# Section 7. Viscous Shock Wave Theory

## §7.1 Introduction

$$\partial_t u + \partial_x f(u) = 0, \qquad u \in \mathbb{R}^n,\ x \in \mathbb{R}^1,\ t > 0. \tag{7.1}$$

(7.1) is obtained as an idealization when the higher order effects such as viscosity, heat conduction, surface tension etc. are neglected, e.g.,

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = 0 \end{cases} \quad \text{ideal gas} \qquad (7.1)'$$

Compressible Navier-Stokes

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0 \\ \partial_t(\rho u) + \partial_x(\rho u^2 + p) = \varepsilon \partial_x^2 u \end{cases} \qquad (7.2)'$$

$\varepsilon > 0$, $\rho = \rho(x,t) > 0$, $u$, velocity.
Any periodic solution of (7.1)' will develop singularity in finite time.

Question: Relationship between (7.1)' and (7.2)'.

Mathematically, vanishing viscosity method, we need to solve (7.1)'.

First, we solve (7.2)' to $(\rho^\varepsilon, u^\varepsilon)$, then try to study the properties of $(\rho^\varepsilon, u^\varepsilon)$ so that one can assert that

$$(\rho^\varepsilon, u^\varepsilon) \longrightarrow (\rho^0, u^0) \quad \text{in appropriate topology.}$$

so that $(\rho^0, u^0)$ is an admissible weak solution to (7.1)'.

$$\partial_t u + \partial_x f(u) = \varepsilon\, \partial_x (B(u)\, \partial_x u) \qquad (7.2)$$
$$B(u) \geq 0,\ \varepsilon > 0,\ B(u) = I.$$

**Remarks:**

(1) It can be justified rather easily that if the system (7.1) has a classical solution $u(x,t)$, $t \in [0,T]$. Then $\exists \,|\, u^{\varepsilon}(x,t)$ smooth, exists on $[0,T]$ such that

$$|u^{\varepsilon}(x,t) - u(x,t)| \leq c\,\varepsilon$$

where $c$ is independent of $\varepsilon$. This is true even for multidimensional case. The proof can be proceeded by standard energy method. Write

$$u^{\varepsilon}(x,t) = u(x,t) + \varepsilon\, u_1(x,t) + \varepsilon^2\, u_2(x,t) + \varphi(x,t).$$

Then derive an equation for the error term $\varphi$,

$$||\varphi|| \leq c\,\varepsilon.$$

This is so called regular perturbation method.

(2) If $\partial_t u + \partial_x f(u) = 0$ has a shock solution

$$u(x,t) = \begin{cases} u_- & x < st \\ u_+ & x > st \end{cases}$$

Can we find a corresponding $\phi\left(\frac{x - st}{\varepsilon}\right)$ to

$$\partial_t u + \partial_x f(u) = \varepsilon\, \partial_x (B(u)\, \partial_x u)?$$

such that

$$\phi(\xi) \to u_\pm \qquad \text{as} \quad \xi \to \pm\infty.$$

Gelfand problem: $(u_-, u_+, s)$ is called admissible if it has a viscous shock profile.

In conclusion, one of the main difficulties is understanding the asymptotic relationship between (7.1) and (7.2) is around the discontinuities at the inviscid flow.

(3) Physical boundary effect

E.g. Let $c$ be a constant,

$$\begin{cases} \partial_t u + c\,\partial_x u = \varepsilon\,\partial_x^2 u \\ u(x, t=0) = u_0(x), \quad x > 0 \\ u(x=0, t) = u_1(t), \quad t > 0 \end{cases} \tag{7.2}''$$

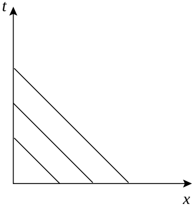
$$\begin{cases} \partial_t u + c\,\partial_x u = 0 \\ u(x, t=0) = u_0(x), \quad x > 0 \\ B.C. \end{cases} \tag{7.1}''$$

Case 1: $c > 0$, we need boundary data, if $u(x=0,t) = u_1(t)$ for (7.1)', then

$$||u^\varepsilon(x,t) - u(x,t)||_{L^\infty} \le c\,\varepsilon$$

No Strong boundary layer!

Case 2: $c < 0$, in this case, the solution of (7.1)' is given uniquely by $u(x,t) = u_0(x - ct)$.




Thus, in particular, $u(x = 0, t) = u_0(-ct)$, in general, it is different from $u_1(t)$, thus

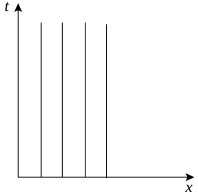
$$u^{\varepsilon}(x,t) \not\rightarrow u(x,t).$$

Therefore, the discrepancy in velocity creates the solution called boundary layer.

$$u^{\varepsilon}(x,t) \backsim b_0\left(\frac{x}{\varepsilon}, t\right) + \varepsilon\, b_1\left(\frac{x}{\varepsilon}, t\right) + u(x,t) + \varepsilon\, u_1(x,t) + O(\varepsilon^2)$$

uniformly non-characteristic boundary layer.

Case 3: $c = 0$, (7.1)" $\partial_t\, u = 0$, the boundary $x = 0$ is one of the characteristic line for the hyperbolic equation. In this case, the boundary is called uniform characteristic boundary.




In this case, there is also a boundary layer, and the boundary layer has width $O(\sqrt{\varepsilon})$.

## §7.2 Viscous Shock Profiles

Consider

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 u, \qquad \varepsilon > 0 \tag{7.3}$$

$$\partial_t u + \partial_x f(u) = 0 \tag{7.4}$$

Assumptions: (7.4) is strictly hyperbolic, each characteristic field is either genuinely nonlinear or linearly degenerate.

$$A(u) = \nabla f(u): \quad \begin{array}{l} \lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u) \\ \gamma_1(u),\ \gamma_2(u),\ \cdots,\ \gamma_n(u) \\ l_1(u),\ l_2(u),\ \cdots,\ l_n(u) \end{array}$$

We normalize it as $l_i^T \gamma_j = \delta_{ij}$.

Lax entropy condition, $(u_-, u_+, s)$ is an $i$-shock if

(1) $\lambda_i(u_+) < s < \lambda_i(u_-)$.

(2) $\lambda_{i-1}(u_-) < s < \lambda_{i+1}(u_+)$.

(For $n \times n$ system, there are $n+1$ characteristics hitting on the shock, $n-1$ characteristics leaving the shock.)

Liu entropy condition: for $u_-$, let the $i$-shock wave curve be given by $\tilde{u}^i(\rho, u_-)$ such that

$$\tilde{u}^i(\rho = 0, u_-) = u_-, \quad s(\rho) = s(\tilde{u}^i(\rho, u_-), u_-), \quad s = (\rho = 0) = \lambda_i(u_-),$$

$$\frac{d}{d\rho}\tilde{u}^i(\rho, u_-)\Big|_{\rho=0} = \gamma_i(u_-), \quad \frac{ds(\rho)}{d\rho}\Big|_{\rho=0} = \frac{1}{2}\nabla\lambda_i(u_-)\cdot\gamma_i(u_-),$$

$$\rho = l_i(u_-)\tilde{u}^i(\rho, u_-) - u_-).$$

Then Liu's entropy condition says:
if $u_+ = \tilde{u}^i(\rho_+, u_-)$, and $(u_+, u_-, s(\rho_+))$ is admissible if $s(\rho) > s(\rho_+)$ for all $\rho$ between 0 and $\rho_+$.

**Remark:** For genuine nonlinear system, Liu's condition $\Longleftrightarrow$ Lax entropy condition.

**Theorem 7.1** (Existence of viscous shock profile)
If $\lambda_i$ is not linearly degenerate at $u_0$. Then $\exists$ a small neighborhood of $u_0$, $N_\delta(u_0)$, such that if $u_\pm \in N_\delta(u_0)$ and $u_+$ is connected to $u_-$ from the right by a connecting orbit iff Liu's entropy condition is satisfied. In other words, the following problem

$$\begin{cases} \phi''(\xi) = \nabla f(\phi)\phi' - s\,\phi' \\ \phi(\pm\infty) = u_\pm \end{cases}$$

has a solution.

**Proof of Theorem 7.1:** The proof is based on center manifold construction.

Step 1: Center Manifold Theorem

Consider the following ODE system

$$\begin{cases} \dfrac{dx}{dt} = Ax + f_1(x, y, z) \\ \dfrac{dy}{dt} = By + f_2(x, y, z) \\ \dfrac{dz}{dt} = Cz + f_3(x, y, z) \end{cases} \quad (7.5)$$

here $A$, $B$, $C$ are constant matrices, and $Re(\lambda(A)) > 0$, $Re(\lambda(C)) < 0$, $Re(\lambda(B)) = 0$, $f_i(0, 0, 0) = 0$, $\nabla f_i(0, 0, 0) = 0$ for $i = 1, 2, 3$. $f_i$ is $C^r$.

Then there exists a $C^r$ manifold

$$M = \{(x, y, z) \mid |y| < \delta,\ x = x(y),\ z = z(y)\}$$

which is invariant for (7.5), and

$$x(0) = 0,\ z(0) = 0,\ \nabla x(0) = 0,\ \nabla z(0) = 0.$$

We integrate the second ODE

$$\phi'' = \nabla f(\phi)\phi' - s\phi'$$

once to obtain

$$\phi' = f(\phi) - f(u_-) - s(u - u_-).$$

We will solve the system

$$\begin{cases} \dfrac{du}{d\xi} & = & f(u) - f(u_-) - s(u - u_-) \\ \dfrac{ds}{d\xi} & = & 0 \\ \dfrac{du_-}{d\xi} & = & 0 \end{cases} \tag{7.6}$$

Set $F(u) = -s(u - u_-) + (f(u) - f(u_-))$, then

$$F(u_-) = F(u_+) = 0.$$

We will assume that $\lambda_p(u_0) = 0$ (otherwise we use a change of variable $y = x - \lambda_p(u_0)t$). Set

$$v = u_-, \quad \omega = u - v, \quad w = (\omega, v, s).$$

We can rewrite system (7.6) as

$$\frac{dw}{d\xi} = \begin{pmatrix} F(v+\omega, v, s) \\ 0 \\ 0 \end{pmatrix} = Tw \tag{7.7}$$

We choose base point as $(0,0,0)$, $(u_0 = 0)$,

$$Tw = dT(0)w + Q(w)$$

where $Q(w) = O(|w|)^2$.

$$dT(0) = \begin{pmatrix} A(0) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$A(0) = \nabla f(0).$$

Clearly, 0 is an eigenvalue of $dT(0)$, moreover, it has multiplicity $n+2$.

It follows that the algebraic eigenspace with $Re\,\lambda \neq 0$ is of dimension $n-1 = \mathbb{Z}$.

The algebraic eigenspace with $Re\,\lambda = 0$ is of dimension $n+2 = Y$. It should be clear from the structure of $dT(0)$ that

$$Y = \text{span}\ \{(\gamma_p(u_0), 0, 0),\ (0, \gamma_i(u_0), 0),\ 1 \leq i \leq n,\ (0, 0, 1)\}$$

$$\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^1 = Y \oplus Z = \ker dT(0) + \text{Range}\ dT(0).$$

We can apply the center manifold theorem to system (7.7) to obtain

**Proposition 7.1:** $\exists\, \delta > 0$, and smooth $\mathrm{map}$ $G : Y \to Z$, such that

$$M = \{y + z |\ y \in Y,\ z \in Z \quad \text{such that } z = G(y),\ |y| < \delta\}$$

is an $n + 2$-dimensional invariant manifold for (7.7) with

$$G(0) = 0, \quad \nabla G(0) = 0.$$

Note that by construction, $M$ contains all the trajectories of (7.7). In particular, all the critical points of (7.7) lies on $M$.

Step 2: Construction of invariant curve so that the connecting orbit problem is reduced to a scalar ODE.

Now we fix $v = u_-$, $s = s$. For any given small constant $\eta$, we consider $y(\eta) = (\eta\,\gamma_p(u_0), u_-, s)$,

$$M(\eta) = \{y(\eta) + z |\ z = G(y(\eta))\}.$$

Therefore, $z = G(y(\eta)) = (g(y(\eta)), 0, 0)$, $g(0) = 0$, $\nabla g(0) = 0$, $z = g(\eta, u_-, s)$.

Then $M(\eta)$ is invariant for the flow defined by (7.7),

$$\omega = \eta\,\gamma_p(u_0) + g(\eta, u_-, s)$$

$$\begin{cases} u = u_- + \eta\,\gamma_p(u_0) + g(\eta, u_-, s) \\ v = u_- \\ s = s \end{cases}$$

**Proposition 7.2** (Existence of invariant curve)

$\exists\, \delta > 0$, such that if

$$|\eta\, \gamma_p(u_0)| + |u - u_-| + |s - \lambda_p(u_0)| < \delta,$$

then

$$u = u_- + \eta\, \gamma_p(u_0) + g(\eta, u_-, s)$$

is invariant so that

$$\frac{du}{d\xi} = -s(u - u_-) + f(u) - f(u_-).$$

Furthermore, if $u_+ \in s_p(\rho, u_-)$, then $\exists\, \eta_+$ such that

$$u_+ = u_- + \eta_+\, \gamma_p(u_0) + g(\eta_+, u_-, s)$$

with $\rho_+ = l_p(u_-)(u_+ - u_-)$.

**Proof of Proposition 7.2**: The first part follows from Proposition 7.1 as the previous discussion. To show the second part, if $u \in s_p(\rho, u_-)$, then $\exists\, s$ such that

$$s(u - u_-) = f(u) - f(u_-),$$

then $\begin{pmatrix} u - u_- \\ u_- \\ s \end{pmatrix}$ is a critical point of (7.7), so we finish the Proof of Proposition 7.2.

Since all the trajectories lie on $M(\eta)$,

$$\partial_\eta\, u \frac{d\eta}{d\xi} = -s(u - u_-) + (f(u) - f(u_-)),$$

so,

$$(l_p(u_-)\cdot\partial_\eta\, u)\frac{d\eta}{d\xi} = (-s(u-u_-)+f(u)-f(u_-))\; l_p(u_-).$$

Thus

$$\begin{aligned}\frac{d\eta}{d\xi} &= (l_p(u_-)\cdot\partial_\eta\, u)^{-1}\;(l_p(u_-)\cdot(-s(u-u_-)+f(u)-f(u_-)))\\ &= \sigma(\eta, u_-, s)\end{aligned}$$

Claim: This scalar ODE is well-defined, $l_p(u_-)\cdot\partial_\eta\, u > 0$.

Proof of Claim:

$$\partial_\eta u = \gamma_p(u_0) + \partial_\eta g(\eta, u_-, s),$$

so,

$$\begin{aligned} l_p(u_-) \cdot \partial_\eta u &= l_p(u_-) \cdot \gamma_p(u_0) + l_p(u_-) \cdot \partial_\eta g(\eta, u_-, s) \\ &= 1 + O(\delta) > 0 \end{aligned}$$

$$\begin{cases} \dfrac{d\eta}{d\xi} = \sigma(\eta, u_-, s) \\ \eta(-\infty) = 0, \quad \eta(+\infty) = \eta_+ \end{cases} \tag{7.8}$$

Fact: (7.8) has solution iff sgn $\sigma(\eta, u_-) =$ sgn $\eta_+$.

Step 3: To check sgn $\sigma(\eta, u_-) =$ sgn $\eta_+$ for any $\eta$ between 0 and $\eta_+$ iff Liu's entropy condition holds.

Some properties of viscous shock profile:

**Proposition 7.3** Assume that $p$-family is genuinely nonlinear, then

(1) $\frac{\partial}{\partial \xi}\lambda_p(\varphi) < 0$ (compressibility of shock).

(2) $O(1)|\partial_\xi \lambda_p(\varphi) \leq |\frac{\partial \varphi}{\partial \xi}| \leq O(1)|\partial_\xi \lambda_p(\varphi)|$.

(3) $\int_{\mathbb{R}} |\varphi'(\xi)| d\xi = O(1)\delta, \;\; \delta = |u_+ - u_-|, \;\; \lambda_p(\phi) - s| = O(1)\delta$.

(4) $c_3\,\delta^2\,\exp(-c_4\,\delta|\xi|) \le |\partial_\xi\,\lambda_p(\phi)| \le c_1\,\delta^2\,\exp(-c_2\,\delta|\xi|).$

(5) $|\varphi''(\xi)| \le O(1)\,\delta|\varphi'(\xi)|.$

(6) $$\phi(\xi,u_-,s) - u_- = O(1)\,\delta\,\exp(\delta\,c_5\,\xi), \quad \text{as} \quad \xi \to -\infty,$$
$$\frac{\partial\varphi}{\partial u_-} - I = O(1)\,\exp(\delta\,c_6\,\xi), \quad \text{as} \quad \xi \to -\infty,$$
$$\frac{\partial\varphi}{\partial s} = O(1)\,\exp(\delta\,c_7\,\xi), \quad \text{as} \quad \xi \to -\infty.$$

(7) $$\varphi(\xi,u_-,s) - u_+ = O(1)\,\delta\,\exp(-\delta\,c_8\,\xi), \quad \text{as} \quad \xi \to +\infty,$$
$$\frac{\partial\varphi}{\partial u_-} - \frac{\partial u_+}{\partial u_-} = O(1)\,\exp(-\delta\,c_9\,\xi), \quad \text{as} \quad \xi \to +\infty,$$
$$\frac{\partial\varphi}{\partial s} - \frac{\partial u_+}{\partial s} = O(1)\,\exp(-\delta\,c_{10}\,\xi), \quad \text{as} \quad \xi \to +\infty.$$

## §7.3 Nonlinear stability of Viscous shock profile

$$\begin{cases} \partial_t u + \partial_x f(u) = \partial_x^2 u & u \in \mathbb{R}^n,\ x \in \mathbb{R}^1, \quad t > 0 \\ u(x, t = 0) = u_0(x) & x \in \mathbb{R}^1 \end{cases}$$

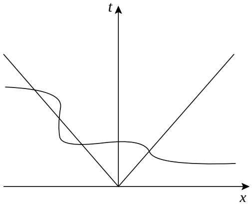
$$\lim_{x \to \pm\infty} u_0(x) = u_\pm.$$

Case 1: $1 \le p \le n$, such that $p$-family is genuinely nonlinear, and $(u_-, u_+, s)$ is a $p$-shock, $\exists\, \varphi(x - st)$ connecting $u_+$ to $u_-$ on the right. Roughly speaking, assume that

$$u_0(x) \quad \text{is “very small perturbation” of } \varphi(x),$$

is there a global solution $u(x, t)$ to (7.3) such that $u(x, t) \to \varphi(x - st)$ as $t \to +\infty$?

Case 2: Let us assume that every family is genuinely nonlinear, so that $\exists\, u_0, u_1, \cdots, u_n$ such that $u_0 = u_-$ and $u_n = u_n$ and $(u_i, u_{i-1}, s_i)$ is an $i$-shock with shock profile $\varphi_i(x - s_i\, t)$. Then $\Phi(x,t)$ is the superposition of $\varphi_i$. Assume that $u_0(x)$ is "very small perturbation" of $\Phi(x, t = 0)$, then is there solution $u(x,t)$ to (7.3) such that $u(x,t) \to \Phi(x,t)$ as $t \to +\infty$?




For $n = 2$, $\Phi(x,t) = (\varphi_2(x - s_2\, t) + \varphi_1(x - s_1\, t)) - u_1$.

**Theorem 7.2** (Stability of viscous shock profile, Goodman)

$\exists\, \delta > 0$, such that if

(1) $\displaystyle\int_{-\infty}^{+\infty} (u_0(x) - \varphi(x))dx = 0.$

(2) $u_0 - \varphi \in H^1(\mathbb{R}^1)$, and $|x|(u_0 - \varphi) \in L^2(\mathbb{R}^1)$.

(3) $||x(u_0 - \varphi)||_{L^2(\mathbb{R})} + ||u_0 - \varphi||_{H^1(\mathbb{R})} + |u_+ - u_-| < \delta.$

Then there exists unique smooth solution $u(x,t) \in c([0,\infty); H^1(\mathbb{R}))$, such that

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}^1} |u(x,t) - \varphi(x - st)| = 0.$$

**Remark:** If $n = 1$, then all the restrictions (1) and (3) can be removed, in that case, $\exists\, x_0$ such that

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |u(x,t) - \varphi(x - st + x_0)| = 0,$$

and $x_0$ is determined uniquely by initial excessive mass.

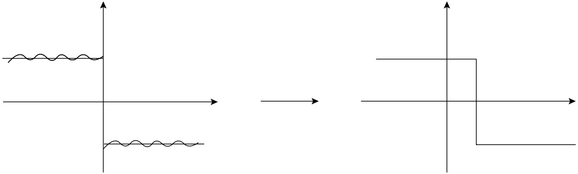
$$\frac{\int_{-\infty}^{\infty} (u_0(x) - \varphi(x))dx}{u_+ - u_-} = x_0$$

$$\int_{-\infty}^{+\infty} (u_0(x) - \varphi(x))dx - x_0(u_+ - u_-) = \int_{-\infty}^{\infty} (u_0(x) - \varphi(x + x_0))dx$$

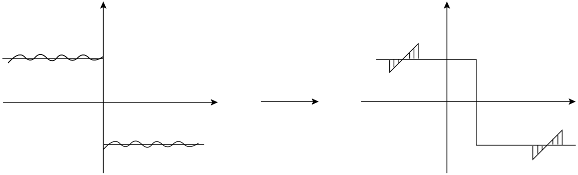
This is very serious issue. It is NOT technical problem.

In fact, a general perturbation of a shock profile introduces not only phase shift, but also the so called diffusion waves.

For $n = 1$, hyperbolic system of conservations laws

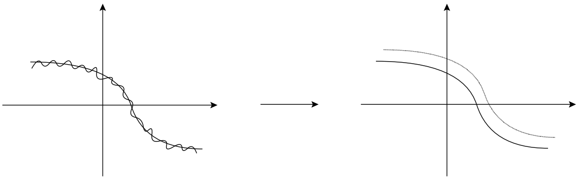


For $n > 1$,



For viscous conservation law:

$n = 1$



$n > 1$,

1985, T.P Liu,

$$u(x,t) = \varphi(x - st + x_0) + \sum_{i \neq p} \theta_i(x,t)\ \gamma_i(u_i) + \text{small perturbation}$$

$$\int_{-\infty}^{\infty} \theta_i(x,t)dx = m_i$$

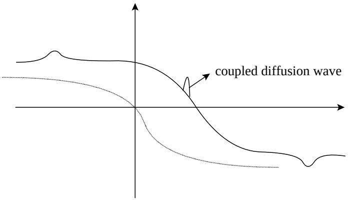
$$\int_{-\infty}^{\infty}(u_0(x)-\varphi(x))dx = x_0(u_+-u_-)+\sum_{i\neq p} m_i\,\gamma_i(u_i), \quad u_i = \begin{cases} u_- & i<p \\ u_+ & i>p \end{cases}$$

$$\partial_t\,\theta_i + \lambda_i(u_i)\;\partial_x\,\theta_i + \frac{1}{2}\nabla\lambda_i(u_i)\cdot\gamma_i(u_i)(\theta_i^2)_x = \theta_{i,xx}$$

This can be solved explicitly by using the self-similar $\dfrac{(x-\lambda_i(u_i)t)}{\sqrt{t}}$.

Let $u(x,t) - u^a(x,t) = w(x,t)$,

$$\int_{-\infty}^{\infty} w(x,t)dx = 0 \qquad \forall\, t$$



Sketch of the Proof of Theorem 7.2:
Energy estimate

$$\begin{cases} \partial_t u + \partial_x f(u) = \partial_x^2 u \\ u(x, t = 0) = u_0(x) \end{cases} \quad (7.9)$$

Method has explore the special feature of the underlying reference solution $\varphi(x - st)$

$$\frac{\partial}{\partial x}(\lambda_p(\varphi(x - st))) < 0.$$

Let us assume $s = 0$ (otherwise work with $x' = x - st$, $\varphi(x')$ is stationary). Let $u(x, t)$ be a solution to (7.9)

$$u(x, t) = \varphi(x) + v(x, t),$$

where perturbation $v(x,t)$ satisfies error problem

$$\begin{cases} \partial_t v + (f(v+\varphi) - f(\varphi))_x = \partial_x^2 v \\ v(x, t=0) = u_0(x) - \varphi(x) = v_0(x) \end{cases}$$

Clearly, $\int_{-\infty}^{\infty} v_0(x)dx = 0, \;\; ||v||_{H^1} \ll 1.$

It suffices to prove that $v$ exists globally in time, and

$$\lim_{t\to+\infty} ||v(\cdot,t)||_{L^\infty} = 0.$$

$$\partial_t v + (\nabla f(\varphi)v)_x + (Q(\varphi,v))_x = \partial_x^2 v,$$

where

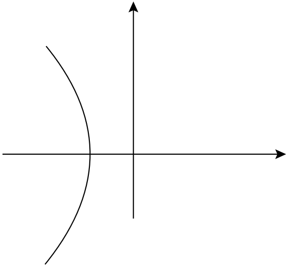
$$Q(\varphi,v) = f(\varphi+v) - f(\varphi) - \nabla f(\varphi)v$$

$$|Q(\varphi,v)| \le O(1)\,|v|^2, \qquad |v| \le 1.$$

Linearized operator (linearized around $v = 0$)

$$\mathcal{L}: \quad \partial_t v + (\nabla f(\varphi) v)_x = \partial_x^2 v.$$

**Remark:** One cannot expert strictly stability since $\lambda = 0$ is always an eigenvalue of $\mathcal{L}$ with $\varphi'$ the eigenfunction.



We illustrate the idea of proof:

$n = 1$ (scalar equation)

$$\frac{1}{2}\,\frac{d}{dt}\int_{-\infty}^{\infty} v^2\,dx + \int_{-\infty}^{\infty} v\;\partial_x(f'(\phi)v)dx = \int_{-\infty}^{\infty} v\;\partial_x^2\,v\;dx.$$

Therefore,

$$\frac{1}{2}\,\frac{d}{dt}\int_{-\infty}^{\infty} v^2(x,t) + \int_{-\infty}^{\infty} (\partial_x\,v)^2\,dx + \int_{-\infty}^{\infty} v\;\partial_x(f'(\phi)v)\;dx = 0$$

$$\begin{aligned}\int_{-\infty}^{\infty} v\;\partial_x(f'(\phi)v)\,dx &= -\int_{-\infty}^{\infty} \partial_x\,v\;f'(\phi)v\;dx \\ &= \int_{-\infty}^{\infty} \partial_x(f'(\phi))\frac{v^2}{2}\;dx\end{aligned}$$

so,

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} v^2(x,t)dx+\int_{-\infty}^{\infty}(\partial_x v(x,t))^2\,dx-\int_{-\infty}^{\infty}|\partial_x f'(\phi)|\,\frac{v^2}{2}\,dx=0$$

Set $v = w_x$,

$$\partial_t w_x + (\nabla f(\varphi) w_x)_x + (Q(\varphi, w_x))_x = \partial_x^2 w_x.$$

Integrating this equation from $-\infty$ to $x$,

$$\begin{cases} \partial_t w + \nabla f(\varphi) w_x + Q(\varphi, w_x) = \partial_x^2 w \\ w(x, t=0) = \int_{-\infty}^{x} v_0(y)dy = w_0(x) \end{cases}$$

This is called integrated error equation.

**Remark:** One needs $w_0(x) \in L^2(\mathbb{R}^1)$,

$$w(-\infty) = w(+\infty) = 0$$

This is equivalent to $\int_{-\infty}^{\infty} v_0(x) = 0.$

The key issue is basic energy estimate

$$||w(\cdot, t)||_{L^2} \leq c_0$$

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} w^2(x,t)dx + \int_{-\infty}^{\infty} w_x^2(x,t)dx + I_1 + I_2 = 0$$

$$
\begin{aligned}
I_1 &\equiv \int_{-\infty}^{\infty} w(\nabla f(\varphi) w_x) = \int_{-\infty}^{\infty} \frac{1}{2} \nabla f(\varphi)\, \partial_x\, w^2\, dx \\
&= -\frac{1}{2} \int_{-\infty}^{\infty} \partial_x(\nabla f(\varphi))\, w^2\, dx \\
&= \frac{1}{2} \int_{-\infty}^{\infty} |\partial_x(\nabla f(\varphi))| w^2\, dx
\end{aligned}
$$

$$
\begin{aligned}
|I_2| &= \left| \int_{-\infty}^{\infty} Q(\varphi, w_x) w\, dx \right| \leq \int_{-\infty}^{\infty} |w|\ |Q(\varphi, w_x)|\, dx \\
&\leq \int_{-\infty}^{\infty} O(1)\ |w|\ |w_x|^2\, dx
\end{aligned}
$$

if $O(1) \sup_{0\leq t\leq T} |w| \leq \frac{1}{2}$, then

$$|I_2| \leq \frac{1}{2}\int_{-\infty}^{\infty} |w_x|^2\, dx$$

so

$$\frac{d}{dt}\,||w(\cdot,t)||_{L^2}^2 + \int_{-\infty}^{\infty} w_x^2\, dx + \int_{-\infty}^{\infty} |\partial_x \nabla f(\varphi)|\, w^2 \leq 0$$

Thus,

$$\sup_{0\leq t\leq T}||w(\cdot,t)||_{L^2}^2 + \int_0^T ||w_x(\cdot,t)||^2\, dt + \int_0^T\int_{-\infty}^{\infty} |\partial_x (\nabla f(\varphi))|\, w^2 dxdt \leq ||w_0(x)||_{L^2}^2$$

Then the $L^2$-estimate for $w_x = v$, $w_{xx} = v_x$ can be done in a standard way.

## How about systems? i.e. $n > 1$

The main difficulties is how to control perturbations in other field. $\varphi$ is a $p$-shock profile,

$$\partial_x \lambda_p(\varphi) < 0 \quad \text{principal family,}$$

we don't know the sign of $\partial_x \lambda_i(\phi)$ for $i \neq p$, transversal family $\theta = L(\phi)w$, where $L$ consists of all left eigenvectors of $A(\phi) = \nabla f(\phi)$. Then

$$\partial_t \theta + \Lambda(\varphi) \partial_x \theta - \partial_x^2 \theta = -LA \frac{\partial R}{\partial x} \theta - LQ(\phi w_x) + 2L \frac{\partial R}{\partial x} \partial_x \theta + L \frac{\partial^2 R}{\partial x^2} \theta$$

$$\begin{array}{rcl} W(x) & = & \text{Diag}\,\{w_1(x), \cdots, w_n(x)\} \\ w_p(x) & \equiv & 1 \\ \forall\, i \neq p & & -\frac{1}{2}\,\partial_x(\lambda_i(\varphi)w_i) = c_i\, w_i\, |\partial_x\, \lambda_p(\varphi)| \end{array}$$

$c_i$ is a positive constant to be determined.

$$w_i(x) = \frac{\lambda_i(\varphi(0))}{\lambda_i(\varphi)}\, w_i(0)\ \exp\left\{ - \int_0^x 2\, c_i\, \frac{|\partial_x\, \lambda_p|}{\lambda_i(\varphi)}\, dx \right\}$$

choose $w_i(0) = 1$.

# §7.4 Nonlinear stability of rarefaction waves

$$\begin{cases} \partial_t u + \partial_x f(u) = \partial_x^2 u \\ u(x, t = 0) = u_0(x) \end{cases}$$

$$\lim_{x \to \pm\infty} u_0(x) = u_\pm$$

Assumption: $(u_-, u_+)$ can be connected by a $p$-centered rarefaction wave $u_R\left(\frac{x}{t}\right)$ whether $\exists\, u(x,t)$ such that $u_0 \sim u_R$, then

$$\lim_{t \to \infty} \sup \left| u(x,t) - u_R\left(\frac{x}{t}\right) \right| = 0$$

Answer is yes.

$$n = 1, \quad \text{convex}, \quad \partial_x \lambda_p(u_R) \geq 0$$

Energy estimate: $2 \times 2$ system, Compressible Navier-Stokes
General system: Szypessy & Zumbrun (ARMA)

## §7.5 Nonlinear stability for Contact discontinuity

1993, Xin

$$\partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 u$$

**Definition 7.1** $u_c$ is metastable if one can construct a viscous constant wave $u_c^\varepsilon(x, t)$

(1) $u_c^\varepsilon(x, t)$ is smooth.

(2) $||u_c^\varepsilon(x, t) - u_c(x, t)||_{L^p(\mathbb{R})} \leq c(\varepsilon t)^{\frac{1}{p}}$.

(3) $u_c^\varepsilon(x, t)$ is nonlinear stable.

1997, Liu, Xin