Tutorial 1 for MATH4220

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1. Review some basic facts in calculus.

   - **The fundamental theorem of calculus**
     Suppose that f is differential in $[a, b]$, we have
     $$f(b) - f(a) = \int_a^b f'(x)dx.$$
   - **Green's Fromula**
     Let D be a bounded plane domain with a piecewise C^1 boundary curve $C = bdyD$. Consider C to be parametrized so that it is traversed once with D on the left. Let $p(x, y)$ and $q(x, y)$ be any C^1 functions defined on $\overline{D} = D \cup C$. Then
     $$\iint_D (q_x - p_y)dxdy = \int_C pdx + qdy.$$
   - **Divergence Theorem:**
     Let D be a bounded spatial domian with a piecewise C^1 boundary surface S. Let $\vec{n}$ be the unit outward normal vector on S. Let $f(x)$ be any C^1 vector field on $\overline{D} = D \cup S$. Then
     $$\iiint_D \nabla \cdot f dx = \iint_S f \cdot \vec{n} dS.$$
   - **Integration by parts**
     Let $D \subset \mathbb{R}^n$ be a bounded domian with a piecewise C^1 boundary surface S. Let $\vec{n} = (x^1, \cdots, x^n)$ be the unit outward normal vector on S. Let $f(x), g(x)$ be any C^1 functions on $\overline{D} = D \cup S$. Then for $i = 1, \cdots, n$
     $$\iiint_D \partial_{x_i} f(x)g(x)dx = \iint_S f(x)g(x)n^i dS - \iiint_D f(x)\partial_{x_i} g(x)dx.$$
   - Mixed derivatives are equal:
     If a function $f(x, y)$ is of class C^2, then $\partial_{xy} u = \partial_{yx} u$.

*Any questions about the tutorial notes, please email me at rzhang@math.cuhk.edu.hk.

- Chain rule.

  The Chain rule deals with functions of functions.

  For example, consider the chain $s, t \mapsto x, y \mapsto u$. Suppose u is a function of x, y of class C^1, and x, y are differential functions of s, t, then

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s}.$$

2. Use **Characteristic Method** to solve the following inhomogeneous PDE

$$a(x,y)\partial_x u + b(x,y)\partial_y u = c(x,y)$$

where $a(x,y), b(x,y), c(x,y)$ are smooth functions.

Solution: The characteristic equation is

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} \tag{1}$$

This is a 1-st order ODE. Suppose the solution is given by

$$f(x,y) = C,$$

or can be expressed explicitly by $y = y(x, C)$ with arbitrary constant C. Define $z(x,C) = u(x, y(x,C))$, then

$$\frac{dz}{dx} = u_x + u_y\frac{dy}{dx} = u_x + \frac{b(x,y(x,C))}{a(x,y(x,C))}u_y = \frac{c(x,y(x,C))}{a(x,y(x,C))} \tag{2}$$

which is a 1-st order linear ODE of z with respect to x by considering C as a parameter. The general solution to above ODE is given by

$$\begin{aligned} z(x,C) =& \int \frac{c(x,y(x,C))}{a(x,y(x,C))}dx + h(C) \\ =:& g(x,C) + h(C) \end{aligned}$$

where h is an arbitrary function. Then we obtain the general solution to PDE

$$u(x,y) = g(x, f(x,y)) + h(f(x,y))$$

where f, g are determined by the above two ODEs (1)(2), respectively, and h is an arbitrary function.