(N_i)) = 0$ for all q. It is then not hard to conclude that property $(\mathcal{P})$ of Remark 3.10(2) holds. Then, T is finite over $\mathbb{R}_+$, and the study of $\inf_{q \leq s \leq 1} \log_m \sum_{i=0}^{m-1} p_i^q m^{-qT_i(s)/s}$ for $q \in (0,1)$ and $\sup_{1 \leq s \leq q} \log_m \sum_{i=0}^{m-1} p_i^q m^{-qT_i(s)/s}$ for $q \in (1,\infty)$ shows that both are uniquely reached at $s = 1$, so $\tau = \tau_\nu > T$ over $(0,1)$ and $\tau = \tau_\nu < T$ over $(1,\infty)$. □

Lemma 8.7. *Let $q \in (0,\widetilde{q}_c)$ be such that $s(q)$ is defined. Suppose that $s > 0$ is such that $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) \geq 0$. With probability 1, the Mandelbrot measure $\mu_{W_s'}$ has the same topological support as μ. If, moreover, $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) > 0$ then, conditional on $\{\mu_{W_s'} \neq 0\}$, the measure $\pi_*\mu_{W_s'}$ is absolutely continuous with respect to ν'. In particular, $\nu'(\pi(K)) > 0$.*

Proof. To begin with we notice that $\sum_{i=0}^{m-1} p_i' T'_{V'_{s,i}}(1-) = \sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s))$. Thus, due to (3.4) our assumption implies $T'_{W'_s}(1-) \geq \dim(\nu') > 0$, hence $\mu_{W'_s}$ is non-degenerate. Moreover, since the weights $W'_{s,i,j}$ and $W_{i,j}$ vanish simultaneously, Proposition A.1 shows that $\mu_{W'_s}$ and μ have almost surely the same topological support.

If, moreover, $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) > 0$, then $T'_{W'_s}(1-) > \dim(\nu')$ and by Theorem 3.1(1)(a), this implies that $\pi_* \mu_{W'_s}$ is almost surely absolutely continuous with respect to $\mathbb{E}(\pi_* \mu_{W'_s}) = \nu'$, so $\nu'(\pi(K)) > 0$. $\square$

Now, for $q \in (0, \widetilde{q}_c) \setminus \{1\}$, if $s(q) < 1$ or if $s(q) = 1$ and $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) > 0$ for all $s \in (0,1)$, let $\widetilde{\nu}_q = \nu'_q$. Otherwise, i.e. if $q \in \widetilde{S}$ ($\widetilde{S}$ is defined in Lemma 8.6) set $\widetilde{\nu}_q = \pi_* \mu_{W'_1}$ (recall that this Mandelbrot measure is defined before Lemma 8.6 and it has the same topological support as μ almost surely by Lemma 8.7). The main result of this section is the following.

Proposition 8.8. *Let $q \in (0, \widetilde{q}_c) \setminus \{1\}$ at which $\tau(q) \neq T(q)$ or $q \in (1, q_c)$ at which $\tau(q) = \tau_\nu(q) = T(q)$. Set $\alpha = \tau'(q)$ if $s(q) < 1$ and $\alpha = \tau'_\nu(q)$ otherwise. Then with probability 1, conditional on $\{\mu \neq 0\}$, $\widetilde{\nu}_q(E(\pi_*\mu, \alpha)) > 0$, and $\dim(\widetilde{\nu}_q) = \tau^*(\alpha)$; consequently, $\dim_H E(\pi_*\mu, \alpha) \geq \tau^*(\alpha)$.*

From now on we fix $q \in (0, \widetilde{q}_c) \setminus \{1\}$ at which $\tau(q) \neq T(q)$ or $\tau(q) = \tau_\nu(q) = T(q)$.

Lemma 8.9. *Suppose that $\widetilde{\nu}_q = \nu'$. Let $\mathscr{S}$ stand for a maximal open interval of points $s > 0$ such that $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) > 0$ and $\mathbb{E}(Y^s) < \infty$. With probability 1, conditional on $\{\mu \neq 0\}$, for ν'-almost every x in $\pi(K)$, for all $s \in \mathscr{S}$,*

$$\lim_{n\to\infty} \frac{-1}{n} \log_m \sum_{v \in \Sigma_n} \Big(\frac{\mu([x_{|n}, v])}{\nu(x_{|n})} \Big)^s = \sum_{i=0}^{m-1} p_i' T_i(s).$$

Proof. By convexity, we only need to check this for each s in a dense countable subset $\mathcal{S}$ of $\mathscr{S}$. Indeed, if this is done, there exists a subset of $\{\mu \neq 0\}$ of probability $\mathbb{P}(\mu \neq 0)$ such that the sequence of concave functions $\frac{-1}{n} \log_m \sum_{v \in \Sigma_n} \Big(\frac{\mu([x_{|n}, v])}{\nu(x_{|n})} \Big)^s$ converge pointwise on $\mathcal{S}$, and this is enough to get the convergence over $\mathscr{S}$.

Fix $s \in \mathcal{S}$. For $n \geq 1$ and x in the topological support of ν', set

$$\begin{aligned} Z_{s,n}(x) &= \Big(\prod_{k=1}^n m^{T_{x_k}(s)} \Big) \sum_{v \in \Sigma_n} \Big(\frac{\mu([x_{|n}, v])}{\nu([x_{|n}])} \Big)^s \\ &= \sum_{v \in \Sigma_n} Y(x_{|n}, v)^s \cdot \prod_{k=1}^n m^{T_{x_k}(s)} V_{x_k, v_k}(x_{|k-1}, v_{|k-1})^s. \end{aligned}$$

Define $V'_{s,i}$, $0 \leq i \leq m-1$, as in (8.4). Since $\mathscr{S}$ is open, we have $\mathbb{E}(Y^{q's}) < \infty$ and $\sum_{i=0}^{m-1} p_i' T_{V'_{s,i}}(q') > -\infty$ for some $q' > 1$, and since $\sum_{i=0}^{m-1} p_i' T'_{V'_{s,i}}(1) = \sum_{i=0}^{m-1} p_i' T_i^*(T_i'(s)) > 0$, we also have $\sum_{i=0}^{m-1} p_i' T_{V'_{s,i}}(q') > 0$ if q' is close enough to 1. By Proposition B.2 applied with $\eta = \nu'$ and $U_i = V'_{s,i}$, the sequence $Z_{s,n}(x)$ converges $\mathbb{P}\otimes\nu'$ almost surely to the same non-degenerate limit $\widetilde{Z}_s(x)$ as the Mandelbrot martingale in random environment

$$\widetilde{Z}_{s,n}(x) = \sum_{v\in\Sigma_n} \prod_{k=1}^{n} m^{T_{x_k}(s)} V_{x_k,v_k}(x_{|k-1}, v_{|k-1})^s.$$

This random variable satisfies the equation

$$\widetilde{Z}_s(x) = \sum_{j=0}^{m-1} V_{x_1,j} \widetilde{Z}_s(\sigma x, j), \tag{8.5}$$

where the $\widetilde{Z}_s(\sigma x, j)$ are independent copies of $\widetilde{Z}_s(\sigma x)$, which are also independent of V_{x_1}.

Equation (8.5) shows that $\mathbb{P}(\{\widetilde{Z}_s(x) = 0\})$ is $\{f_i\}_{0\le i\le m-1; p_i>0}$-stationary (cf. Definition A.2), where f_i stands for the generating function of the random integer N_i. Moreover, we assumed from the beginning that there exists $0 \le i \le m-1$ such that $p_i > 0$ for which $\mathbb{P}(N_i = 1) < 1$. Consequently, Proposition A.3 shows that for ν'-almost every x, $\mathbb{P}(\{\widetilde{Z}_s(x) = 0\})$ is less than 1 (because $\widetilde{Z}_{s,n}(x)$ is non-degenerate) and independent of $s \in \mathcal{S}$.

Also, for each $s \in \mathcal{S}$, the event $\{\widetilde{Z}_s(x) = 0\}$ contains the event $\bigcup_{n\ge1}\{\widetilde{Z}_{s,n}(x) = 0\}$, which due to the definition of $\widetilde{Z}_{s,n}$ is independent of s and is equal to the extinction of the branching process defining the Galton-Watson tree in random environment $T_n(x) = \{v \in \Sigma_n :\ Q(x_{|n}, v) > 0\}$. In addition, the function $\mathbb{P}(\bigcup_{n\ge1}\{T_n(x) = \emptyset\})$ is $\{f_i\}_{0\le i\le m-1; p_i>0}$-stationnary as well, and it cannot be equal to 1 since it is smaller than or equal to $\mathbb{P}(\{\widetilde{Z}_s(x) = 0\})$. Consequently, we conclude that for ν'-almost every x, the event $\{\widetilde{Z}_s(x) > 0 \text{ for all } s \in \mathcal{S}\}$ equals $A_x = \bigcap_{n\ge1}(A_{x,n} := \{\{v \in \Sigma_n :\ Q(x_{|n}, v) > 0\} \neq \emptyset\})$ up to a set of probability 0.

We have

$$\begin{aligned}
&\int \nu'(\{x : Z_s(\omega, x) > 0 \text{ for all } s \in \mathcal{S} \text{ and } \omega \in A_x\})\, \mathbb{P}(\mathrm{d}\omega)\\
&= \mathbb{E}_{\mathbb{P}\otimes\nu'}(\mathbf{1}_{\{Z_s(\omega,x)>0 \text{ for all } s\in\mathcal{S} \text{ and } \omega\in A_x\}})\\
&= \int \mathbb{P}(Z_s(\omega, x) > 0 \text{ for all } s \in \mathcal{S} \text{ and } \omega \in A_x\})\, \nu'(\mathrm{d}x)\\
&= \int \mathbb{P}(A_x)\, \nu'(\mathrm{d}x).
\end{aligned}$$

Notice that the events $A_{x,n}$ are non-increasing so $\mathbb{P}(A_x = \bigcap_{n\geq 1} A_{x,n}) = \lim_{n\to\infty} \mathbb{P}(A_{n,x})$. Consequently,

$$\begin{aligned}
&\int \nu'(\{x : Z_s(\omega, x) > 0 \text{ for all } s \in \mathcal{S} \text{ and } \omega \in A_x\})\, \mathbb{P}(\mathrm{d}\omega) \\
&= \int \lim_{n\to\infty} \mathbb{P}(A_{x,n})\, \nu'(\mathrm{d}x) \\
&= \lim_{n\to\infty} \int \mathbb{P}(A_{x,n})\, \nu'(\mathrm{d}x) \\
&= \lim_{n\to\infty} \mathbb{E}(\nu'(\{x : \{v \in \Sigma_n :\ Q(x_{|n}, v) > 0\} \neq \emptyset\})) \\
&= \lim_{n\to\infty} \mathbb{E}(\nu'(\pi(K_n))) \\
&= \mathbb{E}(\nu'(\lim_{n\to\infty} \pi(K_n))) \\
&= \mathbb{E}(\nu'(\pi(K)).
\end{aligned}$$

Since the inclusion $\{x : Z_s(\omega, x) > 0\} \subset \pi(K(\omega))$ holds by construction, we obtained that $\nu'(\{x \in \pi(K(\omega)) : Z_s(\omega, x) > 0\ \forall\ s \in \mathcal{S}\} = \nu'(\pi(K(\omega)))$ almost surely. In other words, with probability 1, conditional on $\{\mu \neq 0\}$, for ν'-almost every x in $\pi(K)$, for all $s \in \mathcal{S}$ we have $\widetilde{Z}_s(x) > 0$. Finally, $\widetilde{Z}_s(x)$ is the positive limit of $Z_{s,n}(x)$. Since by definition we have

$$\sum_{v\in\Sigma_n} \Big(\frac{\mu([x_{|n}, v])}{\nu([x_{|n}])}\Big)^s = \Big(\prod_{k=1}^{n} m^{T_{x_k}(s)}\Big)^{-1} Z_{s,n}(x)$$

we conclude that

$$\lim_{n\to\infty} \frac{-1}{n} \log_m \sum_{v\in\Sigma_n} \Big(\frac{\mu([x_{|n}, v])}{\nu(x_{|n})}\Big)^s = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} T_i(s) = \sum_{i=0}^{m-1} p'_i T_i(s),$$

due to the ergodic theorem applied to ν'. □

Recall that $X_n(x) = \displaystyle\sum_{v\in\Sigma_n} \frac{\mu([x_{|n}, v])}{\nu(x_{|n})}$ for $x \in \Sigma$ and $n \geq 1$.

Lemma 8.10. *Suppose that $\widetilde{\nu}_q = \nu'$. Let*

$$s_0 = \sup\left\{ s > 0 : \sum_{i=0}^{m-1} p'_i T^*_i(T'_i(s)) > 0 \text{ and } \mathbb{E}(Y^s) < \infty \right\}.$$

With probability 1, for ν'-almost every $x \in \pi(K)$, either $\lim_{n\to\infty} -\frac{1}{n} \log_m X_n(x) = 0$ or $\lim_{n\to\infty} -\frac{1}{n} \log_m X_n(x) = \sum_{i=0}^{m-1} p'_i T'_i(s_0)$ according to whether $s_0 > 1$ or $s_0 \leq 1$.

Proof. We notice that $s_0 = s(q)$ when $s(q) < 1$. Due to the previous lemma, with probability 1, for ν'-almost every $x \in \mathrm{supp}(\pi(K))$, defining

$$\tau_x(s) = \liminf_{n\to\infty} \frac{-1}{n} \log_m \sum_{v\in\Sigma_n} \Big(\frac{\mu([x_{|n}, v])}{\nu([x_{|n}])}\Big)^s,$$

we have

$$\tau_x(s) = \lim_{n\to\infty} \frac{-1}{n} \log_m \sum_{v\in\Sigma_n} \Big(\frac{\mu([x_{|n}, v])}{\nu([x_{|n}])}\Big)^s = \sum_{i=0}^{m-1} p_i' T_i(s) \tag{8.6}$$

over $[0, s_0)$. On the other hand, we naturally have

$$\tau_x(s) \geq \widetilde{T}(s) := \sum_{i=0}^{m-1} p_i' T_i(s) \tag{8.7}$$

for all s. This is due to Lemma C.1 and the fact that

$$\mathbb{E}\Big(\sum_{v\in\Sigma_n} \Big(\frac{\mu([x_{|n}, v])}{\nu([x_{|n}])}\Big)^s\Big) = \prod_{k=1}^{n} m^{-T_{x_k}(s)}.$$

Now let us make a few remarks.

There exist $\alpha_0 < \beta_0$ in $\mathbb{R}$ such that, with probability 1, conditional on $\{\mu \neq 0\}$, $m^{-n\beta_0} \leq \frac{\mu([x_{|n},v])}{\nu([x_{|n}])} \leq m^{-n\alpha_0}$ for all $x \in \pi(K)$, $n \geq 1$ and $v \in \Sigma_n$ such that $\mu([x_{|n}, v]) > 0$. Indeed, for all $x \in \pi(K)$, $n \geq 1$ we already have $(\min\{p_i : p_i > 0\})^n \leq \nu(x_{|n}) \leq (\max\{p_i : p_i > 0\})^n$. Also, we can fix $\eta > 0$ such that $C_\eta = \max(\mathbb{E}(\mathbf{1}_{\{Y>0\}} Y^{-\eta}), \mathbb{E}(Y^\eta)) < \infty$. Then, for any $A > 0$ and $n \geq 1$,

$$\begin{aligned}
&\mathbb{P}\left(\exists\, (u, v) \in \Sigma_n \times \Sigma_n : 0 < \mu([u, v]) \leq m^{-nA} \text{ or } \mu([u, v]) \geq m^{nA}\right) \\
&\leq m^{-n\eta A}\mathbb{E}\Big(\sum_{(u,v)\in\Sigma_n\times\Sigma_n} \mathbf{1}_{\{Q(u,v)>0\}}\mathbf{1}_{\{Y(u,v)>0\}} Q(u, v)^{-\eta} Y(u, v)^{-\eta}\Big) \\
&\qquad + m^{-n\eta A}\mathbb{E}\Big(\sum_{(u,v)\in\Sigma_n\times\Sigma_n} Q(u, v)^{\eta} Y(u, v)^{\eta}\Big) \\
&\leq C_\eta m^{-n\eta A}\left(m^{-nT(-\eta)} + m^{-nT(\eta)}\right).
\end{aligned}$$

Hence, if A is large enough so that $A\eta + \min(T(-\eta), T(\eta)) > 0$, by the Borel-Cantelli lemma we get $m^{-nA} \leq \mu([x_{|n}, v]) \leq m^{nA}$ for all $x \in \pi(K)$, $n \geq 1$ large enough and $v \in \Sigma_n$ such that $\mu([x_{|n}, v]) > 0$.

Recall that $\widetilde{T}(s) := \sum_{i=0}^{m-1} p_i' T_i(s)$. If $s_0 \leq 1$, then $\widetilde{T}'(s_0)$ exists and by definition of s_0 we have $\widetilde{T}^*(\widetilde{T}'(s_0)) = 0$. Moreover, $\widetilde{T}^* \circ \widetilde{T}'$ is strictly decreasing in a neighborhood of s_0 since we have already shown that when they are defined at some s, the functions T_i'' cannot vanish simultaneously there. This, together with (8.7) implies that $\widetilde{\tau}_x^*(\alpha) \leq \widetilde{T}^*(\alpha) < 0$ for all $\alpha < \widetilde{T}'(s_0)$. Then applying Chernoff bound (see e.g. [21]) to the counting measure and the random variable $\frac{\mu([x_{|n},\cdot])}{\nu(x_{|n})}$ on Σ_n shows that

$$\Big\{v \in \Sigma_n : \frac{\mu([x_{|n}, v])}{\nu(x_{|n})} \geq m^{-n\alpha}\Big\} = \emptyset \text{ for all } \alpha < \widetilde{T}'(s_0) \text{ and large enough } n. \tag{8.8}$$

Over its domain, which contains a neighborhood of $[0, 1]$, the mapping $s \mapsto \widetilde{T}^*(\widetilde{T}'(s)) - \widetilde{T}'(s)$ is increasing on the left of 1 and decreasing on the right, and it takes the maximum value 0 at 1. In other words, over its domain, the mapping $\alpha \mapsto \widetilde{T}^*(\alpha) - \alpha$ is strictly increasing on the left of $\widetilde{T}'(1)$ and strictly decreasing on the right of $\widetilde{T}'(1)$, since $q \mapsto T'(q)$ is decreasing.

Now, for $\alpha \in \mathbb{R}$, $n \geq 1$ and $\epsilon > 0$ define

$$f(n,\alpha,\epsilon) = \frac{1}{n}\log_m \# \left\{ v \in \Sigma_n : m^{-n(\alpha+\epsilon)} \leq \frac{\mu([x_{|n}, v])}{\nu([x_{|n}])} \leq m^{-n(\alpha-\epsilon)} \right\}.$$

Fix $\eta > 0$ and $\epsilon > 0$. Again, using Chernoff inequality shows that for any $\alpha \in [\alpha_0, \beta_0]$, there exist $\epsilon_\alpha \in (0,\epsilon)$ and $n_\alpha \geq 1$ such that for all $n \geq n_\alpha$,

$$f(n,\alpha,\epsilon_\alpha) \leq \tau_x^*(\alpha) + \eta \leq \widetilde{T}^*(\alpha) + \eta.$$

Set $\alpha_c = \widetilde{T}'(s_0)$ if $s_0 \leq 1$ and $\alpha_c = \widetilde{T}'(1)$ otherwise. Fix a finite covering $\bigcup_{i=1}^N (\alpha_i - \epsilon_i, \alpha_i + \epsilon_i)$ of $[\alpha_0, \beta_0] \setminus (\alpha_c - \epsilon_c, \alpha_c + \epsilon_c)$, where ϵ_c stands for ϵ_{α_c}, and ϵ_i stands for ϵ_{α_i}, and set $n_0 = \sup\{n_\alpha : \alpha \in \{\alpha_c\} \cup \{\alpha_i : 1 \leq i \leq N\}\}$. Without loss of generality we assume that the α_i belong to $[\alpha_0, \beta_0] \setminus (\alpha_c - \epsilon_c, \alpha_c + \epsilon_c)$. Moreover, due to (8.8), if $s_0 \leq 1$ we can restrict the α_i to be larger than or equal to α_c, and set $\alpha_0 = \alpha_c$. Then, there exists $\gamma > 0$ such that $\widetilde{T}^*(\alpha_i) - \alpha_i \leq \widetilde{T}^*(\alpha_c) - \alpha_c - \gamma$ for all α_i.

For $n \geq n_0$ we have

$$\begin{aligned}
X_n(x) &= \sum_{v \in \Sigma_n} \frac{\mu([x_{|n}, v])}{\nu(x_{|n})} \\
&\leq m^{f(n,\alpha_c,\epsilon_c)} m^{-n(\alpha_c - \epsilon_c)} + \sum_{i=1}^N m^{f(n,\alpha_i,\epsilon_i)} m^{-n(\alpha_i - \epsilon_i)} \\
&\leq m^{n(\eta+\epsilon)} \Big(m^{n(\widetilde{T}^*(\alpha_c) - \alpha_c)} + \sum_{i=1}^N m^{n(\widetilde{T}^*(\alpha_i) - \alpha_i)} \Big) \\
&\leq m^{n(\eta+\epsilon)} m^{n(\widetilde{T}^*(\alpha_c) - \alpha_c)} (1 + N m^{-n\gamma}).
\end{aligned}$$

We conclude that $\liminf_{n\to\infty} -\frac{1}{n}\log_m(X_n(x)) \geq \alpha_c - \widetilde{T}^*(\alpha_c) - \eta - \epsilon$. Since this holds for any positive η and ϵ, we get the desired lower bound: $\widetilde{T}'(s_0)$ if $s_0 \leq 1$, and 0 otherwise.

On the other hand, due to (8.6), Gartner-Ellis theorem (see e.g. [21]) ensures that for all $s \in (0, \min(s_0, 1))$, $\lim_{\epsilon\to 0} \liminf_{n\to\infty} f(n, \widetilde{T}'(s), \epsilon) = \widetilde{T}^*(\widetilde{T}'(s))$. This immediately yields that $\limsup_{n\to\infty} -\frac{1}{n}\log_m(X_n(x)) \leq \widetilde{T}'(s)) - \widetilde{T}^*(\widetilde{T}'(s))$ for all $0 < s < \min(s_0, 1)$, since $X_n(x) \geq m^{nf(n,\alpha,\epsilon) - n(\alpha+\epsilon)}$ for all $\alpha \in \mathbb{R}$ and $\epsilon > 0$. Hence $\limsup_{n\to\infty} -\frac{1}{n}\log_m(X_n(x)) \leq \alpha_c - \widetilde{T}^*(\alpha_c)$. $\square$

Proof of Proposition 8.8. Recall that α stands for $\tau'(q)$ if $s(q) < 1$ and $\tau_\nu'(q)$ if $s(q) = 1$. We write $\pi_*\mu([x_{|n}]) = \nu([x_{|n}]) X_n(x)$.

At first we suppose that $\widetilde{\nu}_q = \nu'$.

If $q \in (0,1)$ and $s(q) < 1$, applying the ergodic theorem to ν' to control the local dimension of ν, and applying Lemma 8.10 to control $X_n(x)$ (in which $s_0 = s(q)$), we obtain that conditional on $\{\pi_*\mu \neq 0\}$, for ν'-almost every $x \in \pi(K)$,

$$\begin{aligned}\dim_{\mathrm{loc}}(\pi_*\mu, x) = \Big(-\sum_{i=0}^{m-1} p_i' \log_m(p_i)\Big) + \widetilde{T}'(s_0) \quad &= \quad -\sum_{i=0}^{m-1} p_i' \log_m(p_i) + \sum_{i=0}^{m-1} p_i' T_i'(s(q)) \\ &= \quad \tau'(q) = \alpha,\end{aligned}$$

by using (6.1) and (6.3). On the other hand,

$$\begin{aligned}\dim(\nu') = -\sum_{i=0}^{m-1} p_i' \log_m(p_i') \quad &= \quad -\sum_{i=0}^{m-1} p_i' \big(q \log_m(p_i) + \tau(q) - qT_i(s(q))/s(q)\big) \\ &= \quad q\tau'(q) - \tau(q) = \tau^*(\alpha)\end{aligned}$$

by using (6.1) and (6.2).

If $s(q) = 1$, then $\nu' = \nu_q$, and this time we apply Lemma 8.10 with $s_0 = s(q) = 1$ to control $X_n(x)$. This yields that conditional on $\{\pi_*\mu \neq 0\}$, for ν'-almost every $x \in \pi(K)$,

$$\dim_{\mathrm{loc}}(\pi_*\mu, x) = \Big(-\sum_{i=0}^{m-1} p_i' \log_m(p_i)\Big) + 0 = \tau_\nu'(q).$$

Moreover, $\dim(\nu_q) = \tau_\nu^*(\tau_\nu'(q)) = \tau_\nu'(q)q - \tau_\nu(q) = \alpha q - \tau(q) = \tau^*(\alpha)$ since $\tau_\nu(q) = \tau(q)$ and $\alpha \in \{\tau'(q^+), \tau'(q^-)\}$.

Thus, at this stage, due to Corollary 8.4 and the conclusions obtained in the previous lines, for all $q \in (0, \widetilde{q}_c) \setminus \widetilde{S}$ and $\alpha = \tau'(q)$, or $\alpha \in \{\tau'(q^+), \tau'(q^-)\}$ if $q > 1$ and the graphs of τ_ν and T cross transversally at $(q, T(q))$, we have established the desired inequality $\dim_H E(\pi_*\mu, \alpha) \geq \tau^*(\alpha)$, almost surely, conditional on $\{\mu \neq 0\}$.

Now suppose that $q \in \widetilde{S}$. Recall that $\widetilde{\nu}_q = \pi_*\mu_{W_1'}$ and by Lemma 8.7 the measure $\mu_{W_1'}$ has almost surely the same topological support as μ. Moreover, it follows from the theory of Mandelbrot measures ([4, 5]) that, with probability 1, conditionally on $\mu \neq 0$, for $\mu_{W_1'}$-almost every (x, y),

$$\begin{aligned}\lim_{n\to\infty} \frac{\mu([x_{|n}, y_{|n}])}{-n\log(m)} \quad &= \quad -\sum_{0\leq i,j\leq m-1} \mathbb{E}(W_{1,i,j}' \log_m(W_{i,j})) \\ &= \quad -\sum_{i=0}^{m-1} p_i' \log_m(p_i) - \sum_{i=0}^{m-1} p_i' \sum_{j=0}^{m-1} \mathbb{E}(V_{1,i,j}' \log_m V_{1,i,j}) \\ &= \quad -\sum_{i=0}^{m-1} p_i' \log_m(p_i),\end{aligned}$$

since $V'_{1,i,j} = V_{i,j}$ and $0 = \sum_{i=0}^{m-1} p'_i T'_i(1) = \sum_{i=0}^{m-1} p'_i \sum_{j=0}^{m-1} \mathbb{E}(V'_{1,i,j} \log_m V_{1,i,j})$. Also,

$$\begin{aligned}\dim(\mu_{W'_1}) &= -\sum_{0\le i,j\le m-1} \mathbb{E}(W'_{1,i,j} \log_m(W'_{1,i,j}))\\ &= -\sum_{i=0}^{m-1} p'_i \log_m(p'_i) - \sum_{i=0}^{m-1} p'_i \sum_{j=0}^{m-1} \mathbb{E}(V'_{1,i,j} \log_m V'_{1,i,j})\\ &= -\sum_{i=0}^{m-1} p'_i \log_m(p'_i) + \sum_{i=0}^{m-1} p'_i T'_i(1)\\ &= -\sum_{i=0}^{m-1} p'_i \log_m(p'_i) = \dim(\nu') = \dim(\mathbb{E}(\pi_*\mu_{W'_1})).\end{aligned}$$

Consequently, for $\pi_*\mu_{W'_1}$-almost every x, $\overline{\dim}_{\mathrm{loc}}(\pi_*\mu, x) \le -\sum_{i=0}^{m-1} p'_i \log_m(p_i) = \tau'_\nu(q) = \alpha$. Moreover, $\dim(\pi_*\mu_{W'_1}) = \dim(\nu') = \tau^*_\nu(\tau'_\nu(q)) = \tau^*(\alpha)$ (the last inequality coming from the equality $\tau_\nu(q) = \tau(q)$ and the fact that $\alpha = \tau'_\nu(q) \in \{\tau'(q^+), \tau'(q^-)\}$. Then, the same arguments as in the proof of Corollary 8.4 where T is replaced by τ_ν and μ_q by $\mu_{W'_1}$ yield $\dim_H E(\pi_*\mu, \alpha) \ge \tau^*(\alpha)$. □

8.3. **Case (III).**

Recall that in the case considered in this section, $\tau_\nu(q) = T(q)$, $\tau'_\nu(q) \ne T'(q)$, and $(\tau'(q^+), \tau'(q^-)) = (\tau'_\nu(q), T'(q))$ or $(\tau'(q^+), \tau'(q^-)) = (T'(q), \tau'_\nu(q))$.

Fix $\lambda \in [0, 1]$. Let $(n_k)_{k\ge1}$ be an increasing sequence of positive integers such that $n_k = o(n_1 + \cdots + n_{k-1})$ as $k \to \infty$, and $n_1 \min(\lambda, 1-\lambda) > 1$ if $\lambda > 0$. For $k \ge 0$, let $N_k = \sum_{i=1}^k n_i$ and $N_{k,\lambda} = N_{k-1} + \lfloor \lambda n_k \rfloor$. We will later further specify the sequence $(n_k)_{k\ge1}$.

For each $n \ge 0$ and $(u, v) \in \Sigma_n \times \Sigma_n$, set

$$\widetilde{W}_\lambda(u,v) = \begin{cases} W_q(u,v) = \left(p_i^q m^{T(q)} V^q_{i,j}(u,v)\right)_{0\le i,j\le m-1} & \text{if } N_{k-1}+1 \le n \le N_{k,\lambda} \text{ for some } k\\ (p_{q,i} V_{i,j}(u,v))_{0\le i,j\le m-1} & \text{otherwise}\end{cases},$$

where $p_{q,i}$ is defined by

$$p_{q,i} = p_i^q m^{\tau_\nu(q)}. \tag{8.9}$$

These random vectors can be used to build a non-homogeneous Mandelbrot measure in the same way as μ and μ_q: for each $n \ge 0$ and $(u, v) \in \Sigma_n \times \Sigma_n$, define

$$\widetilde{Y}_\lambda(u,v) = \lim_{p\to\infty} \widetilde{Y}_{\lambda,p}(u,v),$$

where

$$\widetilde{Y}_{\lambda,p}(u,v) = \sum_{|u'|=|v'|=p} \prod_{k=1}^p \widetilde{W}_{\lambda,u'_k,v'_k}(u\cdot u'_{|k-1}, v\cdot v'_{|k-1}). \tag{8.10}$$

Notice that the limit in defining $\widetilde{Y}_\lambda(u,v)$ exists almost surely since $(\widetilde{Y}_{\lambda,p}(u,v))_{p\geq 0})$ is a non-negative martingale (of expectation 1). Write $\widetilde{Y}_\lambda=\widetilde{Y}_\lambda(\epsilon,\epsilon)$ for convenience. Then,

$$\widetilde{\mu}_\lambda([u]\times[v])=\widetilde{Y}_\lambda(u,v)\prod_{j=1}^n \widetilde{W}_{\lambda,u_j,v_j}(u_{|j-1},v_{|j-1})$$

defines a measure almost surely. Moreover, the same argument as in Proposition A.1 shows that if $\widetilde{\mu}_\lambda$ is not equal to 0 almost surely, then its topological support equals that of μ almost surely. It is the situation which occurs as the following proposition shows. Also, $\widetilde{\mu}_1=\mu_q$, while $\widetilde{\mu}_0$ is a non-degenerate Mandelbrot measure such that $\mathbb{E}(\pi_*\widetilde{\mu}_0)=\nu_q$ and by (3.4) $\dim(\widetilde{\mu}_0)-\dim(\nu_q)=\sum_{i=0}^{m-1}p_{q,i}T_i'(1)$.

Proposition 8.11. $\mathbb{E}(\widetilde{Y}_\lambda)=1$*; consequently* $\widetilde{\mu}_\lambda$ *is not almost surely degenerate, and with probability* 1, $\operatorname{supp}(\widetilde{\mu}_\lambda)=\operatorname{supp}(\mu)$ *conditional on* $\{\mu\neq 0\}$*. Moreover, there exists* $h\in(1,2]$ *such that*

$$M(\lambda,h)=\sup\{\mathbb{E}(\widetilde{Y}_\lambda(u,v)^h):n\geq 0,\ u,v\in\Sigma_n\}<\infty.$$

We postpone the proof of Proposition 8.11 for a while.

For all $k\geq 1$ and $(u,v)\in\Sigma_{N_k}\times\Sigma_{N_k}$, define

$$\widetilde{\mu}_1^T(u,v)=\prod_{i=1}^k\prod_{\ell=N_{i-1}+1}^{N_{i,\lambda}} p_{u_\ell}^q m^{T(q)}V_{u_\ell,v_\ell}(u_{|\ell-1},v_{|\ell-1})^q,$$

$$\mu^T(u,v)=\prod_{i=1}^k\prod_{\ell=N_{i-1}+1}^{N_{i,\lambda}} p_{u_\ell}V_{u_\ell,v_\ell}(u_{|\ell-1},v_{|\ell-1}),$$

$$\widetilde{\mu}_0^{\tau_\nu}(u,v)=\prod_{i=1}^k\prod_{\ell=N_{i,\lambda}+1}^{N_i} p_{q,u_\ell}V_{u_\ell,v_\ell}(u_{|\ell-1},v_{|\ell-1}),$$

$$\mu^{\tau_\nu}(u,v)=\prod_{i=1}^k\prod_{\ell=N_{i,\lambda}+1}^{N_i} p_{u_\ell}V_{u_\ell,v_\ell}(u_{|\ell-1},v_{|\ell-1}).$$

We have

$$\widetilde{\mu}_\lambda([u]\times[v])=\widetilde{\mu}_1^T(u,v)\widetilde{\mu}_0^{\tau_\nu}(u,v)\widetilde{Y}_\lambda(u,v) \tag{8.11}$$

and

$$\mu([u]\times[v])=\mu^T(u,v)\mu^{\tau_\nu}(u,v)Y(u,v). \tag{8.12}$$

Define

$$\alpha=\lambda T'(q)+(1-\lambda)\tau_\nu'(q)\quad\text{and}\quad\alpha'=\sum_{i=0}^{m-1}p_i^q m^{\tau_\nu(q)}T_i'(1). \tag{8.13}$$

Since $\alpha\in[\tau'(q+),\tau'(q-)]$ and $\tau_\nu(q)=\tau(q)=T(q)$, we have

$$\tau^*(\alpha)=\alpha q-\tau_\nu(q)=\lambda T^*(T'(q))+(1-\lambda)\tau_\nu^*(\tau_\nu'(q)).$$

We will prove the following propositions and corollary, which give the desired conclusion.

Proposition 8.12. *With probability 1, conditional on $\mu \neq 0$, for $\widetilde{\mu}_\lambda$-almost every (x, y), the following hold:*

$$\begin{aligned}
&\lim_{k\to\infty} \frac{\log(\widetilde{\mu}_1^T(x_{|N_k}, y_{|N_k}))}{-N_k \log(m)} = \lambda T^*(T'(q)),\\
&\lim_{k\to\infty} \frac{\log(\widetilde{\mu}_0^{\tau_\nu}(x_{|N_k}, y_{|N_k}))}{-N_k \log(m)} = (1-\lambda)\tau_\nu^*(\tau_\nu'(q)) + (1-\lambda)\alpha',\\
&\lim_{k\to\infty} \frac{\log(\mu^T(x_{|N_k}, y_{|N_k}))}{-N_k \log(m)} = \lambda T'(q),\\
&\lim_{k\to\infty} \frac{\log(\mu^{\tau_\nu}(x_{|N_k}, y_{|N_k}))}{-N_k \log(m)} = (1-\lambda)\tau_\nu'(q) + (1-\lambda)\alpha',\\
&\lim_{k\to\infty} \frac{\log(\widetilde{Y}_\lambda(x_{|N_k}, y_{|N_k}))}{-N_k \log(m)} = \lim_{k\to\infty} \frac{\log(Y(x_{|N_k}, y_{|N_k}))}{-N_k \log(m)} = 0;
\end{aligned}$$

in particular, $\dim_{\mathrm{loc}}(\mu, (x, y)) = \alpha + (1-\lambda)\alpha'$ *and* $\dim_{\mathrm{loc}}(\widetilde{\mu}_\lambda, (x, y)) = \tau^*(\alpha) + (1-\lambda)\alpha'$.

We will see in the proof of Proposition 8.11 that $\alpha' \geq 0$.

Proposition 8.13. *Suppose that $\lambda \in (0, 1)$.*

1. *With probability 1, conditional on $\{\mu \neq 0\}$, $\underline{\dim}_{\mathrm{loc}}(\pi_*\widetilde{\mu}_\lambda, x) \geq \tau^*(\alpha)$ for $\pi_*\widetilde{\mu}_\lambda$-almost every x.*
2. *With probability 1, conditional on $\{\mu \neq 0\}$, $\overline{\dim}_{\mathrm{loc}}(\pi_*\widetilde{\mu}_\lambda, x) \leq \tau^*(\alpha)$ and $\overline{\dim}_{\mathrm{loc}}(\pi_*\mu, x) \leq \alpha$ for $\pi_*\widetilde{\mu}_\lambda$-almost every x.*

Corollary 8.14. *With probability 1, conditional on $\{\mu \neq 0\}$, $\dim E(\pi_*\mu, \alpha) = \tau^*(\alpha)$.*

Proof of Proposition 8.11. Recall that $\widetilde{Y}_{\lambda,p}(u, v)$ is defined in (8.10). By definition, for $p \geq 2$,

$$\widetilde{Y}_{\lambda,p}(u, v) = \sum_{0\leq i,j\leq m-1} \widetilde{W}_{\lambda,i,j}(u, v)\widetilde{Y}_{\lambda,p-1}(ui, vj).$$

Let $h \in (1, 2]$. We can use Kahane's original approach [36] to the moments of Mandelbrot martingales to write

$$\widetilde{Y}_{\lambda,p}(u, v)^h \leq \Big(\sum_{0\leq i,j\leq m-1} \widetilde{W}_{\lambda,i,j}(u, v)^{h/2}\widetilde{Y}_{\lambda,p-1}(ui, vj)^{h/2}\Big)^2$$

and then get

$$\begin{aligned}
\mathbb{E}(\widetilde{Y}_{\lambda,p}(u, v)^h) \quad \leq \quad & \sum_{0\leq i,j\leq m-1} \mathbb{E}(\widetilde{W}_{\lambda,i,j}(u, v)^h)\mathbb{E}(\widetilde{Y}_{\lambda,p-1}(ui, vj)^h)\\
&+ \sum_{(i,j)\neq(i',j')} \mathbb{E}(\widetilde{W}_{\lambda,i,j}(u, v)^{h/2}\widetilde{W}_{\lambda,i',j'}(u, v)^{h/2}).
\end{aligned}$$

If h is close enough to 1, there exists $C > 0$ such that

$$\sum_{(i,j)\neq(i',j')} \mathbb{E}(\widetilde{W}_{\lambda,i,j}(u,v)^{h/2}\widetilde{W}_{\lambda,i',j'}(u,v)^{h/2}) \leq C$$

independently on (u, v), by equidistribution of the $W(u, v)$ and the fact that our assumption on the domain of finiteness of T we have $\mathbb{E}(W_{i,j}^{q'}) < \infty$ for all $q' < q_c$ and $0 \leq i, j \leq m-1$. Also, by construction $\mathbb{E}(\widetilde{Y}_{\lambda,p-1}(ui, vj)^h)$ does not depend on (i, j). Thus, if $h < q_c$,

$$\mathbb{E}(\widetilde{Y}_{\lambda,p}(u,v)^h) \leq C + \mathbb{E}(\widetilde{Y}_{\lambda,p-1}(u0,v0)^h) \sum_{0\leq i,j\leq m-1} \mathbb{E}(\widetilde{W}_{\lambda,i,j}(u,v)^h).$$

By definition of $\widetilde{W}_\lambda(u, v)$,

$$\sum_{0\leq i,j\leq m-1} \mathbb{E}(\widetilde{W}_{\lambda,i,j}(u,v)^h) \in \Big\{ \sum_{i=0}^{m-1} p_{q,i}^h m^{-T_i(h)}, m^{hT(q)-T(hq)} \Big\}.$$

Since $T^*(T'(q)) > 0$ by our assumption $q \in (1, q_c)$, for h close enough to 1, we have $hT(q) - T(hq) < 0$ hence $m^{hT(q)-T(hq)} < 1$. On the other hand, since $\tau_\nu(q) = T(q)$,

$$\psi(h) := \sum_{i=0}^{m-1} p_{q,i} m^{-T_i(h)} = \sum_{i=0}^{m-1} p_i^q m^{T(q)} m^{-T_i(h)},$$

and $\psi(1) = \psi(q) = 1$. Since ψ is convex, it follows that $\psi(h) \leq 1$ on $(1, q)$. Consequently, $\sum_{i=0}^{m-1} p_{q,i}^h m^{-T_i(h)} \leq \max\{p_{q,i}^{h-1} : 0 \leq i \leq m-1\}\psi(h) < 1$, since all the positive $p_{q,i}$ belong to $(0, 1)$. Notice also that the derivative of ψ at 1 is non-positive, hence $\alpha' = \sum_{i=0}^{m-1} p_{q,i} T'(1) \geq 0$. Finally, if h is close enough to 1, there exists $c \in (0, 1)$ independent of (u, v) such that $\mathbb{E}(\widetilde{Y}_{\lambda,p}(u,v)^h) \leq C + c\,\mathbb{E}(\widetilde{Y}_{\lambda,p-1}(u0, v0)^h)$. This yields $\mathbb{E}(\widetilde{Y}_{\lambda,p}(u,v)^h) \leq C\mathbb{E}(\widetilde{Y}_{\lambda,0}(u0^p, v0^p)^h)/(1-c) = C/(1-c)$, hence both $\mathbb{E}(\widetilde{Y}_\lambda(u,v)^h) \leq C/(1-c)$ and $\mathbb{E}(\widetilde{Y}_\lambda(u, v)) = 1$. □

Proof of Proposition 8.12. Define $\widetilde{Q}_\lambda(\mathrm{d}\omega, \mathrm{d}x, \mathrm{d}y) = \mathbb{P}(\mathrm{d}\omega)\widetilde{\mu}_{\lambda,\omega}(\mathrm{d}x, \mathrm{d}y)$, $\widetilde{Q}_1(\mathrm{d}\omega, \mathrm{d}x, \mathrm{d}y) = \mathbb{P}(\mathrm{d}\omega)\widetilde{\mu}_{1,\omega}(\mathrm{d}x, \mathrm{d}y)$ and $\widetilde{Q}_0(\mathrm{d}\omega, \mathrm{d}x, \mathrm{d}y) = \mathbb{P}(\mathrm{d}\omega)\widetilde{\mu}_{0,\omega}(\mathrm{d}x, \mathrm{d}y)$ the Peyrière measures associated with $\widetilde{\mu}_\lambda$, $\widetilde{\mu}_1$ and $\widetilde{\mu}_0$ respectively. Also, set $\widetilde{N}_k = \sum_{i=1}^k \lfloor \lambda n_i \rfloor$ and $N_k' = N_k - \widetilde{N}_k$.

It is straightforward to write that under $\widetilde{Q}_\lambda$, the random vectors $\widetilde{W}_\lambda(x_{|n-1}, y_{|n-1})$, $N_{k-1} + 1 \leq n \leq N_{k,\lambda}$, $k \geq 1$, are independent and equidistributed, with the same law as the vectors $W_q(x_{|n-1}, y_{|n-1})$, $n \geq 1$, with respect to $\widetilde{Q}_1(\mathrm{d}\omega, \mathrm{d}x, \mathrm{d}y)$. Moreover, since

$$\widetilde{\mu}_{1,n}([x_{|n}] \times [y_{|n}]) = \prod_{k=1}^n W_{q,x_k,y_k}(x_{|k-1}, y_{|k-1}),$$

the strong law of large numbers yields

$$\lim_{k\to\infty} \frac{\log(\widetilde{\mu}_{1,\widetilde{N}_k}([x_{|\widetilde{N}_k}] \times [y_{|\widetilde{N}_k}]))}{-\widetilde{N}_k \log(m)} = -\mathbb{E} \sum_{0\leq i,j\leq m-1} p_i^q m^{T(q)} V_{i,j}^q \log_m(p_i^q m^{T(q)} V_{i,j}^q) = T^*(T'(q)),$$

$\widetilde{Q}_1$-almost surely. Since $\lim_{k\to\infty}\widetilde{N}_k/N_k=\lambda$, by definition of $\widetilde{\mu}_1^T(x_{|N_k},y_{|N_k}))$ we get the first claim.

The same idea applied with $\mu^T(x_{|N_k},y_{|N_k})$ with respect to $\widetilde{Q}_\lambda$ and $\mu_{\widetilde{N}_k}([x_{|\widetilde{N}_k}]\times[y_{|\widetilde{N}_k}])$ with respect to $\widetilde{Q}_1$ yields

$$\lim_{k\to\infty}\frac{\log(\mu^T(x_{|N_k},y_{|N_k}))}{-\widetilde{N}_k\log(m)}=-\mathbb{E}\sum_{0\le i,j\le m-1}p_i^q m^{T(q)}V_{i,j}^q\log_m(p_iV_{i,j})=T'(q),$$

$\widetilde{Q}_\lambda$-almost surely, i.e. the third claim of the proposition since $\lim_{k\to\infty}\widetilde{N}_k/N_k=\lambda$.

For the second claim, we need to consider $\widetilde{\mu}_0^{\tau_\nu}(x_{|N_k},y_{|N_k})$ and $\widetilde{\mu}_{0,N_k'}([x_{|N_k'}]\times[y_{|N_k'}])$ with respect to $\widetilde{Q}_\lambda$ and $\widetilde{Q}_0$ respectively; then we apply the strong law of large numbers to $\log(\widetilde{\mu}_{0,N_k'}([x_{|N_k'}]\times[y_{|N_k'}]))/N_k'$ under $\widetilde{Q}_0$, and use the fact the $\lim_{k\to\infty}N_k'/N_k=1-\lambda$. The fourth claim follows similarly by considering $\mu^{\tau_\nu}(x_{|N_k},y_{|N_k})$ and $\mu_{N_k'}([x_{|N_k'}]\times[y_{|N_k'}])$ with respect to $\widetilde{Q}_\lambda$ and $\widetilde{Q}_0$ respectively.

For the last two claims, an application of the Markov inequality shows that for any fixed $(u(k),v(k))$ in $\Sigma_{N_k}\times\Sigma_{N_k}$, for $Z\in\{Y,\widetilde{Y}_\lambda\}$ and $\gamma\in\{-1,1\}$, for any $\eta>0$ and $\epsilon>0$,

$$\begin{aligned}&\widetilde{Q}_\lambda(\{(x,y):\mathbf{1}_{\{Z(x_{|N_k},y_{|N_k})>0\}}Z(x_{|N_k},y_{|N_k})^\gamma>m^{N_k\epsilon})\\&\qquad\le m^{-N_k\eta\epsilon}\mathbb{E}\big(\mathbf{1}_{\{Z(u(k),v(k))>0\}}\widetilde{Y}_\lambda(u(k),v(k))Z^{\gamma\eta}(u(k),v(k))\big).\end{aligned}$$

Since, conditional on $\{\mu\neq 0\}$ on non-vanishing, Y has finite negative moments, by Proposition 8.11 and the Hölder inequality we can choose η so that

$$\sup\{\mathbb{E}\big(\mathbf{1}_{\{Z(u(k),v(k))>0\}}\widetilde{Y}_\lambda(u(k),v(k))Z^{\gamma\eta}(u(k),v(k))\big):k\ge1,\ Z\in\{Y,\widetilde{Y}_\lambda\}\}<\infty.$$

Consequently

$$\sum_{k\ge1}\widetilde{Q}_\lambda\left(\Big\{(x,y):\mathbf{1}_{\{Z(x_{|N_k},y_{|N_k})>0\}}Z(x_{|N_k},y_{|N_k})^\gamma>m^{N_k\epsilon}\Big\}\right)<\infty,$$

and the desired claims follow from the Borel-Cantelli lemma.

Finally, the claim about the local dimensions follows from (8.11) and (8.12), and the fact that $\lim_{k\to\infty}N_{k-1}/N_k=1$. □

Proof of Proposition 8.13(1). We will use the following lemma.

Lemma 8.15. *There exist two bounded functions $C(h)$ and $\epsilon(h)$ defined in a right neighborhood $\mathcal{V}$ of 1, with $\lim_{h\to1^+}\epsilon(h)=0$, such that for all $k\ge1$,*

$$\forall\,h\in\mathcal{V},\ \mathbb{E}\Big(\sum_{|u|=N_k}\pi_*\widetilde{\mu}_\lambda([u])^h\Big)\le C(h)N_km^{N_k(-\tau^*(\alpha)(h-1)+\epsilon(h)(h-1)+C(h)(k/N_k))}.$$

We deduce from the previous lemma that for all $\epsilon>0$, for h close enough to $1+$, $\mathbb{E}\left(\sum_{k\ge1}\sum_{|u|=N_k}m^{N_k(h-1)(\tau^*(\alpha)-\epsilon)}\pi_*\widetilde{\mu}_\lambda([u])^h\right)<\infty$.

This implies that with probability 1, conditional on $\{\mu \neq 0\}$, for all $\epsilon > 0$, there exists $h > 1$ such that $\sum_{k\geq 1}\sum_{|u|=N_k} m^{N_k(h-1)(\tau^*(\alpha)-\epsilon)}\pi_*\widetilde{\mu}_\lambda([u])^h < \infty$. Due to Lemma 5.3 and the fact that $\lim_{k\to\infty} N_k/N_{k-1} = 1$, we get $\underline{\dim}_{\rm loc}(\pi_*\widetilde{\mu}_\lambda, x) \geq \tau^*(\alpha)$ for $\pi_*\widetilde{\mu}_\lambda$-a.e. x, which proves part (1) of Proposition 8.13.

If we were able to prove that the same estimate as in the lemma holds for h near $1-$, we could derive the second part of the proposition quite easily (but maybe such a bound does not hold). We have to use another approach, which will be presented after the proof of Lemma 8.15. $\square$

Proof of Lemma 8.15. For $k \geq 1$ and $u \in \Sigma_{N_k}$, by the definition of $\widetilde{\mu}_\lambda$,

$$\pi_*\widetilde{\mu}_\lambda([u]) = \sum_{|v|=N_k} \widetilde{Y}_\lambda(u,v)\prod_{i=1}^{k}\Big(\prod_{\ell=N_{i-1}+1}^{N_{i,\lambda}} p_{u_\ell}^q m^{T(q)} V_{u_\ell,v_\ell}(u_{|\ell-1}, v_{|\ell-1})^q\Big) \cdot \Big(\prod_{\ell'=N_{i,\lambda}+1}^{N_i} p_{q,u'_\ell} V_{u_{\ell'},v_{\ell'}}(u_{|\ell'-1}, v_{|\ell'-1})\Big).$$

Setting, for $k \geq 1$ and $h > 1$ such that $hq < q_c$ (recall that q is fixed)

$$\Lambda(k,h) = \Big(\sum_{i=1}^{k}\lfloor\lambda n_i\rfloor\Big)(T(hq) - hT(q)) + \Big(N_k - \sum_{i=1}^{k}\lfloor\lambda n_i\rfloor\Big)(\tau_\nu(hq) - h\tau_\nu(q)), \tag{8.14}$$

we can write

$$m^{\Lambda(k,h)}\pi_*\widetilde{\mu}_\lambda([u])^h = Z(u)^h \prod_{i=1}^{k}\Big(\prod_{\ell=N_{i-1}+1}^{N_{i,\lambda}} p_{u_\ell}^{hq} m^{T(hq)} m^{-hT_{u_\ell}(q)}\Big)\Big(\prod_{\ell'=N_{i,\lambda}+1}^{N_i} p_{hq,u_{\ell'}}\Big),$$

where

$$Z(u) = \sum_{|v|=N_k} \widetilde{Y}_\lambda(u,v)\prod_{i=1}^{k}\Big(\prod_{\ell=N_{i-1}+1}^{N_{i,\lambda}} m^{T_{u_\ell}(q)} V_{u_\ell,v_\ell}(u_{|\ell-1}, v_{|\ell-1})^q\Big) \cdot \Big(\prod_{\ell'=N_{i,\lambda}+1}^{N_i} V_{u_{\ell'},v_{\ell'}}(u_{|\ell'-1}, v_{|\ell'-1})\Big).$$

Fix $h \in (1,2]$ as in Proposition 8.11 such that $M(\lambda,h) < \infty$ and set

$$C_1(h) = \max_{0\leq i\leq m-1}\sum_{0\leq j\neq j'\leq m-1} \mathbb{E}(m^{T_i(q)h/2}V_{i,j}^{hq/2} m^{T_i(q)h/2}V_{i,j'}^{hq/2})$$

and

$$C_2(h) = \max_{0\leq i\leq m-1}\sum_{0\leq j\neq j'\leq m-1} \mathbb{E}(V_{i,j}^{h/2}V_{i,j'}^{h/2}).$$

Taking h closer to 1 if necessary we have $C(h) = \max(C_1(h), C_2(h)) < \infty$. We notice that $Z(u)$ takes a form similar to $X(u)$ in (9.4), and we can use the same approach as that in the proof of Lemma 9.5(ii) on the positive moments of $X(u)$ (with $t=h<2$ hence $\mathcal{I}_t=\emptyset$) to get

$$\mathbb{E}(Z(u)^h)\le M(\lambda,h)C(h)\sum_{\ell=0}^{N_k} m^{-(\theta^{(1)}_{u_1}+\cdots+\theta^{(\ell)}_{u_\ell})},$$

where $\theta_i^{(\ell)}=hT_i(q)-T_i(hq)$ if $N_{j-1}+1\le\ell\le N_{j,\lambda}$ for some j and $\theta_i^{(\ell)}=T_i(h)$ otherwise. It follows that, if we set $\widetilde{p}_i^{(\ell)}=p_i^{hq}m^{T(hq)}m^{-hT_i(q)}$ whenever$N_{j-1}+1\le\ell\le N_{j,\lambda}$ for some j and $\widetilde{p}_i^{(\ell)}=p_{hq,i}$ otherwise, then

$$\begin{aligned}
&m^{\Lambda(k,h)}\mathbb{E}\sum_{|u|=N_k}\pi_*\widetilde{\mu}_\lambda([u])^h\\
&\le M(\lambda,h)C(h)\sum_{\ell=0}^{N_k}\sum_{|u|=N_k} m^{-(\theta^{(1)}_{u_1}+\cdots+\theta^{(\ell)}_{u_\ell})}\prod_{j=1}^{N_k}\widetilde{p}^{(j)}_{u_j}\\
&= M(\lambda,h)C(h)\sum_{\ell=0}^{N_k}\Big(\prod_{j=1}^{\ell}\sum_{i=0}^{m-1}\widetilde{p}_i^{(j)}m^{-\theta_i^{(j)}}\Big)\Big(\prod_{j'=\ell+1}^{N_k}\sum_{i'=0}^{m-1}\widetilde{p}_{i'}^{(j')}\Big).
\end{aligned}$$

We have for each $1\le j\le N_k$, either $\sum_{i=0}^{m-1}\widetilde{p}_i^{(j)}=\sum_{i=0}^{m-1}p_{hq,i}=1$ or $\sum_{i=0}^{m-1}\widetilde{p}_i^{(j)}=\sum_{i=0}^{m-1}p_i^{hq}m^{T(hq)}m^{-hT_i(q)}$. On the other hand, the computations achieved in the proof of Lemma 8.5 show that the derivative of $h\mapsto\sum_{i=0}^{m-1}p_i^{hq}m^{T(hq)}m^{-hT_i(q)}$ at $h=1$ equals $\log(m)(\dim(\mu_q)-\dim(\mathbb{E}(\pi_*\mu_q)))\le 0$. So $\sum_{i=0}^{m-1}\widetilde{p}_i^{(j)}\le 1+o(h-1)$.

On the other hand, we have $\sum_{i=0}^{m-1}\widetilde{p}_i^{(j)}m^{-\theta_i^{(j)}}=\sum_{i=0}^{m-1}p_i^{hq}m^{T(hq)}m^{-T_i(hq)}=1$ or $\sum_{i=0}^{m-1}\widetilde{p}_i^{(j)}m^{-\theta_i^{(j)}}=\sum_{i=0}^{m-1}p_{hq,i}m^{-T_i(h)}$, and the derivative at 1 of $h\mapsto\sum_{i=0}^{m-1}p_{hq,i}m^{-T_i(h)}$ equals $-\log(m)\sum_{i=0}^{m-1}p_{q,i}T_i'(1)$ which is non-positive by a remark made in the proof of Proposition 8.11. So $\sum_{i=0}^{m-1}\widetilde{p}_i^{(j)}m^{-\theta_i^{(j)}}\le 1+o(h-1)$.

Finally,

$$m^{\Lambda(k,h)}\mathbb{E}\left(\sum_{|u|=N_k}\pi_*\widetilde{\mu}_\lambda([u])^h\right)=O(N_km^{o(h-1)N_k}),$$

where O and o depend only on h. Since it is easily seen from (8.14) that

$$\Lambda(k,h)=N_k\Big(\lambda T^*(T'(q))+(1-\lambda)\tau_\nu^*(\tau_\nu'(q))\Big)(h-1)+o(h-1)N_k+O(k),$$

where O and o still depend only on h, and we know that $\tau^*(\alpha)=\lambda T^*(T'(q))+(1-\lambda)\tau_\nu^*(\tau_\nu'(q))$, we get the desired conclusion. □

Proof of Proposition 8.13(2). Recall that $\alpha'=\sum_{i=0}^{m-1}p_i^q m^{\tau_\nu(q)}T_i'(1)$. If $\alpha'=0$, the result directly follows from Proposition 8.12 since projecting does not increase the upper local dimensions.

Suppose now that $\alpha'>0$. To begin, observe that any $y\in\Sigma$ can be written $y=y_1\widetilde{y}_1\cdot y_2\widetilde{y}_2\cdot y_k\widetilde{y}_k\cdots$ with $y_k\in\Sigma_{\lfloor\lambda n_k\rfloor}$ and $\widetilde{y}_k\in\Sigma_{n_k-\lfloor\lambda n_k\rfloor}$, and denote by φ the mapping $x\mapsto\widetilde{y}_1\widetilde{y}_1\cdots\widetilde{y}_k\cdots$. The mapping φ is β-Hölder continuous for all $\beta\in(0,1-\lambda)$.

It will be convenient to use the version of $\widetilde{\mu}_0$ associated with $\widetilde{Q}_0$ obtained from $\widetilde{Q}_\lambda$ as follows: For any collection $(A_k)_{k\geq 1}$ such that $A_k \subset \Sigma^2_{n_k-\lfloor \lambda n_k\rfloor}$ and $A_k = \Sigma^2_{n_k-\lfloor \lambda n_k\rfloor}$ for k large enough, and any measurable subset A of Ω, $\widetilde{Q}_0(A\times\prod_{k=1}^\infty A_k) = \widetilde{Q}_\lambda(A\times\prod_{k=1}^\infty(\Sigma^2_{\lfloor\lambda n_k\rfloor}\times A_k)$. In other words, $\widetilde{Q}_0$ is the push-forward of $\widetilde{Q}_\lambda$ by the mapping $\Phi : (\omega,x,y)\mapsto(\omega,\varphi(x),\varphi(y))$. Then $\widetilde{Q}_0(\mathrm{d}\omega,\mathrm{d}x',\mathrm{d}y') = \mathbb{P}(\mathrm{d}\omega)\pi_*\widetilde{\mu}_{0,\omega}(\mathrm{d}x')\widetilde{\mu}^x_{0,\omega}(\mathrm{d}y')$ and due to Theorem 3.3(2), with probability 1, conditional on $\widetilde{\mu}_{0,\omega}\neq 0$, for $\pi_*\widetilde{\mu}_{0,\omega}$-almost every x', the measure $\widetilde{\mu}^{x'}_{0,\omega}$ assigns 0 mass to sets of Hausdorff dimension less than α'.

Now write any measurable subset E of $\Omega\times\Sigma^2$ as $\bigcup_{\omega\in\pi_\Omega(E)}\bigcup_{x\in E^\omega}E^{\omega,x}$, where π_Ω is the canonical projection onto Ω, $E^\omega=\pi(\pi_\Omega^{-1}(\{\omega\})\cap E)$, and $E^{\omega,x}=\pi^{-1}_{\Omega\times\Sigma}(\{(\omega,x)\}\cap E)$, where $\pi_{\Omega\times\Sigma}$ is the canonical projection onto $\Omega\times\Sigma$. Also, let $\widetilde{\mu}^x_{\lambda,\omega}$ denote the conditional measure with respect to $(\pi_{\Omega\times\Sigma})_*\widetilde{Q}_\lambda$. It is defined on a measurable set B of full $(\pi_{\Omega\times\Sigma})_*\widetilde{Q}_\lambda$-probability.

Fix $\alpha''\in[0,(1-\lambda)\alpha')$. Let $E=\{(\omega,x,y)\in\pi^{-1}_{\Omega\times\Sigma}(B) : \underline{\dim}_{\mathrm{loc}}(\widetilde{\mu}^x_{\lambda,\omega},y)\leq\alpha''\}$. In particular, $\dim E^{\omega,x}\leq\alpha''$ for all $\omega\in\pi_\Omega(E)$ and $x\in E^\omega$. Suppose that $\widetilde{Q}_\lambda(E)>0$. By definition of $\widetilde{Q}_0$, since $E\subset\Phi^{-1}(\Phi(E))$ and $\widetilde{Q}_\lambda(E)>0$ we have $\widetilde{Q}_0(\Phi(E))>0$. This implies that for all ω in a subset Ω' of $\pi_\Omega(E)$ of positive $\mathbb{P}$-probability, there is $F^\omega\subset\varphi(E^\omega)$ of positive $\pi_*\widetilde{\mu}_{0,\omega}$-measure such that for all $x'\in F^\omega$, $x'=\varphi(x)$ with $x\in E^\omega$ and $\widetilde{\mu}^{\varphi(x)}_{0,\omega}(\varphi(E^{\omega,x}))>0$. However, by the Hölder properties of φ, $\dim\varphi(E^{\omega,x})\leq\beta^{-1}\dim E^{\omega,x}\leq\beta^{-1}\alpha''$ for all $\beta\in(0,1-\lambda)$ hence $\dim\varphi(E^{\omega,x})<\alpha'$. This is a contradiction. Consequently, with probability 1, conditional on $\widetilde{\mu}_{\lambda,\omega}\neq 0$, for $\pi_*\widetilde{\mu}_{\lambda,\omega}$-almost every x, the inequality $\underline{\dim}_H(\widetilde{\mu}^x_{\lambda,\omega})>\alpha''$ holds. Since α'' is arbitrary in $[0,(1-\lambda)\alpha')$, we even have $\underline{\dim}_H(\widetilde{\mu}^x_{\lambda,\omega})\geq(1-\lambda)\alpha'$.

Combining this information with Proposition 8.12, conditional on $\mu\neq 0$, we can find a set $\widetilde{E}$ of full $\widetilde{\mu}_\lambda$-measure such that for all $(x,y)\in\widetilde{E}$,

$$\lim_{k\to\infty}\frac{\log(\widetilde{\mu}_0^{\tau_\nu}(x_{|N_k},y_{|N_k}))}{-N'_k\log(m)}=\tau_\nu^*(\tau_\nu'(q))+\alpha', \tag{8.15}$$

$$\lim_{k\to\infty}\frac{\log(\widetilde{\mu}_1^{T}(x_{|N_k},y_{|N_k}))}{-N_k\log(m)}=\lambda T^*(T'(q))\quad\text{and}\quad\lim_{k\to\infty}\frac{\log(\widetilde{Y}_\lambda(x_{|N_k},y_{|N_k}))}{N_k\log(m)}=0, \tag{8.16}$$

and for any $\widetilde{F}\subset\widetilde{E}$ with $\widetilde{\mu}_\lambda(\widetilde{F})>0$ and for $\pi_*\widetilde{\mu}_\lambda$-almost every $x\in\pi(\widetilde{F})$, $\widetilde{\mu}^x_{\lambda,\omega}(\widetilde{F})>0$ and thus

$$\liminf_{k\to\infty}\frac{\log\#\{v\in\Sigma_{N_k} : ([x_{|N_k}]\times[v])\cap\widetilde{F}\neq\emptyset\}}{N'_k\log(m)}\geq\alpha'. \tag{8.17}$$

Set $\beta_q=T^*(T'(q))$, $\widetilde{\beta}_q=\tau_\nu^*(\tau_\nu'(q))$. For $j\geq 1$ and $\epsilon>0$, let $\widetilde{E}_{j,\epsilon}$ denote the set consisting of the points $(x,y)\in\Sigma\times\Sigma$ such that for any $k\geq j$, the following inequalities hold:

$$\begin{aligned} m^{-N_k'(\widetilde{\beta}_q+\alpha'-\epsilon)} &\geq \widetilde{\mu}_0^{\tau_\nu}(x_{|N_k}, y_{|N_k}) \geq m^{-N_k'(\widetilde{\beta}_q+\alpha'+\epsilon)}, \\ m^{-N_k(\lambda\beta_q-\epsilon)} &\geq \widetilde{\mu}_1^T(x_{|N_k}, y_{|N_k}) \geq m^{-N_k(\lambda\beta_q+\epsilon)}, \\ \widetilde{Y}_\lambda(x_{|N_k}, y_{|N_k}) &\geq m^{-N_k\epsilon}. \end{aligned}$$

By (8.15) and (8.16), $\widetilde{E} \subset \bigcup_{j=1}^\infty \widetilde{E}_{j,\epsilon} = \lim_{j\to\infty} \widetilde{E}_{j,\epsilon}$ for all $\epsilon > 0$. According to (8.17), for any $j \geq 1$ and $\epsilon > 0$, for $\pi_*\widetilde{\mu}_\lambda$-almost every $x \in \pi(\widetilde{E}_{j,\epsilon})$, there exists $k \geq j$ such that there are at least $m^{N_k'(\alpha'-\epsilon)}$ words $v \in \Sigma_{N_k}$ such that $([x_{|N_k}] \times [v]) \cap \widetilde{E}_{j,\epsilon} \neq \emptyset$, so due to (8.11),

$$\widetilde{\mu}_\lambda([x_{|N_k}] \times [v]) \geq m^{-N_k(\lambda T^*(T'(q))+\epsilon)} m^{-N_k'(\tau_\nu^*(\tau_\nu'(q))+\alpha'+\epsilon)} m^{-N_k\epsilon}.$$

Consequently

$$\pi_*\widetilde{\mu}_\lambda([x_{|N_k}]) \geq m^{-N_k\lambda T^*(T'(q))-N_k'\tau_\nu^*(\tau_\nu'(q))} m^{-(2N_k+2N_k')\epsilon}.$$

Since $\lim_{k\to\infty} N_k'/N_k = 1-\lambda$ and $\lim_{k\to\infty} N_{k-1}/N_k = 1$, we can conclude that

$$\overline{\dim}_{\rm loc}(\pi_*\widetilde{\mu}_\lambda, x) \leq \lambda T^*(T'(q)) + (1-\lambda)\tau_\nu^*(\tau_\nu'(q)) + 4\epsilon = \tau^*(\alpha) + 4\epsilon.$$

Letting $j \to \infty$ and then letting $\epsilon \to 0$, we see that $\overline{\dim}_{\rm loc}(\pi_*\widetilde{\mu}_\lambda, x) \leq \tau^*(\alpha)$ for $\pi_*\widetilde{\mu}_\lambda$-almost every x, and similar arguments using again Theorem 3.3(1) and the information provided by Proposition 8.12 about μ as well as (8.12) yield $\overline{\dim}_{\rm loc}(\pi_*\mu, x) \leq \alpha$ for $\pi_*\widetilde{\mu}_\lambda$-almost every x. Notice that due to part (1) of the present proposition we can now claim that the lim inf in (8.17) is in fact a limit and that the the conditional measures of $\widetilde{\mu}_\lambda$ with respect to $\pi_*\widetilde{\mu}_\lambda$ are exact dimensional with dimension $(1-\lambda)\alpha'$. □

Proof of Corollary 8.14. Due to Proposition 8.13, we only need to prove that for $\pi_*\widetilde{\mu}_\lambda$-almost every x, $\underline{\dim}_{\rm loc}(\pi_*\mu, x) \geq \alpha$. Recall that $\alpha \in (\tau'(q+), \tau'(q-))$. Hence by (2.1), for any $\beta < \alpha$,

$$\dim_H\{x \in \mathrm{supp}(\pi_*\mu) : \underline{\dim}_{\rm loc}(\pi_*\mu, x) \leq \beta\} \leq \beta q - \tau(q) < \tau^*(\alpha) = \dim(\pi_*\widetilde{\mu}_\lambda),$$

as a consequence, $\underline{\dim}_{\rm loc}(\pi_*\mu, x) > \beta$ for $\pi_*\widetilde{\mu}_\lambda$-almost every x and we are done. □

8.4. **Case (IV).** We distinguish the three cases of Proposition 6.2.

Notice that by the results obtained in the previous sections we know that $\tau_{\pi_*\mu} = \tau$ over $[0, \widetilde{q}_c)$ conditional on $\{\mu \neq 0\}$. In particular, $\tau_{\pi_*\mu}'(0+) = \tau'(0+)$.

(i) $\tau = T$ **near** $0+$. In this case, we have $\tau'(0+) = T'(0)$, and by continuity the property $\dim(\mu_q) \leq \dim(\mathbb{E}(\mu_q))$ which holds near $0+$ by Lemma 8.5 extends to the Mandelbrot measure μ_0. Also, the approach developed in Section 8.1 still applies to give $\dim_H E(\pi_*\mu, T'(0)) \geq \tau^*(T'(0))$.

(ii) $\tau = \tau_\nu$ **near** $0+$. We have $\tau'(0+) = \tau_\nu'(0)$. Let $p' = (p_i')_{0\leq i\leq m-1}$ be defined as in (6.4) and recall that ν' is the Bernoulli product associated with p'. Since we have $\sum_{i=0}^{m-1} p_i' T_i^*(T_i'(1)) \geq 0$, the approach used in Section 8.2 when $s(q) = 1$ still works and shows that conditional on $\{\mu \neq 0\}$, $\dim_{\mathrm{loc}}(\pi_*\mu, x) = -\sum_{i=0}^{m-1} p'_i \log_m(p_i) = \tau'_\nu(0)$, either at ν'-almost every $x \in \pi(K)$, or at $\pi_*\mu_{W'_1}$-almost every x if $\sum_{i=0}^{m-1} p'_i T_i^* \circ T'_i$ equals 0 over $[0,1]$ ($\mu_{W'_1}$ is the Mandelbrot measure associated with p' and the vectors $V'_{1,i}$ defined in (8.4)). Moreover, by definition of the vector p' we have $\dim(\nu') = \dim_H(\pi(K)) = -\tau(0) = \tau^*(\tau'_\nu(0))$ in the first case and $\dim(\nu') = \dim(\pi_*\mu_{W'_1}) = \dim_H(\pi(K)) = -\tau(0) = \tau^*(\tau'_\nu(0))$ in the second case. This yields $\dim_H E(\pi_*\mu, \tau'_\nu(0)) \geq \tau^*(\tau'_\nu(0))$. We notice that in the second case $\mu_{W'_1}$ coincides with the measure μ' considered in the proof of Corollary 3.5.

(iii) $\tau > \max(\tau_\nu, T)$ **near** $0+$**.** Using the notation of Proposition 6.2, we see that if $s_0 > 0$ we are exactly in the same situation as in Section 8.2, with in addition the fact that $\sum_{i=0}^{m-1} p'_i T_i^*(T'_i(1)) > 0$ is excluded if $s_0 = 1$. This yields $\dim_H E(\pi_*\mu, \tau'(0+)) \geq \tau^*(\tau'(0+))$ in this case. If $s_0 = 0$, consider the Mandelbrot measure $\mu_{W'_0}$ associated with p' and the vectors $V'_{0,i}$ defined in (8.4). Using the theory of Mandelbrot measures ([4, 5]) here again yields, with probability 1, conditional on $\mu \neq 0$, for $\mu_{W'_0}$-almost every (x, y),

$$\lim_{n\to\infty} \frac{\mu([x_{|n}, y_{|n}])}{-n\log(m)} = -\sum_{0\leq i,j\leq m-1} \mathbb{E}(W'_{0,i,j} \log_m(W_{i,j})) = \tau'(0+).$$

Also,

$$\dim(\mu_{W'_0}) = -\sum_{0\leq i,j\leq m-1} \mathbb{E}(W'_{0,i,j} \log_m(W'_{0,i,j})) = \dim(\nu') = \dim(\mathbb{E}(\pi_*\mu_{W'_0}))$$

(notice that $\mu_{W'_0}$ is here again the Mandelbrot measure μ' considered in the proof of Corollary 3.5). Consequently, for $\pi_*\mu_{W'_0}$-almost every x, we have $\overline{\dim}_{\mathrm{loc}}(\pi_*\mu, x) \leq \tau'(0+)$, and $\dim(\pi_*\mu_{W'_0}) = \dim(\nu') = \tau(0) = \tau^*(\tau'(0+))$. Then, an argument similar to that used in the proof of Corollary 8.4 again yields the desired conclusion.

9. Positive moment estimates

We start by establishing two basic lemmas on concave functions in Section 9.1. Then Section 9.2 provides positive moments estimates for $X(x_{|n})$ with respect to $\mathbb{P} \otimes \eta$, where η is a Bernoulli product.

9.1. **Lemmas.** We begin with an elementary observation.

Lemma 9.1. *Let $q > 1$ and $f : [1, q] \to \mathbb{R}$ be a continuous concave function with $f(1) = 0$. Let $k \in \mathbb{N}$. Suppose that $q_1, \dots, q_k \geq 1$ with $\sum_{i=1}^k q_k \leq q$. Then*

(i) $\sum_{i=1}^k f(q_i) \geq f(q)$ *provided that* $\sum_{i=1}^k f(q_i) \leq 0$*;*
(ii) $\sum_{i=1}^k f(q_i) \geq \min\{0, f(q)\}$.

Proof. Clearly (ii) follows from (i). To prove (i), assume that $\sum_{i=1}^k f(q_i) \leq 0$. We show below that $\sum_{i=1}^k f(q_i) \geq f(q)$.

Set $t_i = \dfrac{f(q_i) - f(1)}{q_i - 1} = \dfrac{f(q_i)}{q_i - 1}$ for $1 \le i \le k$, and $t = \dfrac{f(q)}{q-1}$. By concavity we have $t \le t_i$ for every $1 \le i \le k$. Since $\sum_{i=1}^{k} f(q_i) \le 0$, we have $t_i = f(q_i)/(q_i - 1) \le 0$ for some i, and thus $t \le t_i \le 0$. Therefore

$$\begin{aligned}\sum_{i=1}^{k} f(q_i) = \sum_{i=1}^{k} t_i(q_i - 1) &\ge \sum_{i=1}^{k} t(q_i - 1) \\ &\ge t(q-k) \\ &\ge t(q-1) = f(q).\end{aligned}$$

□

Remark 9.2. Under the conditions of Lemma 9.1, it is possible that $0 < \sum_{i=1}^{k} f(q_i) < f(q)$; for instance if we let$f(x) = x - 1$, $q_1 = 2$ and $q = 3$, then $0 < f(q_1) < f(q)$.

Lemma 9.3. *Let $q > 1$ and $f_1, \ldots, f_m$ be continuous concave functions defined on $[1, q]$ satisfying $f_j(1) = 0$ for $1 \le j \le m$. Let $(p'_1, \ldots, p'_m)$ be a probability vector. Suppose that $q_1, \ldots, q_k \ge 1$ with $\sum_{i=1}^{k} q_k \le q$. Then*

$$\sum_{j=1}^{m} p'_j m^{-\sum_{i=1}^{k} f_j(q_i)} \le \max\left\{1, \sum_{j=1}^{m} p'_j m^{-f_j(q)}\right\}. \tag{9.1}$$

Moreover, if $\sum_{j=1}^{m} p'_j m^{-f_j(q)} < 1$, then $\sum_{j=1}^{m} p'_j m^{-\sum_{i=1}^{k} f_j(q_i)} < 1$.

Remark 9.4. Under the conditions of Lemma 9.3, it is possible that

$$1 > \sum_{j=1}^{m} p'_j m^{-\sum_{i=1}^{k} f_j(q_i)} > \sum_{j=1}^{m} p'_j m^{-f_j(q)}.$$

For instance letting $f(x) = x - 1$, $q_1 = 2$ and $q = 3$, then $1 > m^{-f(q_1)} > m^{-f(q)}$.

Proof of Lemma 9.3. We first show that

$$\sum_{j=1}^{m} p'_j m^{-\sum_{i=1}^{k} f_j(q_i)} \le 1 + \sum_{j=1}^{m} p'_j m^{-f_j(q)}. \tag{9.2}$$

Set $\Lambda = \{1 \le j \le m : \ \sum_{i=1}^{k} f_j(q_i) < 0\}$. By Lemma 9.1, $\sum_{i=1}^{k} f_j(q_i) \ge f_j(q)$ for each $j \in \Lambda$. Hence

$$\begin{aligned}\sum_{j=1}^{m} p'_j m^{-\sum_{i=1}^{k} f_j(q_i)} &\le 1 + \sum_{j\in\Lambda} p'_j m^{-\sum_{i=1}^{k} f_j(q_i)} \\ &\le 1 + \sum_{j\in\Lambda} p'_j m^{-f_j(q)} \\ &\le 1 + \sum_{j=1}^{m} p'_j m^{-f_j(q)}.\end{aligned}$$

This proves (9.2).

Next we show that

$$\left(\sum_{j=1}^{m} p'_j m^{-\sum_{i=1}^{k} f_j(q_i)}\right)^n \le 1 + \left(\sum_{j=1}^{m} p'_j m^{-f_j(q)}\right)^n \tag{9.3}$$

for any $n \in \mathbb{N}$, from which (9.1) follows. Indeed setting $p'_{j_1 \ldots j_n} = p'_{j_1} \ldots p'_{j_n}$ and $f_{j_1 \ldots j_n} = f_{j_1} + \ldots + f_{j_n}$, then (9.3) can be re-written as

$$\sum_{1 \le j_1, \ldots, j_n \le m} p'_{j_1 \ldots j_n} m^{-\sum_{i=1}^{k} f_{j_1 \ldots j_n}(q_i)} \le 1 + \sum_{1 \le j_1, \ldots, j_n \le m} p'_{j_1 \ldots j_n} m^{-f_{j_1 \ldots j_n}(q)};$$

but this is just the application of (9.2) to the probability weight $(p'_{j_1 \ldots j_n})$ and the concave functions $f_{j_1 \ldots j_n}$. This finishes the proof of (9.1).

Finally, assume that $\sum_{j=1}^{m} p'_j m^{-f_j(q)} < 1$. By (9.1), $\sum_{j=1}^{m} p'_j m^{-\sum_{i=1}^{k} f_j(q_i)} \le 1$. We need to show that the inequality is strict. Suppose on the contrary that

$$\sum_{j=1}^{m} p'_j m^{-\sum_{i=1}^{k} f_j(q_i)} = 1.$$

Define $g(x) = \sum_{j=1}^{m} p'_j m^{-x \sum_{i=1}^{k} f_j(q_i)}$ for $x \in \mathbb{R}$. Then g is convex. Notice that on a small neighborhood U of 1, $\sum_{j=1}^{m} p'_j m^{-x f_j(q)} < 1$ for $x \in U$. For any fixed $x \in U$, applying (9.1) to the functions $x f_i$, we obtain that $g(x) \le 1$. Hence g takes a local maximum at $x = 1$. However g is convex and analytic on $\mathbb{R}$, it follows that g is constant on $\mathbb{R}$ and therefore

$$\sum_{i=1}^{k} f_j(q_i) = 0$$

for any $1 \le j \le m$. Then by Lemma 9.1(i), $f_j(q) \le 0$ for all $1 \le j \le m$, which contradicts the assumption that $\sum_{j=1}^{m} p'_j m^{-f_j(q)} < 1$. This finishes the proof of the lemma. $\square$

9.2. **Positive moments estimates for X_n.** Let us first recall some notation. We are given $W = (W_{i,j})_{0 \le i,j \le m-1}$, a non-negative random vector with $\mathbb{E}(\sum_{i,j} W_{i,j}) = 1$. Let $q > 1$ and assume that $\mathbb{E}(\sum_{i,j} W_{i,j}^q) < \infty$. Set $p_i = \mathbb{E}(\sum_j W_{i,j})$. Set

$$V_{i,j} = \begin{cases} W_{i,j}/p_i, & \text{if } p_i \ne 0, \\ 1/m, & \text{if } p_i = 0. \end{cases}$$

For $t \in [0, q]$, set

$$T(t) = -\log_m \mathbb{E}(\sum_{i,j} W_{i,j}^t), \quad T_i(t) = -\log_m \mathbb{E}(\sum_j V_{i,j}^t).$$

Then T and T_i $(0 \le i \le m-1)$ are well-defined continuous concave functions on $[0, q]$, with $T(1) = T_i(1) = 0$. Set $\Sigma = \{0, 1, \ldots, m-1\}^{\mathbb{N}}$. Let μ be the (random) Mandelbrot measure on $\Sigma \times \Sigma$ generated by W. Set $Y = \|\mu\|$ to be the total mass of μ and assume that $T(q) > 0$. By Kahane-Peyriere [36] and Durrett-Liggett [22], this is equivalent to the property that $0 < \mathbb{E}(Y^q) < \infty$. For each $(u, v) \in (\Sigma \times \Sigma)_*$, let $Y(u, v)$ be defined as in (2.3). We defined in Section 5

$$X(u) = \sum_{v \in \Sigma_{|u|}} Y(u, v) \prod_{j=1}^{|u|} V_{u_j, v_j}(u|_{j-1}, v|_{j-1}), \quad u \in \Sigma_*. \tag{9.4}$$

and $X_n(x) = X(x_{|n})$ for all $x \in \Sigma$ and $n \geq 1$.

Given any Bernoulli product η on Σ generated by a probability vector $(p'_0, \ldots, p'_{m-1})$, we seek estimates of $\mathbb{E}_{\mathbb{P} \otimes \eta}(X_n^q)$, i.e. $\sum_{|u|=n} \eta([u]) \mathbb{E}(X(u)^q)$.

For short we write $V_{u_1, v_1} = V_{u_1, v_1}(\epsilon, \epsilon)$ and

$$X_1(u, j) = \sum_{v \in \Sigma_{|u|} \colon v_1 = j} Y(u, v) \prod_{k=2}^{|u|} V_{u_k, v_k}(u|_{k-1}, v|_{k-1}), \quad j = 0, \ldots, m-1.$$

Then

$$X(u) = \sum_{j=0}^{m-1} V_{u_1, j} X_1(u, j). \tag{9.5}$$

We emphasize that $X_1(u, j)$ $(j = 0, \ldots, m-1)$ are independent copies of $X(\sigma u)$. Moreover, they are independent of $V_{u_1, j'}$ $(j' = 0, \ldots, m-1)$.

By (9.4) and the assumption that $\mathbb{E}(Y^q) < \infty$, we have $\mathbb{E}(X(u)^q) < \infty$ for each $u \in \Sigma_*$. In particular, $\mathbb{E}(X(u)) = 1$.

For $n \in \mathbb{N}$, set

$$\Delta_n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \ x_1 \leq \cdots \leq x_n\}.$$

Lemma 9.5. *Let $t \in (1, q]$ and $u \in \Sigma_*$. Then*

(i) $\mathbb{E}(X(u)^t) \geq m^{-T_{u_1}(t)} \mathbb{E}(X(\sigma u)^t)$.

(ii) *There exists a positive constant C (depending on q) such that*

$$\mathbb{E}(X(u)^t) \leq m^{-T_{u_1}(t)} \mathbb{E}(X(\sigma u)^t) + C + C \sum_{(q_1, \ldots, q_s) \in \mathcal{I}_t} \prod_{j=1}^{s} \mathbb{E}\left(X(\sigma u)^{q_j}\right), \tag{9.6}$$

where $\mathcal{I}_t$ is defined by

$$\begin{aligned} \mathcal{I}_t := &\left\{ \left(\frac{k_1 t}{\lceil t \rceil}, \ldots, \frac{k_s t}{\lceil t \rceil} \right) \in \Delta_s : \ s, k_i \in \mathbb{N} \cap [2, \infty), \ \sum_{i=1}^{s} k_i \leq \lceil t \rceil \right\} \\ &\cup \left\{ \frac{kt}{\lceil t \rceil} \in \mathbb{R} : \ k \in \mathbb{N}, \ 2 \leq k \leq \lceil t \rceil - 1 \right\}, \end{aligned} \tag{9.7}$$

here $\lceil t \rceil$ is the smallest integer $\geq t$.

Proof. Since $t > 1$, by (9.5) and the super-additivity of $x \mapsto x^t$ on $\mathbb{R}_+$,

$$X(u)^t = \left(\sum_{j=0}^{m-1} V_{u_1,j} X_1(u,j)\right)^t \geq \sum_{j=0}^{m-1} V_{u_1,j}^t X_1(u,j)^t.$$

Taking expectations on both sides, we obtain (i).

To see (ii), by (9.5) and the sub-additivity of $x \mapsto x^{t/\lceil t\rceil}$ on $\mathbb{R}_+$,

$$\begin{aligned} X(u)^t &= \left(\left(\sum_{j=0}^{m-1} V_{u_1,j} X_1(u,j)\right)^{t/\lceil t\rceil}\right)^{\lceil t\rceil} \\ &\leq \left(\sum_{j=0}^{m-1} V_{u_1,j}^{t/\lceil t\rceil} X_1(u,j)^{t/\lceil t\rceil}\right)^{\lceil t\rceil} \\ &= \sum_{k_0+\ldots+k_{m-1}=\lceil t\rceil} \frac{\lceil t\rceil!}{k_0!\cdots k_{m-1}!} \prod_{j=0}^{m-1} (V_{u_1,j} X_1(u,j))^{k_j t/\lceil t\rceil}. \end{aligned}$$

Taking expectations on both sides yields

$$\mathbb{E}(X(u)^t) \leq \sum_{k_0+\ldots+k_{m-1}=\lceil t\rceil} \frac{\lceil t\rceil!}{k_0!\cdots k_{m-1}!} \mathbb{E}\left(\prod_{j=0}^{m-1} V_{u_1,j}^{k_j t/\lceil t\rceil}\right) \prod_{s=0}^{m-1} \mathbb{E}(X(\sigma u)^{k_s t/\lceil t\rceil}),$$

from which (9.6) follows, thanks to the fact that $\mathbb{E}(X(\sigma u)^p) \leq 1$ for $p \in [0,1]$; the involved constant C can be taken as $m^q \sup_{1\leq q'\leq q} \mathbb{E}(\sum_j V_{u_1,j}^{q'})$. Here we use the fact that

$$\begin{aligned} \mathbb{E}\left(\prod_{j=0}^{m-1} V_{u_1,j}^{k_j t/\lceil t\rceil}\right) &\leq \prod_{j=0}^{m-1} (\mathbb{E}(V_{u_1,j}^t))^{k_j/\lceil t\rceil} \\ &\leq \prod_{j=0}^{m-1} \left(\mathbb{E}\left(\sum_{s=0}^{m-1} V_{u_1,s}^t\right)\right)^{k_j/\lceil t\rceil} \\ &= \mathbb{E}\left(\sum_{s=0}^{m-1} V_{u_1,s}^t\right) \leq \sup_{1\leq q'\leq q} \mathbb{E}\left(\sum_j V_{u_1,j}^{q'}\right), \end{aligned}$$

where the first '$\leq$' comes from the Hölder inequality. □

Next we would like to establish an analogue of Lemma 9.5 for $\prod_{j=1}^k \mathbb{E}(X(u)^{t_j})$, where $t_1,\ldots,t_k \in (1,q]$ with $t_1+\ldots+t_k \leq q$. First we introduce some notation. For $(x_1,\ldots,x_n) \in \Delta_n$ and $(y_1,\ldots,y_m) \in \Delta_m$, let $(z_1,\ldots,z_{n+m}) \in \Delta_{n+m}$ be the vector re-ordered from the numbers $x_1,\ldots,x_n,y_1,\ldots,y_m$; and write

$$(x_1,\ldots,x_n) \oplus (y_1,\ldots,y_m) := (z_1,\ldots,z_{n+m}).$$

Clearly, the operation $\oplus$ is commutative. By convention, we write $(x_1,\ldots,x_n) \oplus \emptyset = (x_1,\ldots,x_n)$, where $\emptyset$ denotes the empty set.

For $t_1, \ldots, t_k \in (1, q]$ with $t_1 \leq \cdots \leq t_k$ and $t_1 + \ldots + t_k \leq q$, we write

$$\mathcal{I}_{t_1,\ldots,t_k} = \{w_1 \oplus \cdots \oplus w_k : \ w_i \in \mathcal{I}_{t_i} \cup \{t_i\} \cup \{\emptyset\}\} \backslash \{(t_1, \ldots, t_k)\}, \tag{9.8}$$

where $\mathcal{I}_t$ is defined as in (9.7). The following simple property comes from the definition of $\mathcal{I}_{(\cdot)}$:

Lemma 9.6. *Assume that $\mathcal{I}_{t_1,\ldots,t_k} \neq \emptyset$. Then for any $(q_1, \ldots, q_\ell) \in \mathcal{I}_{t_1,\ldots,t_k}$, $q_1 + \cdots + q_\ell \leq t_1 + \cdots + t_k$; moreover, either $\ell \geq k + 1$ or $q_1 + \ldots + q_\ell \leq t_1 + \ldots + t_k - 1/2$.*

Proof. For any vector $w \in \mathbb{R}^m$, let $\|w\|$ denote the sum of the absolute values of its components. Clearly by (9.7), for any $t > 1$ and $w \in \mathcal{I}_t$, we have $\|w\| \leq t$. Fix $(q_1, \ldots, q_\ell) \in \mathcal{I}_{t_1,\ldots,t_k}$. Then there exist $w_i \in \mathcal{I}_{t_i} \cup \{t_i\} \cup \{\emptyset\}$ $(i = 1, \ldots, k)$ such that $(q_1, \ldots, q_\ell) = w_1 \oplus \cdots \oplus w_k$. Therefore $q_1 + \cdots + q_\ell = \|w_1\| + \cdots + \|w_k\| \leq t_1 + \cdots + t_k$. If $w_i = \emptyset$ for some i, then $q_1 + \cdots + q_\ell \leq (t_1 + \cdots + t_k) - t_i < (t_1 + \cdots t_k) - 1$. If otherwise, we have $w_i \in \mathcal{I}_{t_i} \cup \{t_i\}$ for all $1 \leq i \leq k$, and $w_j \in \mathcal{I}_{t_j}$ for at least one j; in such case, either $\|w_j\| \leq t_j - \frac{t_j}{[t_j]} \leq t_j - \frac{1}{2}$ or the dimension of w_j is ≥ 2, hence either $q_1 + \ldots + q_\ell \leq t_1 + \cdots + t_k - 1/2$ or $\ell \geq k+1$. □

As a direct application of Lemma 9.5, we have

Lemma 9.7. *Let $t_1, \ldots, t_k \in (1, q]$ so that $t_1 \leq \cdots \leq t_k$ and $t_1 + \ldots + t_k \leq q$. Let $u \in \Sigma_*$. Then*

(i) $\prod_{j=1}^k \mathbb{E}(X(u)^{t_j}) \geq m^{-\sum_{i=1}^k T_{u_1}(t_i)} \prod_{j=1}^k \mathbb{E}(X(\sigma u)^{t_j})$.

(ii) *There exists a positive constant C' (depending on q) such that*

$$\begin{aligned} \prod_{j=1}^k \mathbb{E}(X(u)^{t_j}) \leq & m^{-\sum_{i=1}^k T_{u_1}(t_i)} \prod_{j=1}^k \mathbb{E}(X(\sigma u)^{t_j}) \\ & + C' + C' \sum_{(q_1,\ldots,q_\ell) \in \mathcal{I}_{t_1,\ldots,t_k}} \prod_{j=1}^\ell \mathbb{E}\left(X(\sigma u)^{q_j}\right). \end{aligned} \tag{9.9}$$

Proposition 9.8. *Let $q > 1$ such that $T(q) > 0$. Let η be the Bernoulli product measure on Σ generated by a probability vector $(p'_0, \ldots, p'_{m-1})$. Set $A := \max\{1, \sum_{i=0}^{m-1} p'_i m^{-T_i(q)}\}$. Then the following statements hold:*

(i) *There exists a polynomial f_q depending on W and q such that*

$$A^n \leq \sum_{u \in \Sigma_n} \eta([u]) \mathbb{E}(X(u)^q) \leq f_q(n) A^n, \quad \forall n \in \mathbb{N}. \tag{9.10}$$

Moreover, if $q \in (1, 2]$ and $\sum_{i=0}^{m-1} p'_i m^{-T_i(q)} < 1$, then the polynomial f_q can be replaced by a positive constant.

(ii) *More generally, for any $t_1, \ldots, t_k \in (1, q]$ with $t_1 \leq \cdots \leq t_k$ and $t_1 + \ldots + t_k \leq q$, there exists a polynomial $f_{t_1,\ldots,t_k}$ such that*

$$1 \leq \sum_{u \in \Sigma_n} \eta([u]) \prod_{j=1}^k \mathbb{E}(X(u)^{t_j}) \leq f_{t_1,\ldots,t_k}(n) A^n, \quad \forall n \in \mathbb{N}. \tag{9.11}$$

Proof. Since $q > 1$, $\mathbb{E}(X(u)^q) \geq \mathbb{E}(X(u))^q = 1$ for each $u \in \Sigma_*$, and thus

$$\sum_{u \in \Sigma_n} \eta([u]) \mathbb{E}(X(u)^q) \geq 1. \tag{9.12}$$

Similarly

$$\sum_{u \in \Sigma_n} \eta([u]) \prod_{j=1}^{k} \mathbb{E}(X(u)^{t_j}) \geq 1. \tag{9.13}$$

On the other hand, by Lemma 9.5(i),

$$\begin{aligned} \sum_{u \in \Sigma_n} \eta([u]) \mathbb{E}(X(u)^q) &\geq \left(\sum_{i=0}^{m-1} p_i' m^{-T_i(q)} \right) \sum_{u \in \Sigma_{n-1}} \eta([u]) \mathbb{E}(X(u)^q) \\ &\geq \left(\sum_{i=0}^{m-1} p_i' m^{-T_i(q)} \right)^n \mathbb{E}(Y^q) \geq \left(\sum_{i=0}^{m-1} p_i' m^{-T_i(q)} \right)^n. \end{aligned} \tag{9.14}$$

Combining (9.14) with (9.12) yields

$$\sum_{u \in \Sigma_n} \eta([u]) \mathbb{E}(X(u)^q) \geq A^n. \tag{9.15}$$

This completes the proof of the first inequality in (9.10).

To show the second inequality in (9.10), let $t_1, \ldots, t_k \in (1, q]$ with $t_1 \leq \cdots \leq t_k$ and $t_1 + \ldots + t_k \leq q$. By Lemma 9.3,

$$\sum_{j=0}^{m-1} p_j' m^{-\sum_{i=1}^{k} T_j(t_i)} \leq A. \tag{9.16}$$

This together with Lemma 9.7(ii) yields

$$\begin{aligned} \sum_{u \in \Sigma_n} \eta([u]) \prod_{j=1}^{k} \mathbb{E}(X(u)^{t_j}) \leq & A \sum_{u \in \Sigma_{n-1}} \eta([u]) \prod_{j=1}^{k} \mathbb{E}(X(u)^{t_j}) + C' \\ & + C' \sum_{(q_1, \ldots, q_\ell) \in \mathcal{I}_{t_1, \ldots, t_k}} \sum_{u \in \Sigma_{n-1}} \eta([u]) \prod_{j=1}^{\ell} \mathbb{E}\left(X(u)^{q_j}\right), \end{aligned} \tag{9.17}$$

Write $S_n(t_1, \ldots, t_k) := \sum_{u \in \Sigma_n} \eta([u]) \prod_{j=1}^{k} \mathbb{E}(X(u)^{t_j})$. Then (9.17) can be re-written as

$$S_n(t_1, \ldots, t_k) \leq A S_{n-1}(t_1, \ldots, t_k) + C' + C' \sum_{(q_1, \ldots, q_\ell) \in \mathcal{I}_{t_1, \ldots, t_k}} S_{n-1}(q_1, \ldots, q_\ell) \tag{9.18}$$

for $n \in \mathbb{N}$.

We claim that there exists an increasing polynomial function $f_{t_1, \ldots, t_k}$ such that

$$S_n(t_1, \ldots, t_k) \leq f_{t_1, \ldots, t_k}(n) A^n, \qquad \forall n \in \mathbb{N}. \tag{9.19}$$

Clearly the claim is true in the case when $\mathcal{I}_{t_1, \ldots, t_k} = \emptyset$. Indeed in such case, by (9.18),

$$S_n(t_1, \ldots, t_k) \leq A\, S_{n-1}(t_1, \ldots, t_k) + C', \quad \forall n \in \mathbb{N},$$

and thus

$$S_n(t_1,\ldots,t_k) = A^n\ S_0(t_1,\ldots,t_k) + \sum_{j=1}^{n} A^{n-j}\left(S_j(t_1,\ldots,t_k) - A\ S_{j-1}(t_1,\ldots,t_k)\right)$$
$$\leq A^n\ S_0(t_1,\ldots,t_k) + \sum_{j=1}^{n} C' A^{n-j} \leq nA^n(C' + S_0(t_1,\ldots,t_k)).$$

Next we consider the case when $\mathcal{I}_{t_1,\ldots,t_k} \neq \emptyset$. Suppose that for each $(q_1,\ldots,q_\ell) \in \mathcal{I}_{t_1,\ldots,t_k}$, there exists an increasing polynomial function $f_{q_1,\ldots,q_\ell}$ such that

$$S_n(q_1,\ldots,q_\ell) \leq f_{q_1,\ldots,q_\ell}(n)A^n, \qquad \forall n \in \mathbb{N}.$$

Set $g = C' + C' \sum_{(q_1,\ldots,q_\ell)\in\mathcal{I}_{t_1,\ldots,t_k}} f_{q_1,\ldots,q_\ell}$. Then g is an increasing polynomial. By (9.18),

$$S_n(t_1,\ldots,t_k) - A\ S_{n-1}(t_1,\ldots,t_k) \leq g(n-1)A^{n-1}, \qquad \forall n \in \mathbb{N}.$$

Therefore

$$S_n(t_1,\ldots,t_k) - A^n\ S_0(t_1,\ldots,t_k) = \sum_{j=1}^{n} A^{n-j}(S_j(t_1,\ldots,t_k) - A\ S_{j-1}(t_1,\ldots,t_k))$$
$$\leq A^{n-1}\sum_{j=1}^{n} g(j-1) \leq A^{n-1} n g(n),$$

Hence $S_n(t_1,\ldots,t_k)$ is bounded by $f_{t_1,\ldots,t_k}(n)A^n$ with $f_{t_1,\ldots,t_k}(x) := xg(x) + S_0(t_1,\ldots,t_k)$.

According to the arguments in the above two paragraphs, if the claim (9.19) is false at $T_1 := (t_1,\ldots,t_k)$, then $\mathcal{I}_{T_1} \neq \emptyset$ and moreover there exists $T_2 \in \mathcal{I}_{T_1}$ such that (9.19) is false at T_2. Repeatedly applying the arguments, we see that there exist

$$T_n \in \mathcal{I}_{T_{n-1}} \neq \emptyset,\ n = 2, 3, \ldots$$

such that (9.19) is false at T_n. However, by Lemma 9.6, the sequence $(\|T_n\|)_{n=1}^{\infty}$ is non-increasing and is bounded above by q; and moreover, there are infinitely many n such that $\|T_n\| \leq \|T_{n-1}\| - 1/2$ (because the dimension of T_n can not keep strictly increasing for q consecutive integers of n), which leads to a contradiction. This proves the claim (9.19).

Applying (9.19) to the particular case when $k = 1$, we have

$$\sum_{u\in\Sigma_n} \eta([u])\mathbb{E}(X(u)^q) \leq f_q(n)A^n, \qquad \forall n \in \mathbb{N}$$

for some polynomial f_q. This, together with (9.15), yields (9.10). In the meantime, (9.11) follows from (9.19) and (9.13).

Finally, assume that $q \in (1, 2]$ and $\sum_{i=0}^{m-1} p_i' m^{-T_i(q)} < 1$. By the definition (9.7), $\mathcal{I}_q = \emptyset$. Hence applying (9.9) yields

$$\sum_{u\in\Sigma_n} \eta([u])\mathbb{E}(X(u)^q) \leq B \sum_{u\in\Sigma_{n-1}} \eta([u])\mathbb{E}(X(u)^q) + C', \tag{9.20}$$

with $B := \sum_{i=0}^{m-1} p'_i m^{-T_i(q)} < 1$. Iterating (9.20) yields that

$$\sum_{u\in\Sigma_n} \eta([u])\mathbb{E}(X(u)^q) \le C'(1 + B + B^2 + \cdots) = \frac{C'}{1-B}.$$

This finishes the proof of the proposition. □

Corollary 9.9. *Let $q > 1$ such that $T(q) > 0$. Then there exists a polynomial f_q depending on W and q such that*

$$m^{-n\min\{\tau_\nu(q),T(q)\}} \le \mathbb{E}\Big(\sum_{u\in\Sigma_n} \pi_*\mu([u])^q\Big) \le f_q(n) m^{-n\min\{\tau_\nu(q),T(q)\}} \tag{9.21}$$

for all $n \in \mathbb{N}$. Furthermore, if $q \in (1, 2]$ and $\tau_\nu(q) < T(q)$, the polynomial f_q can be replaced by a positive constant.

Proof. Let ν_q denote the Bernoulli product measure on Σ generated by the probability weight $(p'_0, \dots, p'_{m-1})$, where $p'_i := p_i^q / \sum_{j=0}^{m-1} p_j^q$. Then

$$\sum_{u\in\Sigma_n} \pi_*\mu([u])^q = \sum_{u\in\Sigma_n} \nu([u])^q X(u)^q = m^{-n\tau_\nu(q)} \sum_{u\in\Sigma_n} \nu_q([u]) X(u)^q. \tag{9.22}$$

Set $A = \max\{1, \sum_{j=0}^{m-1} p'_j m^{-T_j(q)}\}$. Then $A = \max\{1, m^{\tau_\nu(q)-T(q)}\}$, due to the fact that $\sum_{j=0}^{m-1} p'_j m^{-T_j(q)} = m^{\tau_\nu(q)} \sum_{j=0}^{m-1} p_j^q m^{-T_j(q)} = m^{\tau_\nu(q)-T(q)}$ (cf. (3.3)).

By Proposition 9.8, there is a polynomial function f_q such that

$$A^n \le \sum_{u\in\Sigma_n} \nu_q([u])\mathbb{E}(X(u)^q) \le f_q(n) A^n, \quad \forall n \in \mathbb{N}. \tag{9.23}$$

Now (9.21) follows directly from (9.22) and (9.23). □

Corollary 9.10. *Let $q > 1$ such that $T(q) > 0$. Then $\tau_{\pi_*\mu}(q) \ge \min\{\tau_\nu(q), T(q)\}$.*

Proof. This is a direct consequence of Corollary 9.9 and Lemma C.1. □

10. Results for projections of planar Mandelbrot measures

We begin with a general fact on the relation between the increasing part of the Hausdorff spectrum of a measure obeying the multifractal formalism on Σ and that of its image on $[0, 1]$. Indeed, the natural projection form Σ to $[0, 1]$ does not map cylinders to centered balls, so some care is needed to claim that these spectra do coincide at a given exponent.

Let Π denote the natural from Σ onto $[0, 1]$, namely $\Pi(x) = \sum_{i=1}^\infty x_i m^{-i}$.

Proposition 10.1. *Let ρ be a positive and finite Borel measure on Σ. Let $\alpha \in [0, \tau'_\rho(0+)]$ and suppose that there exists a positive and finite Borel measure ρ_α on Σ such that $\rho_\alpha(E(\rho, \alpha)) > 0$ and $\underline{\dim}_H(\rho_\alpha) \ge \tau^*_\rho(\alpha) > 0$. Then $\dim_H E(\widetilde{\rho}, \alpha) = \tau^*_{\widetilde{\rho}}(\alpha)$, where $\widetilde{\rho} = \Pi_*\rho$ stands for the natural projection of ρ to $[0, 1]$.*

Proof. It is a simple fact that for all $q \geq 0$, $\tau_\rho(q) = \tau_{\widetilde{\rho}}(q)$; hence τ_ρ^* and $\tau_{\widetilde{\rho}}^*$ coincide on $[0, \tau_\rho'(0+)]$.

Setting $\widetilde{\rho}_\alpha = \Pi_* \rho_\alpha$, it is also clear that $\underline{\dim}_H(\widetilde{\rho}_\alpha) = \underline{\dim}_H(\rho_\alpha)$, hence $\underline{\dim}_H(\widetilde{\rho}_\alpha) \geq \tau_{\widetilde{\rho}}^*(\alpha)$. According to (2.1), for all $0 \leq \alpha' < \alpha$, $\dim_H \underline{E}^{\leq}(\widetilde{\rho}, \alpha') \leq \tau_{\widetilde{\rho}}^*(\alpha')$. Moreover, it follows from the fact that τ_ρ^* takes at least one positive value that $\tau_\rho^* = \tau_{\widetilde{\rho}}^*$ is strictly increasing over $(-\infty, \tau_\rho'(0+)] \cap \mathrm{dom}(\tau_\rho^*)$ if this interval is not reduced to a singleton. This implies that $\widetilde{\rho}_\alpha(\bigcup_{0 \leq \alpha' < \alpha} \underline{E}^{\leq}(\widetilde{\rho}, \alpha')) = 0$. Now, let $G_\alpha \subset \Pi(E(\rho, \alpha))$ of full $\widetilde{\rho}_\alpha$-positive measure. Without loss of generality we assume that G_α contains no m-adic number and no element of $\bigcup_{0 \leq \alpha' < \alpha} \underline{E}^{\leq}(\widetilde{\rho}, \alpha')$, i.e. $\underline{\dim}_{\mathrm{loc}}(\widetilde{\rho}, t) \geq \alpha$ for all $t \in G_\alpha$. Fix $t \in G_\alpha$. For $n \geq 1$, denote by $I_n(t)$ the m-adic interval of generation n which contains t. For all $\epsilon > 0$, for n large enough, we have $\widetilde{\rho}(I_n(t)) \geq \rho([(\Pi^{-1}(x))_{|n}]) \geq m^{n(\alpha+\epsilon)}$, hence $\widetilde{\rho}(B(t, m^{-n})) \geq m^{n(\alpha+\epsilon)}$. Consequently, $\overline{\dim}_{\mathrm{loc}}(\widetilde{\rho}, t) \leq \alpha + \epsilon$ for all $\epsilon > 0$. Since we also have $\underline{\dim}_{\mathrm{loc}}(\widetilde{\rho}, t) \geq \alpha$, we get $G_\alpha \subset E(\widetilde{\rho}, \alpha)$ hence the desired lower bound $\dim_H E(\widetilde{\rho}, \alpha) \geq \tau_{\widetilde{\rho}}^*(\alpha)$. $\square$

Now we can state our results for planar Mandelbrot measures. Let $\widetilde{\pi}$ stand for the orthogonal projections from the Euclidean plane $\mathbb{R}^2$ onto its x-axis. Let P be the natural projection from Σ^2 onto $[0,1]^2$, namely $P(x, y) = (\Pi(x), \Pi(y))$. Let μ be a Mandelbrot measure on the symbolic space $\Sigma \times \Sigma$ as in Sections 2 to 9. The measure $\widetilde{\mu}$ and its support $\widetilde{K}$, obtained as the respective images of μ and K by P, are realizations of the measure and the set considered in the introduction. This is due to the fact that, with probability 1, μ assigns zero mass to sets of the form $\{x\} \times \Sigma$ or $\Sigma \times \{y\}$ when $\Pi(x)$ or $\Pi(y)$ is a m-adic point (the verification of this fact is left to the reader). Also, the orthogonal projection of $\widetilde{\mu}$ to the x-axis, namely $\widetilde{\pi}_* \widetilde{\mu}$, is equal to $\Pi_*(\pi_* \mu)$, and the expectation of $\widetilde{\pi}_* \widetilde{\mu}$ is equal to $\widetilde{\nu} = \Pi_* \nu$.

The following properties are easily checked: for any $w \in \Sigma_*$, the equalities $\widetilde{\pi}_* \widetilde{\mu}(\Pi([w])) = \pi_* \mu([w])$ and $\widetilde{\nu}(\Pi([w])) = \nu([w])$ hold. Also, the m-adic intervals can be used to discuss differentiability properties of measures and computations of Hausdorff dimensions, and the mapping Π preserves Hausdorff dimension. This implies the following result.

Theorem 10.2. *Theorem 3.1, Theorem 3.3 and Corollary 3.5 hold if we replace $(\mu, \pi_* \mu, \nu, K)$ therein by $(\widetilde{\mu}, \widetilde{\pi}_* \widetilde{\mu}, \widetilde{\nu}, \widetilde{K})$ and replace $(\mu', \pi_* \mu')$ by $(\widetilde{\mu'}, \widetilde{\pi}_* \widetilde{\mu'})$.*

Using Proposition 10.1, we can also transfer to $\widetilde{\pi}_* \widetilde{\mu}$ the multifractal properties of $\pi_* \mu$.

Theorem 10.3. *Theorem 3.7 hold if we replace $(\mu, \pi_* \mu)$ therein by $(\widetilde{\mu}, \widetilde{\pi}_* \widetilde{\mu})$.*

Finally, we add a general proposition about L^q-spectra. It is certainly not new but difficult to find explicitly written in the literature, except in the context of multiplicative chaos and statistical mechanics (see [14] for instance), where it signs a glassy phase transition. This result is used at the beginning of Section 8.

Proposition 10.4. *Let ρ be a positive and finite Borel measure on Σ. Then $\tau_\rho(q) \geq \frac{q}{q'}\tau_\rho(q')$ for all $q \geq q' > 0$. As a consequence, if $\tau_\rho(q_c) = \tau_\rho'(q_c-)q_c$ at some $q_c > 0$, then $\tau_\rho(q) = \tau_\rho'(q_c-)q$ for all $q \geq q_c$.*

Proof. The first claim follows from writing $\sum_{|u|=n}\rho([u])^q = \sum_{|u|=n}(\rho([u])^{q'})^{q/q'}$ and using the subbadditivity of $x \geq 0 \mapsto x^{q/q'}$. The second claim follows from the concavity of τ_ρ, which implies that $\tau_\rho(q) \leq \tau_\rho'(q_c-)q$ for $q \geq q_c$, while the first claim implies $\tau_\rho(q) \geq \frac{q}{q_c}\tau_\rho(q_c) = \tau_\rho'(q_c-)q$. $\square$

11. Final remarks

As a consequence of our study of the multifractal formalism, we can achieve a part of the multifractal analysis of the number $N_n(x)$ of cylinders of generation n of the form $[x_{|n}, v]$, $v \in \Sigma_n$, which intersect the support K_n of μ_n. Specifically, if $n \geq 1$ and $u \in \Sigma_n$ we set

$$N(u) = \#\{v \in \Sigma_n : Q(u,v) > 0\}.$$

Then $N_n(x) = N(x_{|n})$. This number measures the overlapping amount over $[x_{|n}]$ when one projects K_n onto $\pi(K)$, and equivalently its asymptotic behavior yields the box-counting dimension of $\pi^{-1}(\{x\})$.

Corollary 11.1. (1) *Suppose that $\mathbb{E}(N_i) \leq 1$ for all $0 \leq i \leq m-1$ such that $\mathbb{E}(N_i) > 0$. With probability one, conditional on $\{K \neq \emptyset\}$, for all $x \in \pi(K)$ we have $\dim_B \pi^{-1}(\{x\}) = \lim_{n\to\infty} \frac{\log N_n(x)}{n} = 0$.*

(2) *Suppose that $\mathbb{E}(N_i) > 1$ for at least one $0 \leq i \leq m-1$. Let φ be defined as in (3.6). Let q_0 be the unique point at which φ attains its minimum over $[0,1]$. Define*

$$P : q \mapsto \begin{cases} \log(m)\cdot\varphi(q_0) & \text{if } 0 \leq q \leq q_0 \\ \log(m)\cdot\inf\{\varphi(q/s) : q \leq s \leq 1\} & \text{if } q_0 < q \leq 1 \\ \max(\log\mathbb{E}(N), \log(m)\cdot\varphi(q)) & \text{if } q > 1 \end{cases}. \tag{11.1}$$

If $q_0 < 1$ or $q_0 = 1$ and $\varphi'(1) = 0$, then P is differentiable over $\mathbb{R}_+$, analytic over $[0,q_0)\cup(q_0,\infty)$ and it has a second order phase transition at q_0. Specifically, $P \equiv \log(m)\cdot\varphi(q_0)$ over $[0,q_0)$ and $P \equiv \log(m)\cdot\varphi$ over (q_0,∞).

If $q_0 = 1$ and $\varphi'(1) < 0$, then there exists a unique $q_0' > 1$ such that $P(q_0') = \log\mathbb{E}(N)$, and P is analytic over $[0,q_0')\cup(q_0',\infty)$, with $P \equiv \log\mathbb{E}(N)$ over $[0,q_0')$ and $P \equiv \log(m)\cdot\varphi$ over (q_0',∞). Moreover, P has a first order phase transition at q_0'.

With probability 1, conditional on $\{\pi(K) \neq \emptyset\}$, for all $q \geq 0$ we have

$$\lim_{n\to\infty}\frac{1}{n}\log\sum_{|u|=n}\mathbf{1}_{\{N(u)\geq 1\}}N(u)^q = P(q). \tag{11.2}$$

(3) *If $\alpha \in \{P'(q^-), P'(q^+)\}$ for some $q > 0$ or $\alpha = P'(0+)$, then, with probability 1, conditional on $\{K \neq \emptyset\}$,*

$$\dim_H \left\{ x \in \pi(K) : \dim_B \pi^{-1}(\{x\}) = \lim_{n\to\infty} \frac{\log N_n(x)}{n} = \alpha \right\} = \frac{1}{\log(m)} \inf\{P(q) - \alpha q : q \geq 0\}.$$

Parts (2)-(3) of this corollary follow from the application of Theorem 3.7 to the branching measure, i.e. the Mandelbrot measure μ' associated with

$$W' = (\mathbb{E}(N)^{-1} \mathbf{1}_{\{W_{i,j}>0\}})_{0\leq i,j\leq m-1}.$$

Indeed, $N_n(x) = \mathbb{E}(N)^n \mu'_n([x_n])$ and we can simply use Remark 3.9.

For Part (1), under our assumptions property (11.2) still holds, with P given by (11.1), for the same reason as in item (2). It is then direct to check that $P(q) = \log \mathbb{E}(N)$ for all $q \geq 0$. Consequently, conditional on $\{\pi(K) \neq \emptyset\}$, for any $\epsilon > 0$, for any $q > 0$, if n is large enough,

$$\begin{aligned}
\#\{u \in \Sigma_n : N(u) \geq m^{n\epsilon}\} &\leq m^{-nq\epsilon} \sum_{|u|=n} \mathbf{1}_{\{N(u)>0\}} N(u)^q \\
&\leq m^{-nq\epsilon} m^{n(\log(\mathbb{E}(N))+\epsilon)} \\
&= m^{-n((q-1)\epsilon - \log \mathbb{E}(N))}.
\end{aligned}$$

Choosing $q > 1$ such that $(q-1)\epsilon - \log \mathbb{E}(N) > 0$ yields that for n large enough, $\#\{u \in \Sigma_n : N(u) \geq m^{n\epsilon}\} < 1$ so $\{u \in \Sigma_n : N(u) \geq m^{n\epsilon}\}$ is empty. Thus $\limsup_{n\to\infty} \frac{\log N_n(x)}{n} \leq \epsilon$ for all $x \in \pi(K)$. Since ϵ is arbitrary and $N_n(x) \geq 1$, this yields $\lim_{n\to\infty} \frac{\log N_n(x)}{n} = 0$ for all $x \in \pi(K)$.

Appendix A. Basic facts about extinction probabilities

Proposition A.1. *The events $\{\mu \neq 0\}$ and $\{K := \bigcap_{n\geq 1} \operatorname{supp}(\mu_n) \neq \emptyset\}$ coincide up to a set of probability 0, over which $K = \operatorname{supp}(\mu)$.*

Proof. Recall that we defined $N = \sum_{1\leq i,j\leq m-1} \mathbf{1}_{\{W_{i,j}>0\}}$ and that our assumptions on W imply that $\mathbb{E}(N) > 1$. Consequently, the generating function of N, i.e. $f(x) = \sum_{n\geq 0} \mathbb{P}(N = n)x^n$, has a unique fixed point smaller than 1, which equals the probability of extinction of the associated Galton-Watson process generated by N, i.e. the probability of the event $\{K = \emptyset\} = \bigcup_{n\geq 1}\{K_n = \emptyset\}$. Also, since

$$Y = \|\mu\| = \sum_{1\leq i,j\leq m-1} W_{i,j} Y(i,j),$$

where the $Y(i,j)$ are independent copies of Y, and are also independent of W, the probability of $\{\mu = 0\}$ is also a fixed point of f. Moreover, by construction, $\{K = \emptyset\} \subset \{\mu = 0\}$. Since $\mathbb{P}(\mu \neq 0) < 1$, $\{K = \emptyset\}$ and $\{\mu = 0\} = \{Y = 0\}$ must be equal up to a set of probability 0.

On the other hand, we have $\mathrm{supp}(\mu) \subset K$ almost surely. Moreover, by the previous paragraph and statistical self-similarity, for each $n \geq 1$ and each cylinder $[u, v]$ of the n-th generation, $\{K \cap [u, v] \neq \emptyset\}$ coincides with the event $\{Q(u, v) > 0\} \cap \{Y(u, v) > 0\}$ up to a set of probability 0. Consequently, with probability 1, for all cylinder $[u, v]$, $K \cap [u, v] \neq \emptyset$ implies $\mu([u, v]) > 0$, that is $K \subset \mathrm{supp}(\mu)$. $\square$

For $i = 0, 1, \ldots, m - 1$, define polynomial functions f_i by

$$f_i(x) = \sum_{\ell=0}^{m} \mathbb{P}(N_i = \ell)\, x^\ell,$$

where $N_i = \#\{0 \leq j \leq m : \ V_{i,j} \neq 0\}$.

Definition A.2. *A Borel measurable function $p : \Sigma \to [0, 1]$ is called $\{f_i\}_{i=0}^{m-1}$-stationary, if p satisfies the following condition:*

$$p(\mathbf{i}) = f_{i_1}(p(\sigma \mathbf{i})), \quad \forall\, \mathbf{i} = (i_n)_{n=1}^{\infty} \in \Sigma.$$

Let ν' be a Bernoulli product measure on Σ. Two functions p and p' on Σ are called *equivalent* if $p(\mathbf{i}) = p'(\mathbf{i})$ for ν'-a.e. $\mathbf{i}$; for brevity we write $p = p'$ a.e. if they are equivalent.

Notice that the constant function 1 on Σ is always $\{f_i\}_{i=0}^{m-1}$-stationary.

Proposition A.3. *Assume that there exists at least one i so that $\mathbb{P}(N_i = 1) < 1$; equivalently, there exists i so that $f_i(x) \not\equiv x$. Then there exist at most one $\{f_i\}_{i=0}^{m-1}$-stationary function on Σ which is not equivalent to the constant function 1.*

Proof. Let $\mathcal{G}$ denote the collection of functions $f : \ [0, 1] \to [0, 1]$ so that f is increasing, continuous, convex and $f(1) = 1$. Notice that by convexity, for any $0 < a < 1$ and $f \in \mathcal{G}$,

$$\sup_{0 \leq x, y \leq a} \left| \frac{f(y) - f(x)}{y - x} \right| \leq \frac{f(1) - f(a)}{1 - a} \leq \frac{1}{1 - a}.$$

Therefore, $\mathcal{G}$ is equicontinuous on $[0, a]$ for any $a \in (0, 1)$.

Let $\mathcal{I} = \{i : \ f_i(x) \not\equiv x\}$. By our assumption, $\mathcal{I} \neq \emptyset$. Notice that by convexity, for each $i \in \mathcal{I}$, either $f_i(x) > x$ for any $x \in [0, 1)$, or f has exactly one attractive fixed point in $[0, 1)$. In the first case, $f_i^n(x) \to 1$ uniformly on $[0, a]$ for each $0 < a < 1$, whilst in the second case, $f_i^n(x)$ converges uniformly to the attractive fixed point of f_i, on $[0, a]$ for each $0 < a < 1$.

Now we consider the following two cases separately: (A) $f_i(x) > x$ on $[0, 1)$ for each $i \in \mathcal{I}$; (B) there exists at least one $i \in \mathcal{I}$ such that f has one fixed point in $[0, 1)$.

First suppose that (A) occurs. Then $f_i(x) \geq x$ for any $0 \leq i \leq m - 1$. Pick $i_0 \in \mathcal{I}$ and let $0 \leq a < 1$. Then there exists $n \in \mathbb{N}$ such that $f_{i_0}^n(x) > a$ for any $x \in [0, 1]$. Let p be

$\{f_i\}_{i=0}^{m-1}$-stationary. Then for ν'-a.e. $\mathbf{i} \in \Sigma$, there exists $k \in \mathbb{N}$ such that $\sigma^k \mathbf{i} \in [i_0^n]$, and thus

$$p(\mathbf{i}) = f_{i_1} \circ \ldots \circ f_{i_k} \circ f_{i_0}^n(p(\sigma^{k+n}\mathbf{i})) \geq f_{i_0}^n(p(\sigma^{k+n}\mathbf{i})) \geq a.$$

Since $a \in [0,1)$ is arbitrarily, we see that $p(\mathbf{i}) = 1$ for ν'-a.e. $\mathbf{i}$.

Next suppose that (B) occurs. Assume that p and p' are both $\{f_i\}_{i=0}^{m-1}$-stationary, and not equivalent to the constant function 1. We show below that $p = p'$ a.e.

First we claim that $p(\mathbf{i}) < 1$ and $p'(\mathbf{i}) < 1$ for ν'-a.e. $\mathbf{i}$. Without loss of generality we only prove the first inequality. Suppose on the contrary that $p = 1$ on a Borel set $A \subset \Sigma$ with $\nu'(A) > 0$. By the Poincare recurrence theorem, for ν'-a.e. $\mathbf{i}$, there exists $k = k(\mathbf{i}) \in \mathbb{N}$ such that $\sigma^k \mathbf{i} \in A$; and thus

$$p(\mathbf{i}) = f_{i_1} \circ \ldots \circ f_{i_n}(p(\sigma^n \mathbf{i})) = f_{i_1} \circ \ldots \circ f_{i_n}(1) = 1.$$

This contradicts the assumption that p is not equivalent to the constant function 1. Hence $p(\mathbf{i}) < 1$ for ν'-a.e. $\mathbf{i}$.

By the above claim, we can pick $\delta > 0$ such that there exists a Borel set $A = A_\delta \subset \Sigma$ with $\nu'(A) > 0$ such that

$$p(\mathbf{i}) \leq 1 - \delta \text{ and } p'(\mathbf{i}) \leq 1 - \delta, \quad \forall\, \mathbf{i} \in A.$$

Pick $j_0 \in \mathcal{I}$ so that f_{j_0} has an attracting point in $[0,1)$, say, b. Then

$$\lim_{n\to\infty} f_{j_0}^n([0, 1-\delta]) = b,$$

here and afterwards, $f_{j_0}^n$ denotes the n-th iteration of f_{j_0}. Since for each $n \in \mathbb{N}$, $\nu'([j_0^n] \cap \sigma^{-n}A) > 0$, by the Poincaré recurrence theorem, for ν'-a.e. $\mathbf{i}$, there exist $k_1 < k_2 < \ldots$, such that

$$\sigma^{k_n}\mathbf{i} \in [j_0^n] \cap \sigma^{-n}A.$$

Clearly $\lim_{n\to\infty} p(\sigma^{k_n}\mathbf{i}) = \lim_{n\to\infty} p'(\sigma^{k_n}\mathbf{i}) = b$.

Notice that the family of functions

$$\{f_{i_1} \circ \ldots \circ f_{i_{k_n}}\}_{n\in\mathbb{N}}$$

is a subset of $\mathcal{G}$. Hence it is equi-continuous on $[0,a]$ for any $a < 1$. By the Arzelà-Ascoli theorem, there exists a subsequence (t_ℓ) of (n_k) and a continuous function g on $[0,1)$ such that

$$f_{i_1} \circ \ldots \circ f_{i_{t_\ell}} \text{ converges to } g \text{ uniformly}$$

on any interval $[0,a]$ with $a < 1$. Since $b < 1$,

$$p(\mathbf{i}) = \lim_{\ell\to\infty} f_{i_1} \circ \ldots \circ f_{i_{t_\ell}}(p(\sigma^{t_\ell}\mathbf{i})) = g(b),$$

and similarly $p'(\mathbf{i}) = g(b)$. Hence $p(\mathbf{i}) = p'(\mathbf{i})$. Therefore $p = p'$ a.e. $\square$

Appendix B. Basic properties of Mandelbrot martingales in a Bernoulli environment

Let $U=(U_{i,j})_{(i,j)\in\Sigma_1\times\Sigma_1}$ be a non-negative random vector such that $\sum_{j=0}^{m-1}\mathbb{E}(U_{i,j})=1$ for each $0\le i\le m-1$. Let $(U(u,v))_{(u,v)\in\bigcup_{n\ge 0}\Sigma_n\times\Sigma_n}$ be a sequence of independent copies of U.

For each $n\ge 1$ and $(u,v)\in\Sigma_n\times\Sigma_n$ let

$$Q_U(u,v)=\prod_{k=1}^{n}U_{u_k,v_k}(u_{k-1},v_{k-1})$$

and

$$\widetilde{X}_U(u):=\sum_{|v|=n}Q_U(u,v).$$

Now for each fixed $x\in\Sigma$ and $n\ge 1$ let $\widetilde{\mu}^x_{U,n}$ be the measure on Σ whose density with respect to the measure of maximal entropy is given by $m^nQ_U(x_{|n},v)$ over any cylinder $[v]$ of generation n. The sequence $(\widetilde{\mu}^x_{U,n})_{n\ge 1}$ almost surely converges to an inhomogeneous Mandelbrot measure $\widetilde{\mu}^x_U$.

Let η be a Bernoulli product on Σ associated with a probability vector $(p'_0,\dots,p'_{m-1})$. Then, for η-almost every x the sequence $(\widetilde{\mu}^x_{U,n})_{n\ge 1}$ converges weakly almost surely to a measure ν^x as well, and $(\|\widetilde{\mu}^x_{U,n}\|=\widetilde{X}_U(x_{|n})=\widetilde{X}_{U,n}(x))_{n\ge 1}$ is a Mandelbrot martingale in the random environment given by η, which almost surely converges to $\|\widetilde{\mu}^x_U\|$, which we denote by $\widetilde{X}_U(x)$.

By construction, for each $n\ge 0$ and $J\in\Sigma_n$,

$$\widetilde{\mu}^x_U([J])=\widetilde{X}^{x_{|n},J}(\sigma^nx)\prod_{k=1}^{n}U_{x_k,J_k}(x_{|k-1},J_{|k-1}),$$

where

$$\widetilde{X}^{x_{|n},J}_U(\sigma^nx)=\lim_{p\to\infty}\sum_{K\in\Sigma_p}\prod_{\ell=1}^{p}U_{x_{n+\ell},K_\ell}(x_{|n+\ell-1},J(K_{|\ell-1})).$$

For $0\le i\le m-1$, let

$$T_{U_i}(q)=-\log_m\mathbb{E}\sum_{j=0}^{m-1}U_{i,j}^q\quad(q\ge 0).$$

Suppose that there exists $0\le i\le m-1$ such that $\mathbb{P}(\{U_{i,j}\in\{0,1\}\ \forall\, 0\le j\le m-1\})<1$.

We have the following consequence of a general result by Biggins and Kyprianou [10, Theorem 7.1].

Theorem B.1. *Suppose that $\mathbb{P}(\sum_{j=0}^{m-1}\mathbf{1}_{\{U_{i,j}>0\}}=1)<1$ for some $0\le i\le m-1$. The following properties are equivalent:*

(i) $\mathbb{P}\otimes\eta(\widetilde{X}_U>0)>0$*;*

(ii) $(\widetilde{X}_n)_{n\geq 1}$ *is uniformly integrable with respect to* $\mathbb{P}\otimes\eta$;

(iii) $\sum_{i=0}^{m-1} p'_i T'_{U_i}(1-) = -\mathbb{E}_{\mathbb{P}\otimes\nu}\Big(\sum_{j=0}^{m-1} U_{x_1,j}\log(U_{x_1,j})\Big) > 0$.

We also have the following useful fact. Let Z be an integrable random variable, and let $(Z(u,v))_{(u,v)\in\Sigma_n,\, n\geq 1}$ be a collection of copies of Z such that for each $n\geq 1$ the random variables $Z(u,v)$, $(u,v)\in\Sigma_n$ are independent, and independent of $\sigma(U(u',v') : |u'| = |v'| \leq n-1)$.

Let

$$X_U(x_{|n}) = \sum_{|v|=n} Q_U(x_{|n},v)Z(x_{|n},v).$$

Proposition B.2. *Let* $q\in(1,2]$. *Suppose that* $\mathbb{E}(|Z|^q)<\infty$, $T_{U_i}(q)$ *is finite for all* $0\leq i\leq m-1$ *such that* $p'_i>0$ *and* $\sum_{i=0}^{m-1} p'_i T_{U_i}(q) > 0$. *Then,* $\mathbb{P}\otimes\eta(\widetilde{X}_U>0)>0$, *and with probability 1, for* η*-almost every* x, $\lim_{n\to\infty} X_U(x_{|n}) = \mathbb{E}(Z)\widetilde{X}_U$.

Proof. To begin with we notice by concavity of the mappings T_{U_i}, the fact that all the functions T_{U_i} vanish at 1 together with the assumption $\sum_{i=0}^{m-1} p'_i T_{U_i}(q) > 0$ implies that $\sum_{i=0}^{m-1} p'_i T'_{U_i}(1) > 0$. Consequently, due to Theorem B.1, $\mathbb{P}\otimes\eta(\widetilde{X}_U>0)>0$.

Next, recall the following standard lemma.

Lemma B.3. [2] *Let* $(L_j)_{j\geq 1}$ *be a sequence of centered independent real valued random variables. For every finite* $I\subset\mathbb{N}_+$ *and* $q\in(1,2]$,

$$\mathbb{E}\Big(\Big|\sum_{i\in I} L_i\Big|^q\Big) \leq 2^{q-1}\sum_{i\in I}\mathbb{E}(|L_i|^q).$$

For all $n\geq 1$, we have

$$X_U(x_{|n+1}) - \mathbb{E}(Z)\widetilde{X}_U(x_{|n+1}) = \sum_{|v|=n} Q_U(x_{|n},v)\widetilde{U}(x_{|n},v),$$

where

$$\widetilde{U}(x_{|n},v) = \sum_{j=0}^{m-1} U_{x_{|n+1},v_{n+1}}(x_{|n},v)\big(Z(x_{|n+1},vj)-\mathbb{E}(Z)\big).$$

By construction, conditional on x (recall that we work under $\mathbb{P}\otimes\eta$) the random variables $\widetilde{U}(x_{|n},v)$ are i.i.d and centered, and they are also independent of the $Q_U(x_{|n},v)$ invoked in $X_U(x_{|n+1}) - \mathbb{E}(Z)\widetilde{X}_U(x_{|n+1})$ (with respect to $\mathbb{P}$). Consequently, conditioning with respect to the $Q_U(x_{|n},v)$ makes it possible to apply Lemma B.3 to $\{L_v = \widetilde{U}(x_{|n},v)\}_{v\in\Sigma_n}$ weighted by the constants $Q_U(x_{|n},v)$ and finally to get, for $q\in(1,2]$:

$$\text{(B.1)}\qquad \mathbb{E}(|X_U(x_{|n+1}) - \mathbb{E}(Z)\widetilde{X}_U(x_{|n+1})|^q) \leq 2^{q-1}\sum_{|v|=n}\mathbb{E}(Q_U(x_{|n},v)^q)\mathbb{E}(|\widetilde{U}(x_{|n},v_0)|^q),$$

where v_0 is any element of Σ_n.

The branching property yields $\mathbb{E}(Q_U(x_{|n}, v)^q) = \prod_{k=1}^n m^{-T_{U_{x_k}}(q)}$, and applying triangular inequality and a convexity inequality yields $\mathbb{E}(|\widetilde{U}(x_{|n}, v_0)|^q) \leq 2^q \mathbb{E}(|Z|^q) m^{-T_{x_{n+1}}(q)}$, which is bounded by a constant independent of x since $T_{U_i}(q)$ is finite for all $0 \leq i \leq m-1$ such that $p'_i > 0$. Also, the strong law of large numbers yields

$$\lim_{n\to\infty} n^{-1} \log \prod_{k=1}^n m^{-T_{U_{x_k}}(q)} = -\sum_{i=0}^{m-1} p'_i T_{U_i}(q) < 0$$

for η-almost every x. Consequently, the estimate (B.1) shows that for η-almost every x, $\sum_{n\geq 1} \big(\mathbb{E}(|X_U(x_{|n+1}) - \mathbb{E}(Z)\widetilde{X}_U(x_{|n+1})|^q)\big)^{1/q} < \infty$ which implies that $X_U(x_{|n+1})$ converges $\mathbb{P}$-almost surely to the same limit as $\mathbb{E}(Z)\widetilde{X}_U(x_{|n+1})$, that is $\mathbb{E}(Z)\widetilde{X}_U(x)$. $\square$

Appendix C. A useful lemma

Lemma C.1. *Let $(Z_n)_{n\geq 1}$ be a sequence of non-negative random variables on a probability space $(\Omega, \mathbb{P})$. Then almost surely*

$$\limsup_{n\to\infty} \frac{\log Z_n}{n} \leq \limsup_{n\to\infty} \frac{\log \mathbb{E}(Z_n)}{n}.$$

Proof. Let $b > a > \limsup_{n\to\infty} \frac{\log \mathbb{E}(Z_n)}{n}$. Then by Markov's inequality,

$$\mathbb{P}(\{(1/n)\log z_n \geq b\}) \leq \mathbb{E}(Z_n)e^{-bn} \leq e^{n(a-b)}$$

when n is large enough. Hence $\sum_{n=1}^\infty \mathbb{P}\{(1/n)\log z_n \geq b\} < \infty$. The Borel-Cantelli lemma implies that $\limsup_{n\to\infty} \frac{\log Z_n}{n} \leq b$ almost surely. Letting b tend to $\limsup_{n\to\infty} \frac{\log \mathbb{E}(Z_n)}{n}$ yields the desired result. $\square$

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