Suggested Solution to Assignment 3

Exercise 3.1

2. Let $v(x,t) = u(x,t) - 1$. Then $v(x,t)$ will satisfy

$$v_t = kv_{xx},\ v(x,0) = -1,\ v(0,t) = 0.$$

Hence,

$$\begin{aligned} v(x,t) &= -\frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}}]dy \\ &= -\mathscr{E}rf(\frac{x}{\sqrt{4kt}}). \\ u(x,t) &= v(x,t) + 1 = 1 - \mathscr{E}rf(\frac{x}{\sqrt{4kt}}). \qquad \square \end{aligned}$$

3. By the method of even reflection, we can translate the original problem for the half-line to the problem for the whole line and then using the formula for the latter to obtain

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty [e^{-(x-y)^2/4kt} + e^{-(x+y)^2/4kt}]\phi(y)dy.$$

For the details, please see your textbook. $\square$

4. (a) With the rule for differentiation under an integral sign and the property of source function, $v(x,t)$ satisfies

$$v_t = kv_{xx},\ v(x,0) = f(x).$$

(b) By (a), $w(x,t)$ satisfies

$$w_t = kw_{xx},\ w(x,0) = f'(x) - 2f(x).$$

(c) By the definition of f,

$$\begin{aligned} f'(x) - 2f(x) &= \begin{cases} 1 - 2x, & x > 0; \\ -1 - 2x, & x < 0. \end{cases} \\ f'(-x) - 2f(-x) &= \begin{cases} -1 + 2x, & x > 0; \\ 1 + 2x, & x < 0. \end{cases} \\ &= -[f'(x) - 2f(x)]. \end{aligned}$$

Hence, $f'(x) - 2f(x)$ is an odd function.

(d) Since $w(x,0)$ is an odd function, using the conclusion in Exercise 2.4.11, w is an odd function of x.

(e) By (a), $v(x,t)$ satisfies DE and IC. By (d), $v(x,t)$ satisfies BC. Thus we have proved that $v(x,t)$ satisfies (1) for $x > 0$. Hence, using the assumption for the uniqueness, the solution of (1) is given by

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^\infty e^{-(x-y)^2/4kt} f(y)dy,$$

where

$$f(y) = \begin{cases} y, & y > 0; \\ y + 1, & y < 0. \end{cases} \qquad \square$$

5. (a)Let $f(x) = x$ for $x > 0, = x+2/h$ for $x < 0$, $v(x)$ be the functions in Exercise 4 and define $w = v_x - hv$, then $w_t = kw_{xx}$ and $w(x,0) = f'(x) - hf(x)$. By the definition of $f(x)$, we can know that $f'(x) - hf(x)$ is an odd function. Thus, w is an odd function. Using the same argument in Exercise 4(e), we can obtain the solution.

   (b)Let $f(x) = \phi(x)$ for $x > 0, = F(x)$ for $x < 0$ (where $F(x)$ need to be determined), $v(x)$ be the functions in Exercise 4 and define $w = v_x - hv$, then $w_t = kw_{xx}$ and $w(x,0) = f'(x) - hf(x)$. By the definition of $f(x)$, in order that $f'(x) - hf(x)$ is an odd function, we have to solve $F'(x) - hF(x) = -\phi'(-x) + h\phi(-x)$ for $x < 0$. Solving the ODE, we obtain $F(x) = (F(-1) + \phi(1))e^{h(1+x)} - \phi(-x) - 2\int_{-x}^{1} e^{h(x+y)}\phi'(y)dy$ for $x < 0$. Thus, for $F(x)$ defined as above, w is an odd function. Using the same argument in Exercise 4(e), we can obtain the solution.

Exercise 3.2

1. By the method of even extension, we have

$$
\begin{aligned}
v(x,t) &= \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{even}}(y)dy \\
&= \begin{cases} \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy, & x \geq ct; \\ \frac{1}{2}[\phi(x+ct) + \phi(-x+ct)] + \frac{1}{2c}[\int_0^{x+ct} \psi(y)dy + \int_0^{-x+ct} \psi(y)dy], & 0 < x < ct. \end{cases}
\end{aligned}
$$

   It is similar for $t < 0$.

2. We can do this problem by even extension, then we obtain the solution to this problem $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(s)ds$, where $\psi_{\text{ext}}(s) = V$ for $a < s < 2a$, $-2a < s < -a$, and zero otherwise. Substitute $t = 0, a/c, 3a/2c, 2a/c, 3a/c$ into this formula and we omit it. □

3. If the string is fixed at the end $x = 0$, then we have the homogeneous Dirichlet condition $u(0,t) = 0$. Therefore the vibrations $u(x,t)$ of the string for $t > 0$ is given the odd reflection formula with initial date $f(x)$ and $cf'(x)$, that is,

$$
u(x,t) = \begin{cases} f(x+ct) & x \geq ct \\ f(x+ct) - f(ct-x) & 0 < x < ct. \end{cases}.
$$

   For details see the formulas (1)-(3) in section 3.2 of the book. □

5. Using the odd reflection method or formulas(2) and (3), we have

$$
u(x,t) = \begin{cases} 1, & x > 2|t|; \\ 0, & x < 2|t|. \end{cases}
$$

   Hence the singularity is on the lines $x = 2|t|$. □

6. Since $u_t(0,t) + au_x(0,t) = 0$, we can consider the function $w(x,t)$ defined on the whole line

$$
w(x,t) = \begin{cases} u_t(x,t) + au_x(x,t) & x > 0; \\ 0, & x = 0; \\ -u_t(-x,t) - au_x(-x,t), & t < 0. \end{cases}
$$

   Here, $u_t(0,t) + au_x(0,t) = 0$ enables $w(x,t)$ is continuous and differentiable around $x = 0$. Since $w(x,t)$ is a linear combination of derivatives of $u(x,t)$, it also satisfies the wave equation, that is,

$$
w_{tt} = c^2 w_{xx}.
$$

By direct calculation,

$$w(x,0)=\phi(x)=\begin{cases}V, & x>0;\\ 0, & x=0;\\ -V, & x<0.\end{cases}$$

$$\begin{aligned}w_t(x,0)&=u_{tt}(x,0)+au_{xt}(x,0)=c^2u_{xx}(x,0)+au_{xt}(x,0)\\&=c^2\partial^2_{xx}(0)+a\partial_x(V)=0.\end{aligned}$$

Then the d'Alembert's formula implies

$$w(x,t)=\frac{1}{2}[\phi(x+ct)+\phi(x-ct)]=\begin{cases}V, & x>ct,\\ V/2, & x=ct,\\ 0, & -ct<x<ct,\\ -V/2 & x=-ct,\\ -V & x<-ct.\end{cases}$$

Let $\varphi(s)=u(x+as,t+s)$, and then $\varphi'(s)=u_t+au_x=w(x+as,t+s)$, $\varphi(-t)=u(x-at,0)=0$ and $\varphi(0)=u(x,t)$. Hence,

$$u(x,t)=\int_{-t}^{0}w(x+as,t+s)ds.$$

Denote $A=\{(x_1,t_1);0\le t_1\le t\}=\{(x_0,t_0);x_0=ct_0,0\le t_0\le t\}\cap\{(x_0,t_0);x-x_0=a(t-t_0),0\le t_0\le t\}$(i.e. (x_1,t_1) is the point where the line $x_0=ct_0$ intersects the line $x-x_0=a(t-t_0)$ when $0\le t_0\le t$) and $B=\{(x_2,t_2);0\le t_1\le t\}=\{(x_0,t_0);x_0=-ct_0,0\le t_0\le t\}\cap\{(x_0,t_0);x-x_0=a(t-t_0),0\le t_0\le t\}$. Hence, when $x\ge at$, $A=B=\emptyset$ and

$$u(x,t)=\int_{-t}^{0}Vds=Vt;$$

when $ct\le x\le at$, $t_1=\dfrac{at-x}{a-c}$, $t_2=\dfrac{at-x}{a+c}$ and

$$u(x,t)=\int_{t_1-t}^{0}Vds+\int_{-t}^{t_2-t}-Vds=V\frac{x-ct}{a-c}-V\frac{at-x}{a+c}=V\frac{2ax-(a^2+c^2)t}{a^2-c^2};$$

when $0\le x\le ct$, $A=\emptyset$, $t_2=\dfrac{at-x}{a+c}$ and

$$u(x,t)=\int_{-t}^{t_2-t}-Vds=-V\frac{at-x}{a+c}.\qquad\square$$

8. In the diamond-shaped region $(0,0)$, we have $ct<x<l-ct$, $0<t<l/(2c)$ and $v(x,t)=1/2\phi(x+ct)+1/2\phi(x-ct)+1/(2c)\int_{x-ct}^{x+ct}\psi(s)ds$. In the diamond-shaped region (m,m), $m\ge1$, we have $ct-ml<x<ct-(m-1)l$, $ml-ct<x<(m+1)l-ct$, $(m-1/2)l/c<t<(m+1/2)l/c$, and $v(x,t)=1/2\phi(x+ct-ml)+1/2\phi(x-ct+ml)+1/(2c)[\int_{x-ct+ml}^{l}\psi(s)ds+\int_0^{x+ct-ml}\psi(s)ds+\int_{-l}^{0}-\psi(-s)ds]$ if m is even, $=-1/2\phi(-x-ct+(m+1)l)-1/2\phi(-x+ct-(m-1)l)+1/(2c)[\int_0^l\psi(s)ds+\int_{x-ct+(m-1)l}^{0}-\psi(-s)ds+\int_{-l}^{x+ct-(m+1)l}-\psi(-s)ds]$ if m is odd. In the diamond-shaped region $(m,m-1)$, $m\ge1$, we have $0<x<ct-(m-1)l$, $x<ml-ct$, $(m-1)l/c<t<ml/c$ and $v(x,t)=-1/2\phi(-x-ct+ml)+1/2\phi(x-ct+ml)+1/(2c)[\int_{x-ct+ml}^{l}\psi(s)ds+\int_{-l}^{x+ct-ml}-\psi(-s)ds]$ if m is even, $=1/2\phi(x+ct-(m-1)l)-1/2\phi(-x+$

$ct-(m-1)l)+1/(2c)[\int_0^{x+ct-(m-1)l}\psi(s)ds+\int_{x-ct+(m-1)l}^{0}-\psi(-s)ds]$ if m is odd. In the diamond-shaped region $(m-1,m)$, $m\geq 1$, we have $ct-(m-1)l<x<l$, $x>ml-ct$, $(m-1)l/c<t<ml/c$ and $v(x,t)=1/2\phi(x+ct-ml)-1/2\phi(-x+ct-(m-2)l)+1/(2c)[\int_0^{x+ct-ml}\psi(s)ds+\int_{x-ct+(m-2)l}^{0}-\psi(-s)ds]$ if m is even, $=-1/2\phi(-x-ct+(m+1)l)+1/2\phi(x-ct+(m-1)l)+1/(2c)[\int_{x-ct+(m-1)l}^{l}\psi(s)ds+\int_{-l}^{x+ct-(m+1)l}-\psi(-s)ds]$ if m is odd.

10. $u(x,t)=1/2[cos(x+3t)+cos(x-3t)]$

Exercise 3.3

1. Using the method of reflection and the formula (2) in Section 3.3, we have

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{odd}}(y)dy + \int_0^t\int_{-\infty}^{\infty} S(x-y,t-s)f_{\text{odd}}(y,s)dyds\\
&= \int_0^{\infty}[S(x-y,t)-S(x+y,t)]\phi(y)dy\\
&+ \int_0^t\int_0^{\infty}[S(x-y,t-s)-S(x+y,t-s)]f(y,s)dyds,
\end{aligned}$$

where $f_{\text{odd}}(y,s)$ is the odd extension of $f(y,s)$ w.r.t the variable y, and

$$S(x,t)=\frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}},\ t>0. \qquad \square$$

2. Let $V(x,t)=v(x,t)-h(t)$. Then $V(x,t)$ will satisfy

$$V_t-kV_{xx}=f(x,t)-h'(t) \qquad \text{for } 0<x<\infty,\ 0<t<\infty,$$

$$V(0,t)=0, \qquad V(x,0)=\phi(x)-h(0).$$

Using the result above, we have

$$\begin{aligned}
V(x,t) &= \int_0^{\infty}[S(x-y,t)-S(x+y,t)][\phi(y)-h(0)]dy\\
&+ \int_0^t\int_0^{\infty}[S(x-y,t-s)-S(x+y,t-s)][f(y,s)-h'(t)]dyds,\\
v(x,t) &= h(t)+\int_0^{\infty}[S(x-y,t)-S(x+y,t)][\phi(y)-h(0)]dy\\
&+ \int_0^t\int_0^{\infty}[S(x-y,t-s)-S(x+y,t-s)][f(y,s)-h'(t)]dyds,
\end{aligned}$$

where $f_{\text{odd}}(y,s)$ and $S(x,t)$ are shown above. $\square$

3. Let $W(x,t)=w(x,t)-xh(t)$. Then $W(x,t)$ will satisfy

$$W_t-kW_{xx}=-xh'(t) \qquad \text{for } 0<x<\infty,\ 0<t<\infty,$$

$$W_x(0,t)=0, \qquad W(x,0)=\phi(x)-xh(0).$$

Using the method of reflection of even functions, we have

$$\begin{aligned} W(x,t) &= \int_{-\infty}^{\infty} S(x-y,t)\phi_{\text{even}}(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f_{\text{even}}(y,s)dyds \\ &= \int_0^{\infty} [S(x-y,t)+S(x+y,t)][\phi(y)-yh(0)]dy \\ &+ \int_0^t \int_0^{\infty} [S(x-y,t-s)+S(x+y,t-s)][-yh'(s)]dyds, \\ w(x,t) &= W(x,t)+xh(t), \end{aligned}$$

where $f_{\text{even}}(y,s)$ is the even extension of $f(y,s)$ in the variable y, and

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}, \ t>0. \qquad \square$$

Exercise 3.4

1. By the Theorem 1 in Section 3.4, we have

$$u(x,t) = \frac{1}{2c}\iint\limits_{\Delta} ys\ dyds = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} ys\ dyds = \frac{xt^3}{6}. \qquad \square$$

2. By the Theorem 1 in Section 3.4, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\iint\limits_{\Delta} e^{ay}\ dyds = \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} e^{ay}\ dyds \\ &= \begin{cases} \dfrac{e^{ax}}{a^2c^2}\left(\dfrac{e^{act}+e^{-act}}{2}-1\right), & a\neq 0; \\ \frac{1}{2}t^2, & a=0. \end{cases} \qquad \square \end{aligned}$$

3. By the Theorem 1 in Section 3.4, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2}[\sin(x+ct)+\sin(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}(1+s)ds + \frac{1}{2c}\iint\limits_{\Delta} \cos y\ dyds \\ &= \sin x\cos(ct) + (x+1)t + \frac{1}{c^2}\cos x[1-\cos(ct)]. \qquad \square \end{aligned}$$

4. Let u_1 be the solution of the wave equation

$$u_{tt} = c^2u_{xx} + f,\ u(x,0)=0,\ u_t(x,0)=0,$$

u_2 be the solution of the wave equation

$$u_{tt} = c^2u_{xx},\ u(x,0)=\phi(x),\ u_t(x,0)=0,$$

u_3 be the solution of the wave equation

$$u_{tt} = c^2u_{xx},\ u(x,0)=0,\ u_t(x,0)=\psi(x).$$

Then $u=u_1+u_2+u_3$ is the unique solution for the original problem since the equation and conditions are linear and the uniqueness of the wave equation. Note that $u_1,\ u_2,\ u_3$ are terms for f, ϕ and ψ respectively. Hence the solution of the original problem can be written in the sum of three terms, one each for f, ϕ and ψ. $\square$

5. We write $u(x,t) = \frac{1}{2c}\int_0^t \int_{x-ct+cs}^{x+ct-cs} f(y,s)dyds$. Then by direct calculation, we have

$$u_x = \frac{1}{2c}\int_0^t [f(x+ct-cs) - f(x-ct+cs)]ds,\ u_{xx} = \frac{1}{2c}\int_0^t [f'(x+ct-cs) - f'(x-ct+cs)]ds,$$

$$u_t = \frac{1}{2}\int_0^t [f(x+ct-cs) + f(x-ct+cs)]ds,\ u_{tt} = f(x) + \frac{c}{2}\int_0^t [f'(x+ct-cs) - f'(x-ct+cs)]ds.$$

Hence, we have

$$u_{tt} = c^2 u_{xx} + f$$

$$u(x,0) = \frac{1}{2c}\int_0^0 \int_{x+cs}^{x-cs} f(y,s)dyds \equiv 0,$$
$$u_t(x,0) = \frac{1}{2}\int_0^0 [f(x-cs) + f(x+cs)]ds \equiv 0. \qquad \square$$

8. For arbitrary C^2 function ψ, $\mathscr{S}\psi = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy$. We have

$$[\mathscr{S}\psi]_{tt} = \frac{c}{2}[\psi'(x+ct) - \psi'(x-ct)] = c^2[\mathscr{S}\psi]_{xx}.$$

$$[\mathscr{S}(0)\psi] = \frac{1}{2c}\int_x^x \psi(y)dy = 0,\ [\mathscr{S}_t(0)\psi] = \frac{1}{2}[\psi(x) + \psi(x)] = \psi(x).$$

So we conclude that

$$\mathscr{S}_{tt} - c^2\mathscr{S}_{xx} = 0,\ \mathscr{S}(0) = 0,\ \mathscr{S}_t(0) = I. \qquad \square$$

9. According to the definition of $u(x,t)$ and the result above, we have

$$u_t = \mathscr{S}(0)f(t) + \int_0^t \mathscr{S}_t(t-s)f(s)ds = \int_0^t \mathscr{S}_t(t-s)f(s)ds,$$
$$u_{tt} = \mathscr{S}_t(0)f(t) + \int_0^t \mathscr{S}_{tt}(t-s)f(s)ds = f(t) + \int_0^t \mathscr{S}_{tt}(t-s)f(s)ds,$$
$$u_{xx} = \int_0^t \mathscr{S}_{xx}(t-s)f(s)ds.$$

So we conclude that

$$u_{tt} - c^2u_{xx} = f,\ u(x,0) = \int_0^0 \mathscr{S}(-s)f(s)ds = 0,\ u_t(0) = \int_0^0 \mathscr{S}_t(-s)f(s)ds = 0 \qquad \square$$

11. By the definition of u, $u(x,0) = 0$ since $x > 0 = ct$ and $u(0,t) = h(t)$ since $x = 0 < ct$. For $x < ct$, $u_{tt} = h''(t - x/c) = c^2u_{xx}$. For $x > t$, $u_{tt} \equiv 0 \equiv c^2u_{xx}$.

12. For $x_0 > ct_0 > 0$, integrate over Δ, where Δ is the region bounded by three lines

$$L_0 = [(x_0 - ct_0, 0), (x_0 + ct_0, 0)], L_1 = [(x_0 + ct_0, 0), (x_0, t_0)], L_2 = [(x_0, t_0), (x_0 - ct_0, 0)]$$

(see figure 6 in Page 76), by Green's theorem, we have

$$\iint_\Delta f dxdt = \iint_\Delta u_{tt} - c^2u_{xx}dxdt = \int_{L_0+L_1+L_2} -c^2u_xdt - u_tdx$$

On $L_0, dt = 0, u_t(x) = \psi(x), \int_{L_0} -c^2u_xdt - u_tdx = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx$.
On $L_1, x + ct = x_0 + ct_0 \implies dx + cdt = 0, -c^2u_xdt - u_tdx = cu_xdx + cu_tdt = cdu$.

$$\int_{L_1} = c\int_{L_1} du = cu(x_0, t_0) - c\phi(x_0 + ct_0)$$

By the same reasoning, $\int_{L_2} = -c\int_{L_2} du = -c\phi(x_0 - ct_0) + cu(x_0, t_0)$. Summing the three terms, we have for

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi + \frac{1}{2c}\iint\limits_{\Delta} f, \qquad \text{if } x > ct > 0. \tag{1}$$

For $x_0 < ct_0$, integrate over Δ', where Δ' is the reflected region bounded by four lines

$$L_0 = [(ct_0 - x_0, 0), (x_0 + ct_0, 0)], L_1 = [(x_0 + ct_0, 0), (x_0, t_0)],$$

$$L_2 = [(x_0, t_0), (0, t_0 - x_0/c)],\ L_3 = [(0, t_0 - x_0/c), (ct_0 - x_0, 0)]$$

(see figure 2 in Page 72), by Green's theorem, we have

$$\iint\limits_{\Delta'} f dxdt = \iint\limits_{\Delta'} u_{tt} - c^2u_{xx}dxdt = \int_{L_0+L_1+L_2+L_3} -c^2u_xdt - u_tdx$$

On $L_0, dt = 0, u_t(x) = \psi(x)$. Hence, we have

$$\begin{aligned}
\int_{L_0} -c^2u_xdt - u_tdx &= -\int_{ct_0-x_0}^{x_0+ct_0} \psi(x)dx,\\
\int_{L_1} = c\int_{L_1} du &= cu(x_0, t_0) - c\phi(x_0 + ct_0),\\
\int_{L_2} = -c\int_{L_2} du &= -ch(t_0 - x_0/c) + cu(x_0, t_0),\\
\int_{L_3} = c\int_{L_3} du &= c\phi(ct_0 - x_0) - ch(t_0 - x_0/c).
\end{aligned}$$

Summing the four terms, we have

$$u(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] - \frac{1}{2c}\int_{ct-x}^{x+ct} \psi + h(t - \frac{x}{c}) + \frac{1}{2c}\iint\limits_{\Delta'} f, \text{ if } 0 < x < ct. \tag{2}$$

13. By the result above, $f \equiv 0$, $\phi(x) \equiv x$, $\psi(x) \equiv 0$ and $h(t) = t^2$ imply that

$$\begin{aligned}
u(x,t) &= \begin{cases} \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi + \frac{1}{2c}\iint\limits_{\Delta} f & x \ge ct > 0 \\ \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] - \frac{1}{2c}\int_{ct-x}^{x+ct}\psi + h(t - \frac{x}{c}) + \frac{1}{2c}\iint\limits_{\Delta'} f & 0 < x < ct \end{cases}\\
&= \begin{cases} x & x \ge ct > 0 \\ x + (t - \frac{x}{c})^2 & 0 < x < ct \end{cases} \qquad \square
\end{aligned}$$

14. Let $v(x,t) = u(x,t) - xk(t)$. Then v satisfies

$$v_{tt} - c^2v_{xx} = -xk''(t),$$

$$v(x,0) = -xk(0),\ v_t(x,0) = -xk'(0),\ v_x(0,t) = 0.$$

Then $v_x(0,t) = 0$ enables us to have an even extension. So the solution of v is

$$v(x,t) = \frac{1}{2}[\phi_{\text{even}}(x+ct) + \phi_{\text{even}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{\text{even}} + \frac{1}{2c}\iint_{\Delta} f_{\text{even}},$$

where ϕ_{even}, ψ_{even} and f_{even} are the even extensions of ϕ, ψ and f respectively. Finally, we can have

$$u = \begin{cases} 0 & x \geq ct; \\ -c\int_0^{t-x/c} k(s)ds & x \leq ct. \end{cases} \qquad \square$$

Exercise 3.5

1. Since

$$\frac{1}{\sqrt{4\pi}}\int_0^\infty e^{-p^2/4}dp = 1/2,$$

we have

$$\Big|\frac{1}{\sqrt{4\pi}}\int_0^\infty e^{-p^2/4}\phi(x+\sqrt{kt}p)dp - \frac{1}{2}\phi(x+)\Big| \leq \frac{1}{\sqrt{4\pi}}\int_0^\infty e^{-p^2/4}|\phi(x+\sqrt{kt}p) - \phi(x+)|dp$$
$$\frac{1}{\sqrt{4\pi}}\int_{p_0}^\infty e^{-p^2/4}|\phi(x+\sqrt{kt}p) - \phi(x+)|dp + \frac{1}{\sqrt{4\pi}}\int_0^{p_0} e^{-p^2/4}|\phi(x+\sqrt{kt}p) - \phi(x+)|dp$$

For $\forall\epsilon > 0$, choose p_0 large enough such that $\int_{p_0}^\infty e^{-p^2/4}dp$ is small enough and then

$$\frac{1}{\sqrt{4\pi}}\int_{p_0}^\infty e^{-p^2/4}|\phi(x+\sqrt{kt}p) - \phi(x+)|dp \leq C\ max|\phi|\ \int_{p_0}^\infty e^{-p^2/4}dp < \frac{\epsilon}{2};$$

after this, we can choose t is small enough such that

$$|\phi(x+\sqrt{kt}p) - \phi(x+)| < \epsilon$$

and then

$$\frac{1}{\sqrt{4\pi}}\int_0^{p_0} e^{-p^2/4}|\phi(x+\sqrt{kt}p) - \phi(x+)|dp \leq \left(\frac{1}{\sqrt{4\pi}}\int_0^{p_0} e^{-p^2/4}dp\right)\ \epsilon = \frac{\epsilon}{2}.$$

Hence,

$$\frac{1}{\sqrt{4\pi}}\int_0^\infty e^{-p^2/4}\phi(x+\sqrt{kt}p)\ dp \to \frac{1}{2}\phi(x+) \quad \text{as } t\searrow 0;$$

similarly we can prove that

$$\frac{1}{\sqrt{4\pi}}\int_0^{-\infty} e^{-p^2/4}\phi(x+\sqrt{kt}p)\ dp \to -\frac{1}{2}\phi(x-) \quad \text{as } t\searrow 0. \qquad \square$$

2. Since $\phi(x)$ is bounded, by the same argument in Theorem 1, we can show that (1) is an infinitely differentiable solution for $t > 0$. In addition, by Exercise 1,

$$\lim_{t\searrow 0} u(x,t) = \frac{1}{2}[\phi(x+) + \phi(x-)]$$