Section 4. Instrinoic Description of the Trajectory of a SRSG (Viscosity Solution)

Question: Assume (2.7) admits a SRSG. Let $u = u(x,t)$ be an arbitrary weak solution (function) to (2.7), such that $u(x, t = 0) = u_0(x)$. When $u(\cdot, t) = S_t\, u_0$?

Consider $u \in C([0,\infty), \mathcal{D})$. For fixed $\xi \in \mathbb{R}^1$, $\tau \geq 0$, set

$$\lim_{x\to\xi^\pm} u(x,\tau)\,\text{exists},\; u^- = \lim_{x\to\xi^-} u(x,\tau),\; u^+ = \lim_{x\to\xi^+} u(x,\tau) = u(\xi,\tau)$$

$$\begin{cases} \partial_t\, w + \partial_x\, f(w) = 0, & t > 0 \\ w(x, t=0) = \begin{cases} u^-, & x < 0 \\ u^+, & x > 0 \end{cases} \end{cases}$$

For fixed $(\xi,\tau) \in \mathbb{R}^1 \times \mathbb{R}^1_+$, we define

$$U^{\#}_{(u;\xi,\tau)}(x,t) = \begin{cases} w(x-\xi, t-\tau) & |x-\varepsilon| \leq \bar{\lambda}(t-\tau) \\ u(x,\tau) & \end{cases}$$

for $x \in \mathbb{R}^1$, $t > \tau$; also

$$U^{b}_{(u;\xi,t)}(x,t) = W(x,t;\ \xi,\tau)$$

with W solving

$$\begin{cases} \partial_t W + \tilde{A}\,\partial_x W = 0 \\ W(x,t=\tau) = u(x,\tau) \end{cases}$$

where $\tilde{A} = \nabla f(u(\xi,\tau))$.

Definition 4.1 (Viscosity Solution). Let $u : [0, T] \longrightarrow \mathcal{D}$ be continuous with respect to t in the L^1-norm. Then $u(x,t)$ is said to be a viscosity solution to (2.7) if $\exists\, C$ such that at each point $(\xi, \tau) \in \mathbb{R}^1 \times [0, T]$ and for every $\rho > 0$, $\varepsilon > 0$ sufficiently small, one has

$$(1) \qquad \frac{1}{\varepsilon} \int_{(\xi-\rho)+\bar{\lambda}\varepsilon}^{(\xi+\rho)-\bar{\lambda}\varepsilon} \Big| u(x, \tau+\varepsilon) - U^{\#}(x, \tau+\varepsilon) \Big| \, dx \\ \leq C \; T.V.\{u(\tau); (\xi-\rho, \xi) \cup (\xi, \xi+\rho)\}$$

$$(2) \qquad \frac{1}{\varepsilon} \int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} \Big| u(x, \tau+\varepsilon) - U^{b}(x, \tau+\varepsilon) \Big| \, dx \\ \leq C \left(T.V.\{u(\tau); (\xi-\rho, \xi+\rho)\}\right)^2$$

Theorem 4.1 Assume that (2.7) admits a SRSG S. Then a continuous map: $u : [0, T] \longrightarrow \mathcal{D}$ in L^1-norm is a viscosity solution to the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & (4.1) \\ u(x, t = 0) = u_0(x) & (4.2) \end{cases}$$

iff

$$u(\cdot, t) = S_t u_0$$

Proposition 4.1 All the trajectories of SRSG must be viscosity solutions, i.e. $\exists\, c > 0$ at any given point $(\xi, \tau) \in \mathbb{R}^1 \times \mathbb{R}^1_+$, and for sufficiently small ρ and ε, one has

$$\text{(V1)} \qquad \begin{aligned} & \frac{1}{\varepsilon} \int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} \left| S_{\tau+\varepsilon}\, \bar{u} - U^{\#}_{(u;\ \xi,\tau)}(x, \tau + s) \right| dx \\ \leq \quad & C\, T.V.\, \{u(\tau) = S_\tau\, \bar{u} :\ (\xi - \rho, \xi) \cup (\xi, \xi + \rho)\} \end{aligned}$$

$$\text{(V2)} \qquad \begin{aligned} & \frac{1}{\varepsilon} \int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} \left| S_{\tau+\varepsilon}\, \bar{u} - U^{b}_{(u;\ \xi,\tau)}(x, \tau + s) \right| dx \\ \leq \quad & C\, (T.V.\, \{u(\tau) = S_\tau\, \bar{u} :\ (\xi - \rho, \xi + \rho)\})^2 \end{aligned}$$

Proof of Proposition 4.1

Step 1: Verification of (V2).

Given $(\xi, \tau) \in \mathbb{R}^1 \times \mathbb{R}^1_+$, $\tau > 0$, $\rho > 0$, $\varepsilon > 0$.

Fix $\varepsilon' > 0$, we choose a piecewise constant function $\bar{\nu} \in \mathcal{D}$

(1) $\bar{\nu}(\xi) = u(\xi, \tau) = u(\xi+, \tau)$

(2) $\displaystyle\int_{\xi-\rho}^{\xi+\rho} |\bar{\nu}(x) - u(x, \tau)| dx \le \varepsilon'$

(3) $T.V.\bar{\nu} \le T.V.\, u(\cdot, \tau)$

Define $\nu(x,t)$ to be the solution to

$$\begin{cases} \partial_t \nu + \tilde{A}\,\partial_x \nu = 0, & \tilde{A} = \nabla f(u(\xi,\tau)), \quad t > \tau \\ \nu(x, t=\tau) = \bar{\nu}(x) \end{cases} \tag{4.3}$$

the solution to (4.3) can be written explicitly. Let the eigenvalues of $\tilde{A}$ be $\tilde{\lambda}_1, \cdots, \tilde{\lambda}_n$ with corresponding eigenvectors $\tilde{\gamma}_1, \cdots, \tilde{\gamma}_n$.

$$\nu(x,t) = \sum_{i=1}^{n} < \tilde{l}_i, \quad \bar{\nu}(x - \tilde{\lambda}_i(t-\tau)) > \tilde{\gamma}_i$$

$\tilde{l}_i$ is the left eigenvector of $\tilde{A}$, with normalization $\tilde{L}\tilde{R} = I$.
On the other hand, by the definition,

$$U^b_{(u;\xi,\tau)} = \sum_{i=1}^{n} < \tilde{l}_i, \qquad u\left(x - \tilde{\lambda}_i(t-\tau), \tau\right) > \tilde{\gamma}_i.$$

Thus,

$$
\begin{aligned}
& \int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} \left| \nu(x,\tau+\varepsilon) - U^{b}_{(u;\xi,\tau)}\,(x,\tau+\varepsilon) \right| dx \\
\leq \; & \int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} \sum_{i=1}^{n} \left| < \tilde{l}_i,\; \bar{\nu}\left(x - \tilde{\lambda}_i((\varepsilon+\tau)-\tau)\right) - u\left(x - \tilde{\lambda}_i\,\varepsilon, \tau\right) > \tilde{\gamma}_i \right| \\
\leq \; & O(1) \int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} \left| \bar{\nu}\left(x - \tilde{\lambda}_i\,\varepsilon\right) - u\left(x - \tilde{\lambda}_i\,\varepsilon, \tau\right) \right| dx \\
\leq \; & O(1) \int_{\xi-\rho}^{\xi+\rho} \left| \bar{\nu}(y) - u(y,\tau) \right| dy \\
\leq \; & O(1)\,\varepsilon'
\end{aligned}
$$

so,

$$
\begin{aligned}
& \frac{1}{\varepsilon} \int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} \left| S_{\tau+\varepsilon}\, \bar{u} - U^{b}_{(u;\xi,\tau)}\, (x, \tau+\varepsilon) \right| dx \\
\leq\ & \frac{1}{\varepsilon} \int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} \left\{ \left| S_{\varepsilon}\, u(\tau) - S_{\varepsilon}\, \bar{\nu} \right| + \left| S_{\varepsilon}\, \bar{\nu} - \nu(x, \tau+\varepsilon) \right| \right. \\
& \left. + \left| \nu(x, \tau+\varepsilon) - U^{b}_{(u;\xi,\tau)}\, (x, \tau+\varepsilon) \right| \right\} dx \\
\leq\ & \frac{1}{\varepsilon} \int_{\xi-\rho}^{\xi+\rho} \left| u(\tau) - \bar{\nu} \right| dx \\
& + \frac{1}{\varepsilon} \int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} \left| S_{\varepsilon}\, \bar{\nu} - \nu(x, \tau+\varepsilon) \right| dx + \frac{O(1)\, \varepsilon'}{\varepsilon}
\end{aligned}
$$

Since $\bar{\nu}$ is piecewise constant, and also $\nu(x,t)$ is also piecewise constant:

<u>Claim</u>:

$$
\begin{aligned}
& \frac{1}{\varepsilon}\int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} |S_\varepsilon\,\bar{\nu} - \nu(x,\tau+\varepsilon)|\,dx \\
\leq\ & \frac{L}{\varepsilon}\int_{\tau}^{\tau+\varepsilon} \overline{\lim}_{h\to 0+} \frac{\int_{\xi-\rho+\bar{\lambda}(h+t-\tau)}^{\xi+\rho-\bar{\lambda}(h+t-\tau)} |S_h\,\nu(t) - \nu(x,t+h)|dx}{h}\,dt \\
\leq\ & O(1)(T.V\,\{\bar{\nu} : (\xi-\rho,\ \xi+\rho)\})^2
\end{aligned}
$$

Indeed, since $\nu(x,t)$ is piecewise constant with finitely many polygonal lines as its discontinuities. Let $x = x_\alpha(t)$ is one of discontinuity. Then $\exists\, i$, such that

$$(\tilde{A} - \tilde{\lambda}_i I)(\nu(x_\alpha(t)+,t) - \nu(x_\alpha(t)-,t)) = 0$$

Applying Case III of Lemma 3.1,

$$\begin{aligned} & \frac{1}{h} \int_{\xi-\rho+\bar{\lambda}(h+t-\tau)}^{\xi+\rho-\bar{\lambda}(h+t-\tau)} |S_h\,\nu(t) - \nu(x,t+h)|\,dx \\ \leq\ & O(1) \sum_{\alpha=1}^{m} |\nu(x_\alpha(t)+,t) - \nu(x_\alpha(t)-,t)|\,(|\nu(x_\alpha(t)+,t) - \bar{\nu}(\xi)| \\ & +|\bar{\nu}(\xi) - \nu(x_\alpha(t)-,t)|) \end{aligned}$$

However, by construction,

$$|\nu(x_\alpha(t)+, t) - \bar{\nu}(\xi)| < T.V.\{\bar{\nu} : (\xi - \rho,\ \xi + \rho)\}$$

Step 2: Verification of (V1). In a similar way, for any given $(\xi, \tau) \in \mathbb{R}^1 \times \mathbb{R}^1_+$, $\tau > 0$ and ρ, ε sufficiently small.

Fix any given ε', choose $\bar{\nu}(x)$, piecewise constant, with finite many jumps such that

(1) $\bar{\nu}(\xi\pm) = u(\xi\pm, \tau)$

(2) $\displaystyle\int_{\xi-\rho}^{\xi+\rho} |\bar{\nu}(x) - u(x, \tau)| dx \leq \varepsilon'$

(3) $T.V.\{\bar{\nu} : (\xi - \rho, \xi) \cup (\xi, \xi + \rho)\} \leq$
$T.V.\{u(\tau); (\xi - \rho, \xi) \cup (\xi, \xi + \rho)\}$

Set $\nu(x,t)$ to be

$$\nu(x,t) = \begin{cases} \bar{\nu}(x) & |x-\xi| \geq \bar{\lambda}(t-\tau) \\ w(x-\xi, t-\tau) & |x-\xi| < \bar{\lambda}(t-\tau) \end{cases}$$

Set $I_\varepsilon = \{x| \quad \bar{\lambda}\,\varepsilon < |x-\xi| < \rho - \bar{\lambda}\,\varepsilon\}$.

Then by construction,

$$\begin{aligned} & \int_{\xi-\rho+\bar{\lambda}\,\varepsilon}^{\xi+\rho-\bar{\lambda}\,\varepsilon} \left|\nu(x,\tau+\varepsilon) - U^{\#}_{(u;\xi,\tau)}\,(x,\tau+\varepsilon)\right| dx \\ \leq \; & \int_{I_0} |\bar{\nu}(x) - u(x,\tau)| dx \\ \leq \; & \varepsilon' \end{aligned} \tag{4.4}$$

On the other hand,

$$
\begin{aligned}
&\int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} |S_\varepsilon \bar{\nu} - S_\varepsilon u(\tau)|\, dx \\
\leq\ & L \int_{\xi-\rho}^{\xi+\rho} |\bar{\nu} - u(\xi,\tau)| dx \qquad (4.5) \\
\leq\ & L\,\varepsilon'
\end{aligned}
$$

Now, apply Lemma 3.2,

<u>Claim</u>:

$$
\begin{aligned}
&\frac{1}{\varepsilon} \int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} |\nu(x,\tau+\varepsilon) - S_\varepsilon \bar{\nu}| dx \\
\leq\ & \frac{1}{\varepsilon} \int_{\tau}^{\tau+\varepsilon} \overline{\lim}_{h\to 0+} \frac{1}{h} \int_{\xi-\rho+\bar{\lambda}(t-\tau+h)}^{\xi+\rho-\bar{\lambda}(t-\tau+h)} |\nu(x,t+h) - S_h\,\nu(t)| dx\, dt \\
\leq\ & O(1)\, T.V.\{\bar{\nu} : (\xi-\rho,\xi) \cup (\xi,\xi+\rho)\}
\end{aligned}
$$

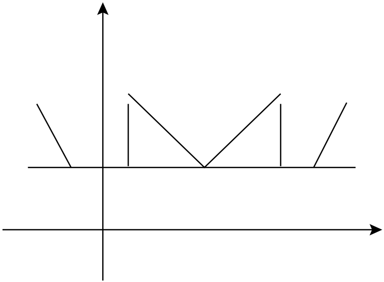
If the claim is verified, then

$$\begin{aligned} & \frac{1}{\varepsilon} \int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} \left| S_{\tau+\varepsilon}\,\bar{u} - U^{\#}_{(u;\xi,\tau)}(x,\tau+\varepsilon) \right| dx \\ \leq \;& \frac{1}{\varepsilon} \int_{\xi-\rho+\bar{\lambda}\varepsilon}^{\xi+\rho-\bar{\lambda}\varepsilon} \Big(\Big| \nu(x,\tau+\varepsilon) - U^{\#}_{(u;\xi,\tau)}(x,\tau+\varepsilon) \Big| \\ & + |\nu(x,\tau+\varepsilon) - S_\varepsilon\,\bar{\nu}| + |S_\varepsilon\,\bar{\nu} - u(x,\tau+\varepsilon)| \Big)\, dx \end{aligned}$$

Note that inside the cone: $|x-\xi| \leq \bar{\lambda}(t-\tau)$

$$\nu(x,\tau+h) - S_h\,\nu(\tau) = 0$$

But outside, $|x - \xi| = \bar{\lambda}|t - \tau|$.



$\nu(x, t + h)$ are piecewise constant with discontinuity at $x = x_\alpha(t)$ $(\alpha = 1, \cdots, m)$, $\dot{x}_\alpha = \lambda_\alpha$, and include the two lines $|x - \xi| = \bar{\lambda}(t - \tau)$.

But then, one can apply the Case 1 of Lemma 3.1,

$$\begin{aligned} &\int_{\xi-\rho+\bar{\lambda}(t-\tau+h)}^{\xi+\rho-\bar{\lambda}(t-\tau+h)} |\nu(x,t+h) - S_h\,\nu(t)|dx \\ \leq\ & O(1) \sum_{\alpha=1}^{m} |\nu(x_\alpha(t+h)+, t+h) - \nu(x_\alpha(t+h)-, t+h)| \end{aligned}$$

for $|x_\alpha(t) - \xi| > \bar{\lambda}(t-\tau)$

$$\leq O(1)\ T.V.\ \{\bar{\nu} : (\xi-\rho, \xi) \cup (\xi, \xi+\rho)\}$$

Conversely, we have the following proposition:

Proposition 4.2 Any viscosity solution $u(x,t)$ must be a trajectory of a SRSG. In fact, let $u \in [0,\infty) \times \mathcal{D} \longmapsto \mathcal{D}$ be continuous in L^1-norm, $u(0) = \bar{u}$. Assume that $\exists$ countable set $\mathcal{N} \subset \mathbb{R}^1_+$, $\mathcal{N} = \{\tau_1, \cdots\}$ such that for each $\tau \in \mathbb{R}^1_+ \setminus \mathcal{N}$ and $\xi \in \mathbb{R}^1$, $\exists$ a Radon measure μ_τ on $\mathbb{R}$ with the properties:

$$\text{(V1)} \qquad \frac{1}{\varepsilon}\int_{\xi-\rho+\bar{\lambda}\,\varepsilon}^{\xi+\rho-\bar{\lambda}\,\varepsilon} \left| u(x,\tau+\varepsilon) - U^{\#}_{(u;\xi,\tau)}(x,\tau+\varepsilon)\right| dx \leq \mu_\tau\{(\xi-\rho,\xi)\cup(\xi,\xi+\rho)\}$$

$$\text{(V2)} \qquad \frac{1}{\varepsilon}\int_{\xi-\rho+\bar{\lambda}\,\varepsilon}^{\xi+\rho-\bar{\lambda}\,\varepsilon} \left| u(x,\tau+\varepsilon) - U^{b}_{(u;\xi,\tau)}(x,\tau+\varepsilon)\right| dx \leq \left(\mu_\tau(\xi-\rho,\xi+\rho)\right)^2$$

for all but sufficiently small ρ and ε. Then

$$u(x,t) = (S_t\,\bar{u})(x)$$

i.e., u must be a trajectory of the SRSG.

Proof of Proposition 4.2

Step 1: For any given $\delta_0 \ll 1$, R sufficiently large and T fixed > 0, it holds that

$$\int_{-R+\bar{\lambda}\,T}^{R-\bar{\lambda}\,T} |S_{T-t}\,u(t) - S_T\,\bar{u}|dx \leq \delta_0 \left(t + \sum_{\tau_i < t} 2^{-i} \right), \; t \in [0, T] \; (\bigstar)$$

Let $A = \{t \in [0, T], \text{ such that } (\bigstar) \text{ holds}\}$.

(1) $A \neq \phi$, since $O \in A$.

(2) Define $\tau = \sup A$, then $\tau \in A$, since the left hand side of $(\bigstar)$ is continuous and the right hand side is lower semi-continuous.

(3) If $\tau = T$, then $T \in A$, thus

$$\int_{-R+\bar{\lambda}\,T}^{R-\bar{\lambda}\,T} |u(x,t) - S_T\,\bar{u}|dx \leq \delta_0 \left(T + \sum_{\tau_i < T} 2^{-i} \right).$$

We are done!

Step 2: $\tau = T$. If not, $\tau < T$.

Case 1: $\tau = \tau_i \in \mathcal{N}$ for some i.

Then by the continuity of SRSG, $\exists\, \delta > 0$ such that for $t \in [\tau, \tau + \delta]$, then

$$||S_{T-t}\, u(t) - S_T\, \bar{u}|| \leq ||S_{T-\tau}\, u(\tau) - S_T\, \bar{u}|| + \delta_0 \cdot 2^{-i}$$

Thus for $t \in [\tau_i, \tau_i + \delta]$,

$$\begin{aligned} ||S_{T-t}\, u(t) - S_T\, \bar{u}|| &\leq ||S_{T-\tau_i}\, u(\tau) - S_T\, \bar{u}|| + \delta_0 \cdot 2^{-i} \\ &\leq \delta_0 \left(\tau_i + \sum_{\tau_j < \tau_i} 2^{-j} \right) + \delta_0 \cdot 2^{-i} \\ &< \delta_0 \left(t + \sum_{\tau_j < t} 2^{-j} \right) \end{aligned}$$

Therefore, $t \in A$.

Step 3: $\tau = \max A \notin \mathcal{N}$, then by assumption, $\exists$ Radon measure on $\mathbb{R}$, such that (V1) and (V2) are satisfied. For convenience, we assume that

(1) $\mu_\tau((-R, R)) > 0 \quad$ (or $\mu((-R + \bar{\lambda}\, T, R - \bar{\lambda}\, T)) > 0$).

(2) $\mu_\tau(I) \geq T.V.\ \{u(\tau); I\}$ for all Borel sets I.

Now, we are going to show $\tau + \varepsilon \in A$ for some small $\varepsilon > 0$.

Step 3.1: Since $S_t\, u$ is a viscosity solution by Proposition 4.1, $\exists\, c > 0$ such that

$$\begin{aligned}
\frac{1}{\varepsilon} \int_{\xi-\rho+\bar{\lambda}\,\varepsilon}^{\xi+\rho-\bar{\lambda}\,\varepsilon} \left| S_\varepsilon\, u(\tau) - U^{\#}_{u;\xi,\tau}\,(\tau+\varepsilon) \right| dx &\leq c\, \mu((\xi-\rho, \xi) \cup (\xi, \xi+\rho)) \\
\frac{1}{\varepsilon} \int_{\xi-\rho+\bar{\lambda}\,\varepsilon}^{\xi+\rho-\bar{\lambda}\,\varepsilon} \left| S_\varepsilon\, u(\tau) - U^{b}_{u;\xi,\tau}\,(\tau+\varepsilon) \right| dx &\leq c(\mu(\xi-\rho, \xi+\rho))^2
\end{aligned}$$

for all ε, $\rho \ll 1$, where $\mu = \mu_\tau$.

Step 3.2: Since $\mu((-R,R)) < +\infty$, so the set

$$B = \left\{ \xi_i \in (-R,R),\ \mu(\xi_i) \geq \frac{\delta_0}{5\mu((-R,R))\ L(1+c)} \right\}$$

Then, B cannot be infinite, either $B = \phi$ or B is finite.

$$B = \{\xi_i,\ i = 1, \cdots, N\}, \qquad N < +\infty$$

Now for each $\xi_i \in B$, we choose ρ_i, ε_i such that

(i) $\mu((\xi_i - \rho_i, \xi_i) \cup (\xi_i, \xi_i + \rho_i)) < \dfrac{\delta_0}{2L(1+c)N}$,

(ii) $\dfrac{1}{\varepsilon} \displaystyle\int_{\xi_i - \rho_i + \bar{\lambda}\varepsilon}^{\xi_i + \rho_i - \bar{\lambda}\varepsilon} \left| u(\tau + \varepsilon) - U^{\#}_{u;\xi,\tau}(\tau + \varepsilon) \right| dx \leq$
$\mu((\xi_i - \rho_i, \xi_i) \cup (\xi_i, \xi_i + \rho_i))$,
$0 < \varepsilon < \varepsilon_i$,

(iii) $\frac{1}{\varepsilon}\int_{\xi_i-\rho_i+\bar{\lambda}\,\varepsilon}^{\xi_i+\rho_i-\bar{\lambda}\,\varepsilon} \left|S_\varepsilon\, u(\tau) - U^{\#}_{u;\xi,\tau}(\tau+\varepsilon)\right| dx \leq$
$c\mu(]\xi_i-\rho_i,\xi_i[\cup]\xi_i,\xi_i+\rho_i[), \quad 0<\varepsilon<\varepsilon_i,$

(iv) $(\xi_i-\rho_i,\xi_i+\rho_i)\cap(\xi_j-\rho_j,\xi_j+\rho_j)=\phi, \quad i\neq j.$

Now since $[-R+\bar{\lambda}\,\tau, R-\bar{\lambda}\,\tau]\setminus\left(\bigcup_{i=1}^{N} I_i\right)$ with $I_i=(\xi_i-\rho_i,\xi_i+\rho_i)$ is a compact set. Then

Claim: $\exists$ a family of open intervals $I'_j=(\xi'_j-\rho'_j,\xi'_j+\rho'_j)$, $1\leq j\leq M$, with the following properties:

(1) $\bigcup_{j=1}^{M} I'_j$ covers $[-R+\bar{\lambda}\,\tau, R-\bar{\lambda}\,\tau] \setminus \left(\bigcup_{j=1}^{N} I_j\right)$,

(2) $\mu(I'_j) < \dfrac{\delta_0}{4L(1+c)\ \mu((-R,R))}$,

(3) $I'_j \cap I'_m \cap I'_k = \phi$ if $j,\ m,\ k$ are all distinct,

(4) $\displaystyle \frac{1}{\varepsilon} \int_{\xi'_j-\rho'_j+\bar{\lambda}\,\varepsilon}^{\xi'_j+\rho'_j-\bar{\lambda}\,\varepsilon} \left| u(\tau+\varepsilon) - U^b_{u;\xi'_j,\tau}\,(\tau+\varepsilon)\right| dx \leq$
$(\mu((\xi'_j-\rho'_j, \xi'_j+\rho'_j)))^2,\ \ 0<\varepsilon<\varepsilon'_j,$

(5) $\displaystyle \frac{1}{\varepsilon} \int_{\xi'_j-\rho'_j+\bar{\lambda}\,\varepsilon}^{\xi'_j+\rho'_j-\bar{\lambda}\,\varepsilon} \left| S_\varepsilon\, u(\tau) - U^b_{u;\xi'_j,\tau}\,(\tau+\varepsilon)\right| dx \leq$
$c(\mu((\xi'_j-\rho'_j, \xi'_j+\rho'_j)))^2,\ \ 0<\varepsilon<\varepsilon'_j.$

Clearly, (3) $\Longrightarrow \sum_{j=1}^{m} \mu(I_j') \leq 2\mu((-R, R))$.

Now, we choose $\varepsilon^* > 0$ such that

(i) $0 < \varepsilon^* < \min\{\varepsilon_1, \cdots, \varepsilon_N;\ \varepsilon_1', \cdots, \varepsilon_M'\}$

(ii) $$\left(\bigcup_{i=1}^{N}(\xi_i - \rho_i + \bar{\lambda}\,\varepsilon^*,\ \xi_i + \rho_i - \bar{\lambda}\,\varepsilon^*)\right) \cup \left(\bigcup_{j=1}^{M}(\xi_j' - \rho_j' + \bar{\lambda}\,\varepsilon^*,\ \xi_j' + \rho_j' - \bar{\lambda}\,\varepsilon^*)\right) \supset (-R + \bar{\lambda}\,\tau, R - \bar{\lambda}\,\tau)$$

Step 3.3: Claim:

$$\int_{-R+\bar{\lambda}\,T}^{R-\bar{\lambda}\,T} \left| S_{T-(\tau+\varepsilon)}\, u(\tau+\varepsilon) - S_T\, \bar{u} \right| \leq \delta_0 \left(\tau + \varepsilon + \sum_{\tau_i < \tau+\varepsilon} 2^{-i} \right)$$

$$\begin{aligned}
& \int_{-R+\bar{\lambda}\,T}^{R-\bar{\lambda}\,T} |S_{T-(\tau+\varepsilon)}\, u(\tau+\varepsilon) - S_T\, \bar{u}|dx \\
\leq\ & \int_{-R+\bar{\lambda}\,T}^{R-\bar{\lambda}\,T} |S_{T-(\tau+\varepsilon)}\, u(\tau+\varepsilon) - S_{T-\tau}\, u(\tau)|dx \\
& + \int_{-R+\bar{\lambda}\,T}^{R-\bar{\lambda}\,T} |S_{T-\tau}\, u(\tau) - S_T\, \bar{u}|dx \\
=\ & \int_{-R+\bar{\lambda}\,T}^{R-\bar{\lambda}\,T} |S_{T-(\tau+\varepsilon)}\, u(\tau+\varepsilon) - S_{T-(\tau+\varepsilon)}\, S_\varepsilon\, u(\tau)|dx \\
& + \int_{-R+\bar{\lambda}\,T}^{R-\bar{\lambda}\,T} |S_{T-\tau}\, u(\tau) - S_T\, \bar{u}|dx \\
\leq\ & L \int_{-R+\bar{\lambda}(\tau+\varepsilon)}^{R-\bar{\lambda}(\tau+\varepsilon)} |u(\tau+\varepsilon) - S_\varepsilon\, u(\tau)|dx + \delta_0 \left(\tau + \sum_{\tau_i < \tau} 2^{-i} \right)
\end{aligned}$$

$$
\begin{aligned}
& \int_{-R+\bar\lambda(\tau+\varepsilon)}^{R-\bar\lambda(\tau+\varepsilon)} |u(\tau+\varepsilon) - S_\varepsilon\, u(\tau)| dx \\
< \;& \sum_{i=1}^{N} \int_{\xi_i-\rho_i+\bar\lambda\,\varepsilon}^{\xi_i+\rho_i-\bar\lambda\,\varepsilon} |u(\tau+\varepsilon) - S_\varepsilon\, u(\tau)| dx \\
& + \sum_{j=1}^{M} \int_{\xi'_j-\rho'_j+\bar\lambda\,\varepsilon}^{\xi'_j+\rho'_j-\bar\lambda\,\varepsilon} |u(\tau+\varepsilon) - S_\varepsilon\, u(\tau)| dx \\
\leq \;& \sum_{i=1}^{N} \int_{\xi_i-\rho_i+\bar\lambda\,\varepsilon}^{\xi_i+\rho_i-\bar\lambda\,\varepsilon} \left| u(\tau+\varepsilon) - U^{\#}_{u;\xi_i,\tau}(\tau+\varepsilon)\right| + \left| S_\varepsilon\, u(\tau) - U^{\#}_{u;\xi,\tau}(\tau+\varepsilon)\right| dx \\
& + \sum_{j=1}^{M} \int_{\xi'_j-\rho'_j+\bar\lambda\,\varepsilon}^{\xi'_j+\rho'_j-\bar\lambda\,\varepsilon} \left| u(\tau+\varepsilon) - U^{b}_{u;\xi'_j,\tau}(\tau+\varepsilon)\right| + \left| S_\varepsilon\, u(\tau) - U^{b}_{u;\xi'_j,\tau}(\tau+\varepsilon)\right| dx \\
\leq \;& \varepsilon \sum_{i=1}^{N} (1+c)\; \mu\; ((\xi_i+\rho_i, \xi_i) \cup (\xi_i, \xi_i+\rho_i)) + \varepsilon \sum_{j=1}^{M} (1+c)\; (\mu(I'_j))^2 \\
\leq \;& \varepsilon(1+c) \frac{\delta_0}{2L(1+c)N} N + \varepsilon(1+c) \frac{\delta_0}{4L(1+c)\;\mu\;((-R,R))} 2\mu((-R,R)) \\
= \;& \frac{\varepsilon\,\delta_0}{L}
\end{aligned}
$$

Thus we have shown

$$\int_{-R+\bar{\lambda}(\tau+\varepsilon)}^{R-\bar{\lambda}(\tau+\varepsilon)} |u(\tau+\varepsilon) - S_\varepsilon\, u(\tau)| dx \le \frac{\varepsilon\,\delta_0}{L},$$

so,

$$\int_{-R+\bar{\lambda}\,T}^{R-\bar{\lambda}\,T} |S_{T-(\tau+\varepsilon)}\, u(\tau+\varepsilon) - S_T\,\bar{u}| dx \le \delta_0 \left(\tau + \varepsilon + \sum_{\tau_i<\tau+\varepsilon} 2^{-i} \right).$$