Suggested Solution to Quiz 2

April 10, 2018

1. (**7 points**) Consider the eigenvalue problem

$$\begin{cases} -X''(x) = \lambda X(x), & 0 < x < \pi \\ X'(0) = 0, \quad X(\pi) = 0 \end{cases}$$

   (a) (2points) Can the eigenvalue problem have complex eigenvalue? Why?

   (b) (2points) Can the eigenvalue problem have non-positive eigenvalue? Why?

   (c) (3points) Solve the eigenvalue problem.

Solution:

(a) (2points) No.

Let λ be the eigenvalue of the problem and $X(x)$ the corresponding eigenfunction. Multiply the equation $-X''(x) = \lambda X(x)$ by $\overline{X(x)}$ and integrate with respect to x, then we get

$$-\int_0^\pi X''(x)\overline{X(x)}dx = \lambda \int_0^\pi |X(x)|^2 dx$$

With the help of the boundary conditions, we have

$$-\int_0^\pi X''(x)\overline{X(x)}dx = -X'(x)\overline{X(x)}\Big|_0^\pi + \int_0^\pi |X'(x)|^2 dx = \int_0^\pi |X'(x)|^2 dx$$

Therefore,

$$\lambda = \frac{\int_0^\pi |X'(x)|^2 dx}{\int_0^\pi |X(x)|^2 dx} \in \mathbb{R}$$

(b) (2points) No.

By (a), we know that $\lambda \geq 0$. If $\lambda = 0$, then we must have $X'(x) \equiv 0$ on $[0, \pi]$ which implies that $X(x) = Constant$. Since $X(\pi) = 0$, then $X(x) \equiv 0$ which is impossible. Therefore $\lambda > 0$.

(c) Since $\lambda > 0$, let $\lambda = \beta^2$ for $\beta > 0$. The general solution to the ODE $-X'' = \lambda X$ is

$$X(x) = A\cos(\beta x) + B\sin(\beta x), \qquad \text{(1point)}$$

then

$$X'(x) = -\beta A \sin(\beta x) + \beta B \cos(\beta x).$$

Apply the boundary conditions, we have

$$\begin{aligned} 0 = X'(0) \Rightarrow& \qquad 0 = \beta B \\ 0 = X(\pi) \Rightarrow& \qquad 0 = A\cos(\beta\pi) + B\sin(\beta\pi) \qquad \text{(1point)} \end{aligned}$$

Thus $B = 0$ and $\beta = \frac{1}{2} + n$ for $n = 0, 1, \cdots$. Therefore the eigenvalues and corresponding eigenvectors are

$$\lambda_n = (\frac{1}{2} + n)^2, \ X_n(x) = \cos(\frac{\pi}{2} + n\pi), n = 0, 1, \cdots \qquad \text{(1point)}$$

2. (**3 points**) Solve the following problem

$$\begin{cases} \partial_t^2 u = \partial_x^2 u, & 0 < x < \pi, t \geq 0 \\ \partial_x u(0,t) = \partial_x u(\pi,t) = 0, & \\ u(x,t=0) = 0, \partial_t u(x,t=0) = \cos^2 x, & 0 < x < \pi \end{cases}$$

Solution: Use the separation of variables method, let $u(x,t) = X(x)T(t) \neq 0$, then the PDE gives

$$\frac{X''}{X} = \frac{T''}{T} = -\lambda,$$

which implies that λ is a constant. Moreover, boundary conditions yield $X'(0) = X'(\pi) = 0$. Then we obtain the following eignevalue problem

$$\begin{cases} X''(x) = -\lambda X(x), & 0 < x < \pi \\ X'(0) = 0, \ X'(\pi) = 0. & \end{cases}$$

Note that the eigenvalue of above problem is real and further nonnegative. In fact, mulitipying $X'' = -\lambda X$ by $\overline{X}$ and using the boundary conditions, we have

$$\lambda = \frac{\int_0^\pi |X'(x)|^2 dx}{\int_0^\pi |X(x)|^2 dx} \geq 0.$$

Then solving the eigenvalue problem, the eigenvales and corresponding eigenvectors are

$$\lambda_n = n^2, \ X_n(x) = \cos(n\pi), n = 0, 1, \cdots \qquad \text{(1point)}$$

Then solve

$$T''(t) = -n^2 T(t)$$

we have

$$\begin{aligned} T_0 &= A_0 + B_0 t \\ T_n &= A_n \cos(nt) + B_n \sin(nt), n = 1, 2, \cdots \end{aligned} \qquad \text{(0.5point)}$$

with constants A_n, B_n to be determined. Hence

$$u(x,t) = \sum_{n=0}^{\infty} X_n(x)T_n(t) = A_0 + B_0 t + \sum_{n=1}^{\infty} A_n \cos(nt)\cos(nx) + B_n \sin(nt)\cos(nx). \qquad \text{(0.5point)}$$

Then initial conditions give

$$\begin{aligned} u(x,t=0) = 0 &= A_0 + \sum_{n=1}^{\infty} A_n \cos(nx), \\ \partial_t u(x,t=0) = \cos^2 x = \frac{1+\cos(2x)}{2} &= B_0 + \sum_{n=1}^{\infty} nB_n \cos(nx) \end{aligned} \qquad \text{(0.5point)}$$

which impliy that $A_n = 0$ for $n = 0, 1, \cdots$ and $B_0 = \frac{1}{2}, B_2 = \frac{1}{4}, B_n = 0$ for $n \neq 0, 2$. Therefore

$$u(x,t) = \frac{t}{2} + \frac{1}{4}\sin(2t)\cos(2x). \qquad \text{(0.5point)}$$