Section 2. Uniqueness Theory

We study

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & x \in \mathbb{R}^1, \quad t > 0 \\ u(x, t = 0) = u_0(x) & \end{cases}$$

where $u(x,t) \in \mathbb{R}^n$, $f(u) \in \mathbb{R}^n$, $f \in C^3$, $u \in \Omega$.

$A(u) = \nabla_u f(u)$ is an $n \times n$ matrix.

Strictly hyperbolic: the eigenvalues of A are simple, and

$$\lambda_1(u) < \lambda_2(u) < \cdots < \lambda_n(u) \qquad \text{for} \qquad u \in \Omega. \tag{2.1}$$

The corresponding right and left eigenvectors are

$$\gamma_1(u), \cdots, \gamma_n(u); \quad l_1(u), \cdots, l_n(u).$$

More precisely, $A(u)\,\gamma_i(u) = \lambda_i(u)\,\gamma_i(u), \quad l_i(u)\,A(u) = \lambda_i(u)\,l_i(u).$

Denote $A(u, \tilde{u}) = \int_0^1 A(\theta u + (1-\theta)\tilde{u})d\theta$
and assume

$$\lambda_1(u, \tilde{u}) < \lambda_2(u, \tilde{u}) < \cdots < \lambda_n(u, \tilde{u}) \tag{2.2}$$

clearly $\lim_{\tilde{u} \to u} A(u, \tilde{u}) = A(u)$.

Corresponding right and left eigenvectors are

$$\gamma_1(u,\tilde{u}),\cdots,\gamma_n(u,\tilde{u})$$
$$l_1(u,\tilde{u}),\cdots,l_n(u,\tilde{u})$$

Assume

$$|\lambda_i(u,\tilde{u})| \leq \bar{\lambda} \qquad \forall\ u,\tilde{u}\in\Omega. \tag{2.3}$$

We will use the following notation

$$\gamma_i(u)\cdot\phi(u) = \nabla\phi(u)\cdot\gamma_i(u) = \lim_{\varepsilon\to 0}\frac{\phi(u+\varepsilon\gamma_i(u))-\phi(u)}{\varepsilon}. \tag{2.4}$$

Assumption

$$\text{either} \qquad \gamma_i(u) \cdot \lambda_i(u) \neq 0 \qquad \forall\ u \in \Omega \tag{2.5}$$
$$\text{or} \qquad \gamma_i(u) \cdot \lambda_i(u) \equiv 0 \qquad \forall\ u \in \Omega \tag{2.6}$$

In the case of (2.5), we normalize so that

$$\gamma_i(u) \cdot \lambda_i(u) > 0,\ \forall\ u \in \Omega \tag{2.5$'$}$$

Remark: If (2.5) is true, then the i-th characteristic field is genuinely nonlinear, and if (2.6) is true, then the i-th characteristic field is said to be linearly degenerate.

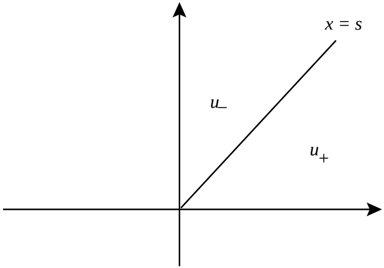
In gas dynamics, the two sound wave families are genuinely nonlinear, the entropy wave family is linearly degenerate.

Shock Wave: A triple (u_-, u_+, S) is called a p-shock, if R-H condition

$$s(u_- - u_+) = f(u_+) - f(u_-)$$

and the entropy condition (Lax condition)

$$\lambda_p(u_+) < s < \lambda_p(u_-)$$



Rarefaction Waves: Look for integral curve of the $\gamma_i(u)$, which is

$$\begin{cases} \dot{u} = \gamma_i(u) \\ u(0) = \bar{u} \end{cases}$$

Denote this integral curve as $\exp\{\sigma\gamma_i\}_{\bar{u}}$, σ is the parameter on the integral curve.

This will yield rarefaction wave solution, by choosing $\sigma = \frac{x}{t}$.

Riemann Solution:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & (2.7) \\ u_0(x) = \begin{cases} u_- & x < 0 \\ u_+ & x > 0 \end{cases} & (2.8) \end{cases}$$

Remark: (2.8) is dilation invariant

$$u_0(\alpha x) = u_0(x) \qquad \forall\ \alpha > 0, \quad x \neq 0.$$

The equation (2.7) is also dilation invariant, if $u(x,t)$ solves (2.7), then $u(\alpha x, \alpha t), \alpha > 0$ also solves (2.7).

If one has uniqueness, then the solution must be self-similar.

$$u(x,t) = U\left(\frac{x}{t}\right).$$

Lax Theory:

$$\text{If} \qquad |u_- - u_+| \ll 1, \quad \exists$$
$$u_- = u_0, u_1, u_2, \cdots, u_n = u_+$$

u_{i-1} is connected to u_i by i-elementary wave (either an i-shock, or an i-rarefaction wave, or an i-contact wave)

$$U^R = U^R\left(\frac{x}{t}\right)$$

Strength of the i-wave

$$\begin{array}{lll} \varepsilon_i = \lambda_i(u_i) - \lambda_i(u_{i-1}) & \text{if} & \gamma_i \cdot \lambda_i(u) \neq 0 \\ \varepsilon_i \quad \text{for} \quad u_i = \exp\{\varepsilon_i \gamma_i\}_{u_{i-1}} & \text{if} & \gamma_i \cdot \lambda_i(u) \equiv 0 \end{array}$$

Wave potential and interaction potential:

Definition 2.1: *Let* $u : \mathbb{R}^1 \to \Omega \in \mathbb{R}^n$ *is a piecewise constant function with bounded support having jupms at* $x_1, \cdots, x_N$.

Wave potential is defined to be

$$V(u) = \sum_{\alpha=1}^{N} \sum_{i=1}^{n} |\varepsilon_i^\alpha|$$

Interaction potential

$$Q(u) = \sum_{((\alpha,i),(\beta,j))\in\mathcal{A}} |\varepsilon_i^\alpha \, \varepsilon_j^\beta|$$

ε_i^α is the strength of the i-th elementary wave in Riemann solution corresponding to the data $(u(x_\alpha-), u(x_\alpha+))$.

$$\begin{aligned} & \mathcal{A} \text{ (approaching pairs)} \\ = \;& \{((\alpha, i), (\beta, j))_i \quad \text{either} \quad x_\alpha < x_\beta,\ i > j, \\ & \qquad\qquad\qquad\quad \text{or} \quad i = j,\ \gamma_i \cdot \lambda_i \neq 0,\ \min(\varepsilon_i^\alpha, \varepsilon_j^\beta) < 0 \Big\} \end{aligned}$$

Proposition 2.1: *(Existence of approximate solution by front tracking) There exist uniform constants $C_0, \delta_0 > 0$ such that $u_0 \in L^1(\mathbb{R}; \mathbb{R}^n)$ is a piecewise constant with*

$$V(u_0) + C_0\, Q(u_0) < \delta_0$$

Then $\forall\ \varepsilon > 0, \exists$ an approximate solution $u = u(x,t)$ to (2.7) such that

$$u(x, t = 0) = u_0(x) \qquad V(u(\cdot, t)) + C_0\, Q(u(\cdot, t)) < \delta.$$

(1) The function $u(x,t)$ is piecewise constant with discontinuities occurring along finitely many polygonal lines in the (x,t) plane.

There are two types of such lines:

Type I: $\{x = x_\alpha(t)\}_{\alpha=1,\cdots,N}$.
Type II: $\{x = y_\beta(t)\}_{\beta=1,\cdots,N'}$.

(2) Type I discontinuities: Except for finite many interaction times, for α, the values $u^- = u(x_\alpha^-, t)$, $u^+ = u(x_\alpha^+, t)$ are either connected by a shock, in which case, it holds true that

$$f(u^+) - f(u^-) = \dot{x}_\alpha(t)(u^+ - u^-)$$
$$\dot{x}_\alpha(t) = \lambda_{k_\alpha}(u^+, u^-) \qquad (\lambda_{k_\alpha}(u^+) < \lambda_{k_\alpha}(u^-), \quad \text{entropy condition})$$

or else, the two states u^-, u^+ lie on the same integral curve of a family of eigenvectors γ_{k_α}, so that

$$\dot{x}_\alpha(t) = \lambda_{k_\alpha}(u^+) \qquad u^+ = \exp\{\varepsilon_\alpha \gamma_{k_\alpha}\}u^-$$

for some ε_α. If the k_α-family is genuinely nonlinear, then, also $\varepsilon_\alpha \in [0, \varepsilon]$.

Type II: All lines $\{x = y_\beta(t)\}$ have same speed (constant)

$$\dot{y}_\beta(t) - \bar{\lambda}$$

The strength of all jumps occurring on these lines are uniformly small

$$\sum_{\beta=1}^{N'} |u(y_\beta^+, t) - u(y_\beta^-, t)| < \varepsilon$$

Proposition 2.2: *Let δ_0 and C_0 be the uniform constants in Proposition 2.1. Set*

$$\mathcal{D} = \text{closure } \left\{ \nu \in L^1(\mathbb{R}, \mathbb{R}^n), \ \nu \text{ is piecewise constants} \right\}$$
$$V(\nu) + C_0\, Q(\nu) < \delta_0$$

the closure is in $\mathcal{L}^1$-topology.

Then $\forall\ u_0 \in \mathcal{D}$, there exists a solution $u = u(x, t)$ to the Cauchy problem (2.7) - (2.8) such that $u(\cdot, t) \in \mathcal{D}, \ \forall\, t$. Furthermore, $u = u(x, t)$ can be obtained as a limit of approximate solutions constructed Proposition 2.1 by the front tracking method.

Remark: $\mathcal{D}$ is invariant for the solution operator if $u_0 \in \mathcal{D}$, then $u(\cdot, t) \in \mathcal{D}, \ \forall\, t \geq 0$.

Sketch of proof of Proposition 2.2

If $u_0 \in \mathcal{D}$, then one can find $u_0^\nu \in \mathcal{D}$, u_0^ν piecewise constant. Then Proposition 2.1 implies that $u^\nu(x, t)$ exists corresponding to the initial data u_0^ν. Therefore $u^\nu(\cdot, t) \in \mathcal{D}$, i.e.

$$V(u^\nu(\cdot, t)) + C_0\, Q(u^\nu(\cdot, t)) \leq \delta_0.$$

Then

$$T.V.u^\nu(\cdot, t) \leq \delta_0.$$

Helley principle implies $u^\nu(\cdot, t) \to u(\cdot, t)$ in L^1.

Moreover, $V(u(\cdot, t)) + C_0\, Q(u(\cdot, t)) \leq \delta_0$.

For the Cauchy problem

$$\begin{cases} u_t + f(u)_x = 0 & u \in \mathbb{R}^n \\ u(x, t = 0) = u_0(x) \end{cases}$$

$T.V.u_0 \leq \delta_0$.

We have two theories:

Theory 1 (Glimm theory). $\exists\ u_1(x,t)$ satisfies (2.7) in the sense of distribution, also it satisfies entropy condition. $u(x,t)$ is obtained by the Glimm's Random choice method.

Theory 2 (Front tracking method). $\exists\ u_2(x,t)$ to (2.7) and (2.8) generate by Proposition 2.2.

Is the Glimm solution unique? Does u_1 equal to u_2?