Tutorial 6 for MATH4220

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1. Derive the solution formula for

$$\begin{cases} \partial_t u - k\partial_x^2 u = f(x,t), & x > 0, t > 0 \\ u(x, t=0) = \phi(x), & x > 0 \\ \partial_x u(x=0,t) = h(t), & t > 0. \end{cases}$$

Solution: Let $w(x,t) = u(x,t) - h(t)x$, then w satisfies

$$\begin{cases} \partial_t w - k\partial_x^2 w = f(x,t) - h'(t)x, & x > 0, t > 0 \\ w(x, t=0) = \phi(x) - h(0)x, & x > 0 \\ \partial_x w(x=0,t) = 0, & t > 0. \end{cases}$$

By reflection method, the solution of above problem is given by

$$\begin{aligned} w(x,t) = &\int_{-\infty}^{\infty} \{S(x-y,t) + S(x+y,t)\}\{\phi(y) - h(0)y\}dy \\ &+ \int_0^t \int_{-\infty}^{\infty} \{S(x-y,t-s) + S(x+y,t-s)\}\{f(y,s) - h'(s)y\}dyds. \end{aligned}$$

Hence

$$\begin{aligned} u(x,t) =& h(t)x + \int_{-\infty}^{\infty} \{S(x-y,t) + S(x+y,t)\}\{\phi(y) - h(0)y\}dy \\ &+ \int_0^t \int_{-\infty}^{\infty} \{S(x-y,t-s) + S(x+y,t-s)\}\{f(y,s) - h'(s)y\}dyds. \end{aligned}$$

2. Derive solution formula for

$$\begin{cases} \partial_t^2 u - c^2\partial_x^2 u = f(x,t), -\infty < x < \infty, t > 0 \\ u(x, t=0) = \phi(x), -\infty < x < \infty \\ \partial_t u(x, t=0) = \psi(x), -\infty < x < \infty \end{cases}$$

by the method using Green's Theorem.

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Solution: Green's Theorem implies that

$$\begin{aligned}\iint_\Delta f(y,s)dyds = \iint_\Delta u_{tt} - c^2 u_{xx} dyds = \iint_\Delta \partial_x(-c^2 u_x) - \partial_t(-u_t)dyds \\ = \int_{\partial\Delta} -u_t dy - c^2 u_x ds\end{aligned}$$

Note that $\Delta = \{(y,s) : 0 < s < t, x - c(t-s) < y < x + c(t-s)\}$ and $\partial\Delta = L_1 + L_2 + L_3$ with counterclockwise direction where $L_1 = \{(y,0) : x - ct < y < x + ct\}, L_2 = \{(y,s) : 0 < s < t, y = x + c(t-s)\}$ and $L_3 = \{(y,s) : 0 < s < t, x - c(t-s) = y\}$. Then

$$\int_{L_1} -u_t dy - c^2 u_x ds = \int_{x-ct}^{x+ct} -u_t(y,0)dy = \int_{x-ct}^{x+ct} -\psi(y)dy$$

$$\begin{aligned}\int_{L_2} -u_t dy - c^2 u_x ds = \int_{L_2} c u_t ds + c u_x dy = c\int_{L_2} udu \\ = c(u(x,t) - u(x+ct,0)) = cu(x,t) - c\phi(x+ct)\end{aligned}$$

where we have used the facts that $dy = -cds$ on L_2 and $du = u_x dy + u_t ds$.

$$\begin{aligned}\int_{L_3} -u_t dy - c^2 u_x ds = \int_{L_3} -c u_t ds - c u_x dy = c\int_{L_3} -udu \\ = -c(u(x-ct,0) - u(x,t)) = cu(x,t) - c\phi(x-ct)\end{aligned}$$

where we have used the facts that $dy = cds$ on L_3 and $du = u_x dy + u_t ds$.

Hence we have

$$u(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\iint_\Delta f(y,s)dyds.$$

3. Derive solution formula for

$$\begin{cases} v_{tt} - c^2 v_{xx} = f(x,t), x > 0, t > 0 \\ v(x,0) = \phi(x), v_t(x,0) = \psi(x), x > 0 \\ v(0,t) = h(t), t > 0 \end{cases}$$

with compatibility conditions $\phi(0) = h(0)$ and $\psi(0) = h'(0)$.

Solution: First, consider the following two problems:

$$\begin{cases} v^1_{tt} - c^2 v^1_{xx} = f(x,t), x > 0, t > 0 \\ v^1(x,0) = \phi(x), v^1_t(x,0) = \psi(x), x > 0 \\ v^1(0,t) = 0, t > 0 \end{cases} \tag{1}$$

and

$$\begin{cases} v^2_{tt} - c^2 v^2_{xx} = 0, x > 0, t > 0 \\ v^2(x,0) = 0, v^2_t(x,0) = 0, x > 0 \\ v^2(0,t) = h(t), t > 0 \end{cases} \tag{2}$$

then $v = v^1 + v^2$ is the solution to original inhomogeneous IBVP.

For problem (1), by reflextion method, the solution formula is given by

$$v_1 = \begin{cases} \frac{1}{2}(\phi(x+ct)+\phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy + \int_0^t\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds, & x > ct \\ \frac{1}{2}(\phi(x+ct)-\phi(ct-x)) + \frac{1}{2c}\int_{ct-x}^{x+ct}\psi(y)dy & \\ +(\int_0^{t-\frac{x}{c}}\int_{c(t-s)-x}^{x+c(t-s)} + \int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)})f(y,s)dyds, & x < ct. \end{cases}$$

For problem (2), the solution has the form of $v_2 = F(x+ct)+G(x-ct)$. The initial conditions imply that for $x > 0$

$$F(x) + G(x) = 0, F'(x) - G'(x) = 0$$

then $F(x) = -G(x) = C$ with constant C for $x > 0$. Let $\tilde{F} = F-C, \tilde{G} = G+C$, then $\tilde{F}(x) = \tilde{G}(x) = 0$ for $x > 0$, and $v_2 = F(x+ct)+G(x-ct) = \tilde{F}(x+ct)+\tilde{G}(x-ct)$. While the boundary condition implies that for $t > 0$

$$\tilde{F}(ct) + \tilde{G}(-ct) = h(t)$$

Notice that $\tilde{F}(x) = 0$ for $x > 0$, thus $\tilde{G}(-ct) = h(t)$, i.e. $\tilde{G}(x) = h(-\frac{x}{c})$ for $x < 0$. Hence the general solution to (2) is

$$v_2 = \begin{cases} 0, & x > ct \\ 0 + \tilde{G}(x-ct) = h(t-\frac{x}{c}), & x < ct \end{cases}$$

Therefore,

$$v = \begin{cases} \frac{1}{2}(\phi(x+ct)+\phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy + \iint_\Delta f(y,s)dyds, & x > ct \\ \frac{1}{2}(\phi(x+ct)-\phi(ct-x)) + \frac{1}{2c}\int_{ct-x}^{x+ct}\psi(y)dy + \iint_D f(y,s)dyds + h(t-\frac{x}{c}), & x < ct \end{cases}$$

where Δ and D are characteristic domains as shown in v_1.