Suggested Solution to Quiz 2

April 6, 2017

1. (**3 points**) Can the eigenvalue problem

$$\begin{cases} -X''(x) = \lambda X(x), \quad 0 < x < 1 \\ X(0) = 0, \quad X'(1) = 0 \end{cases}$$

have nonpositive eigenvalues? Prove your statements.

Solution: No, the above eigenvalue problem only have positive eigenvalues.

In fact, let λ be the eigenvalue of the problem and $X(x)$ the corresponding eigenfunction. Multiply the equation $-X''(x) = \lambda X(x)$ by $X(x)$ and integrate with respect to x, then we get

$$-\int_0^1 X''(x)X(x)dx = \lambda \int_0^1 X^2(x)dx$$

With the help of the boundary conditions, we have

$$-\int_0^1 X''(x)X(x)dx = -X'(x)X(x)\Big|_0^1 + \int_0^1 |X'(x)|^2 dx = \int_0^1 |X'(x)|^2 dx$$

Therefore,

$$\lambda = \frac{\int_0^1 |X'(x)|^2 dx}{\int_0^1 X^2(x)dx} \geq 0$$

If $\lambda = 0$, then we must have $X'(x) \equiv 0$ on $[0, 1]$ which implies that $X(x) = Constant$. Since $X(0) = 0$, then $X(x) \equiv 0$ which is impossible. Therefore $\lambda > 0$.

2. (**3 points**) Find the Fourier sine series of $f(x) = x$ on $(0, \pi)$. Then find the sum

$$\sum_{n=0}^{\infty} \frac{1}{n^2}$$

by using Parseval's identity.

Solution: The Fourier sine series of $f(x) = x$ on $(0, \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} A_n \sin nx$$

where the coefficients are

$$\begin{aligned} A_n &= \frac{2}{\pi} \int_0^{\pi} x \sin nx dx \\ &= -\frac{2}{n\pi} x \cos nx \Big|_0^{\pi} + \frac{2}{n\pi} \int_0^{\pi} \cos nx dx \\ &= \frac{2}{n}(-1)^{n+1}, \quad n = 1, 2, \cdots \end{aligned}$$

Hence

$$x = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx.$$

The Parseval's equality is

$$\int_0^\pi |f(x)|^2 dx = \sum_{n=0}^{\infty} |A_n|^2 \int_0^\pi |X_n(x)|^2 dx$$

Here $f(x) = x$ and $X_n(x) = \sin nx, n = 1, 2, \cdots$, thus we have

$$\frac{\pi^3}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2} \frac{\pi}{2}$$

Hence

$$\sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

3. (**4 points**) Solve the following problem

$$\begin{cases} \partial_t u = \partial_x^2 u, & 0 < x < \pi, t > 0 \\ u(0,t) = 0, \quad u(\pi, t) = 0, & t > 0 \\ u(x, t = 0) = x, & 0 < x < \pi \end{cases}$$

Solution: Use the separation of variables method, let $u(x,t) = X(x)T(t)$, then the PDE gives

$$\frac{X''}{X} = \frac{T'}{T} = -\lambda.$$

and boundary conditions yield $X(0) = X(\pi) = 0$. Then solve the following eignevalue problem firstly

$$\begin{cases} X''(x) = -\lambda X(x), & 0 < x < \pi \\ X(0) = 0, \ X(\pi) = 0, \end{cases}$$

we have $\lambda = n^2$ and corresponding eigenfunctions are $X_n(x) = \sin nx$ for $n = 1, 2, \cdots$. Then $T'(t) = -n^2 T(t)$ gives that $T_n(t) = A_n e^{-n^2 t}$ with constants A_n to be determined. Hence

$$u(x,t) = \sum_{n=1}^{\infty} X_n(x) T_n(t) = \sum_{n=1}^{\infty} A_n e^{-n^2 t} \sin nx.$$

Then initial condition yield

$$u(x, t = 0) = x = \sum_{n=1}^{\infty} A_n \sin nx$$

which is the Fourier sine series of x on $(0, \pi)$, thus the constants $A_n = \frac{2}{\pi} \int_0^\pi x \sin nx dx = \frac{2}{n}(-1)^{n+1}$. Therefore

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} e^{-n^2 t} \sin nx.$$