Tutorial 8 for MATH4220

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1. Discuss in details of the negative eigenvalues for

$$\begin{cases} -X'' = \lambda X, & 0 < x < l \\ X'(0) = a_0 X(0), & X'(l) = -a_l X(l) \end{cases}$$

with $a_0 < 0, a_l > 0, a_0 + a_l > 0$.

Solution: Let $\lambda = -\beta^2, \beta > 0$, then

$$X(x) = A \sinh \beta x + B \cosh \beta x$$

with A, B constants to be determined. Note that

$$X'(x) = A\beta \cosh \beta x + B\beta \sinh \beta x$$

Apply the boundary conditons, then

$$\begin{aligned} X'(0) = a_0 X(0) \Rightarrow & \quad A\beta = a_0 B \\ X'(l) = -a_l X(l) \Rightarrow & \quad A\beta \cosh \beta l + B\beta \sinh \beta l = -a_l (A \sinh \beta l + B \cosh \beta l). \end{aligned}$$

It follows from simple computications that

$$\frac{a_0}{\beta} = -\frac{\beta \sinh \beta l + a_l \cosh \beta l}{\beta \cosh \beta l + a_l \sinh \beta l}$$

or equivalently

$$\tanh \beta l = -\frac{(a_0 + a_l)\beta}{\beta^2 + a_0 a_l}.$$

Let $f(\beta) = \tanh \beta l$ and $g(\beta) = -\frac{(a_0+a_l)\beta}{\beta^2+a_0 a_l}$. To find the negative eigenvalues is to find the intersection points of $f(\beta)$ and $g(\beta)$ for $\beta > 0$.

Note that $f(0) = g(0) = 0$, and $f(+\infty) = 1, g(+\infty) = 0$ and $g(\sqrt{-a_0 a_l}-) = +\infty$, $g(\sqrt{-a_0 a_l}+) = -\infty$. Then it follows from simple computications that

$$f'(\beta) = \frac{l}{\cosh^2 \beta l} > 0$$

$$g'(\beta) = \frac{(a_0 + a_l)(\beta^2 - a_0 a_l)}{(\beta^2 + a_0 a_l)^2} > 0.$$

Then $f(\beta)$ is an increasing function on $(0, +\infty)$ with value from 0 to 1, and $g(\beta)$ is an increasing function on $(0, \sqrt{-a_0 a_l})$ with value from 0 to $+\infty$ and on $(\sqrt{-a_0 a_l}, +\infty)$ with value from $-\infty$ to 0. Hence the intersection point of f and g only appears on $(0, \sqrt{-a_0 a_l})$. Note that $f'(0) = l$ and $g'(0) = -\frac{a_0+a_l}{a_0 a_l}$, then

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(a) If $l = f'(0) > g'(0) = -\frac{a_0+a_l}{a_0 a_l}$, equivalently, $a_0 + a_l < -la_0 a_l$, there is only one root to $f(\beta) = g(\beta)$ for $\beta > 0$, thus there is only one negative eigenvalue.

(b) If $l = f'(0) < g'(0) = -\frac{a_0+a_l}{a_0 a_l}$, equivalently, $a_0 + a_l > -la_0 a_l$, there is no root to $f(\beta) = g(\beta)$ for $\beta > 0$, thus there is no negative eigenvalue.

2. Find the solution to the following problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, 0 < x < l, t \in \mathbb{R} \\ u_x(0,t) = -u(0,t), u_x(l,t) = -u(l,t), \\ u(x, t=0) = \phi(x), \partial_t u(x, t=0) = \psi(x) \end{cases}$$

Solution: Use the separation of variables method.

Step1: Let $u(x,t) = T(t)X(x)$, we have

$$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda,$$

which implies that λ is a constant. Moreover, the boundary conditions show that

$$X'(0) = -X(0), X'(l) = -X(l).$$

Step2: Consider the eigenvalue problem

$$\begin{cases} X''(x) = -\lambda X(x), \\ X'(0) = -X(0), X'(l) = -X(l). \end{cases}$$

First, claim that λ is real. In fact, multiply $X'' = -\lambda X$ by $\overline{X}$ and integrate from 0 to l, then

$$\int_0^l X''(x)\overline{X(x)}dx = -\lambda \int_0^l |X(x)|^2 dx.$$

Note that

$$\begin{aligned} \int_0^l X''(x)\overline{X(x)}dx &= X'(x)\overline{X(x)}\Big|_0^l - \int_0^l |X'(x)|^2 dx \\ &= -|X(l)|^2 + |X(0)|^2 - \int_0^l |X'(x)|^2 dx. \end{aligned}$$

Then

$$\lambda = \frac{|X(l)|^2 - |X(0)|^2 + \int_0^l |X'(x)|^2 dx}{\int_0^l |X(x)|^2 dx} \in \mathbb{R}.$$

If $\lambda = -\beta^2, \beta > 0$, then the general solution to $X'' = -\lambda X$ is

$$X(x) = a\cosh(\beta x) + b\sinh(\beta x)$$

with constants a, b. Then

$$X'(x) = a\beta\sinh(\beta x) + b\beta\cosh(\beta x).$$

The boundary conditions give that

$$\begin{aligned} X'(0) &= -X(0), && \Rightarrow b\beta = -a, \\ X'(l) &= -X(l), && \Rightarrow a\beta\sinh(\beta l) + b\beta\cosh(\beta l) = -a\cosh(\beta l) - b\sinh(\beta l), \end{aligned}$$

then

$$-\beta = -\frac{\beta\cosh\beta l + \sinh\beta l}{\beta\sinh\beta l + \cosh\beta l}$$

or equivalently,

$$\beta^2 \sinh\beta l = \sinh\beta l,$$

since $\beta > 0$, then $\beta = 1$. Thus there is only one negative eigenvalue $\lambda_0 = -1$ and the corresponding eigenfunction is $X_0 = \cosh\beta l - \sinh\beta l = e^{-x}$.

If $\lambda = 0$, then $X(x) = ax + b$. Then boundary conditions show that

$$\begin{aligned} X'(0) &= -X(0), && \Rightarrow a = -b, \\ X'(l) &= -X(l), && \Rightarrow a = -(al + b), \end{aligned}$$

which imply that $a = b = 0$, thus there is no zero eigenvalue.

If $\lambda = \beta^2, \beta > 0$, then the general solution to $X'' = -\lambda X$ is

$$X(x) = a\cos(\beta x) + b\sin(\beta x)$$

with constants a, b. Then

$$X'(x) = -a\beta\sin(\beta x) + b\beta\cos(\beta x)$$

The boundary conditions give that

$$\begin{aligned} X'(0) &= -X(0), && \Rightarrow b\beta = -a, \\ X'(l) &= -X(l), && \Rightarrow -a\beta\sin(\beta l) + b\beta\cos(\beta l) = -a\cos(\beta l) - b\sin(\beta l), \end{aligned}$$

then

$$-\beta = -\frac{\beta\cos\beta l + \sin\beta l}{-\beta\sin\beta l + \cos\beta l}$$

or equivalently,

$$-\beta^2 \sin\beta l = \sin\beta l,$$

which shows that $\sin\beta l = 0$. Thus the eigenvalues and eigenfunctions are

$$\lambda_n = (\frac{n\pi}{l})^2, X_n(x) = -\frac{n\pi}{l}\cos(\frac{n\pi}{l}x) + \sin(\frac{n\pi}{l}x), n = 1, 2, \cdots$$

<u>Step3:</u> Solve the equation $T'' = -\lambda c^2 T$, then

$$\begin{aligned} \lambda_0 &= -1, && T_0(t) = A_0\sinh ct + B_0\cosh ct \\ \lambda_n &= (\frac{n\pi}{l})^2 > 0, && T_n(t) = A_n\sin(\frac{cn\pi}{l}t) + B_n\cos(\frac{cn\pi}{l}t), n = 1, 2, \cdots \end{aligned}$$

where $A_n, B_n, n = 0, 1, \cdots$ are constants to be determined.

Step4: Finally, the solution is given by

$$
\begin{aligned}
u(x,t) =& \sum_{n=0}^{\infty} X_n(x)T_n(t) \\
=& (A_0 \sinh ct + B_0 \cosh ct)e^{-x} + \\
& \sum_{n=1}^{\infty} \Big\{ A_n \sin(\frac{cn\pi}{l}t) + B_n \cos(\frac{cn\pi}{l}t) \Big\} \Big\{ -\frac{n\pi}{l} \cos(\frac{n\pi}{l}x) + \sin(\frac{n\pi}{l}x) \Big\}.
\end{aligned}
$$

While the initial data yield that

$$
\begin{aligned}
\phi(x) = u(x,0) = B_0 e^{-x} + \sum_{n=1}^{\infty} B_n \Big\{ -\frac{n\pi}{l} \cos(\frac{n\pi}{l}x) + \sin(\frac{n\pi}{l}x) \Big\}, \\
\psi(x) = \partial_t u(x,0) = A_0 c e^{-x} + \sum_{n=1}^{\infty} A_n \frac{cn\pi}{l} \Big\{ -\frac{n\pi}{l} \cos(\frac{n\pi}{l}x) + \sin(\frac{n\pi}{l}x) \Big\}.
\end{aligned}
$$