Suggested Solution to Assignment 6

Exercise 6.1

2. Note that in the spherical coordinates (r, θ, ϕ),

$$\Delta_3 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

Thus,

$$u_{rr} + \frac{2}{r}u_r = \Delta_3 u = k^2 u.$$

Let $u = v/r$, we get

$$u_r = \frac{v_r}{r} - \frac{v}{r^2}, \qquad u_{rr} = \frac{v_{rr}}{r} - \frac{2v_r}{r^2} + \frac{2v}{r^3}.$$

Hence, by the equation of u, $v_{rr} = k^2 v$, which implies $v(r) = Ae^{-kr} + Be^{kr}$, where A, B are constants. Therefore, $u(r) = A\frac{1}{r}e^{-kr} + B\frac{1}{r}e^{kr}$, where A, B are constants. □

4. We have known that $-c_1 r^{-1} + c_2$ is a solution, where c_1 and c_2 satisfy the equation:

$$-c_1 a^{-1} + c_2 = A, \qquad -c_1 b^{-1} + c_2 = B.$$

Hence,

$$u(x,t) = ab\frac{A-B}{b-a}r^{-1} + A + b\frac{B-A}{b-a}, \text{ where } r = \sqrt{x^2+y^2+z^2},$$

is a solution. Therefore, it is the unique solution by the Uniqueness Theorem of the Dirichlet problem for the Laplace's equation. □

6. Firstly, we find a solution depending only on r. Let $u(r)$, where $r = \sqrt{x^2+y^2}$, is a solution. As before, we have

$$u = \frac{1}{4}r^2 + c_1 \ln r + c_2, \text{ where } c_1,\ c_2 \text{ are constants.}$$

By the boundary conditions, we get

$$\frac{1}{4}a^2 + c_1\ln a + c_2 = 0, \qquad \frac{1}{4}b^2 + c_1 \ln b + c_2 = 0,$$

Hence,

$$u(x,y) = \frac{1}{4}(r^2 - a^2) - \frac{b^2 - a^2}{4(\ln b - \ln a)}(\ln r - \ln a),$$

is the unique solution by the Uniqueness Theorem. □

7. Firstly, we look for a solution depending only on $r = \sqrt{x^2+y^2+z^2}$. Let $u(r)$ be a solution, then as before,

$$u_{rr} + \frac{2}{r}u_r = 1,$$

from which we have

$$u = \frac{1}{6}r^2 + \frac{c_1}{r} + c_2, \text{ where} c_1,\ c_2 \text{ are constants.}$$

Thus, by the boundary conditions, we get

$$\frac{1}{6}a^2 + \frac{c_1}{a} + c_2 = 0, \qquad \frac{1}{6}b^2 + \frac{c_1}{b} + c_2 = 0.$$

Hence,

$$u(x,y) = \frac{1}{6}(r^2 - a^2) + ab\frac{a+b}{6}(\frac{1}{r} - \frac{1}{a}),$$

is the unique solution by the Uniqueness Theorem. □

9. (a) Firstly, we look for a solution depending only on $r = \sqrt{x^2+y^2+z^2}$. Let $u(r)$ be a solution, then as before,

$$u_{rr} + \frac{2}{r}u_r = 0,$$

from which we have $u_r = \frac{c}{r} + d$, where c, d are constants. Thus, by the boundary conditions, we have

$$c + d = 100, \qquad c = 4\gamma.$$

Therefore, $u = \frac{4\gamma}{r} + 100 - 4\gamma$. u is unique by the maximal principle.

(b) The hottest temperature is $100\,^\circ\mathrm{C}$, the coldest is $100 - 2\gamma$.

(c) By assumption, we have $100 - 2\gamma = 20$, therefore, $\gamma = 40$.

11. Integrating the equation $\Delta u = f$ and using the divergence theorem,

$$\iiint_D f \; dxdydz = \iiint_D \Delta u \; dxdydz = \iint_{\mathrm{bdy}(D)} \frac{\partial u}{\partial n} \; dS = \iint_{\mathrm{bdy}(D)} g \; dS.$$

Hence, there is no solution unless

$$\iiint_D f \; dxdydz = \iint_{\mathrm{bdy}(D)} g \; dS. \qquad \square$$

Exercise 6.2

1. By the boundary conditions, we can guess $u_x(x,y) = x - a$ and $u_y(x,y) = -y + b$. Luckily these also satisfy the equation. Hence,

$$u(x,y) = \frac{1}{2}x^2 - \frac{1}{2}y^2 - ax + by + c, \text{ where } c \text{ is any constant},$$

are solutions. Actually, we can prove that they are all solutions by the Hopf maximum principle. $\square$

2. Let $(m,n) \neq (m',n')$, then

$$\begin{aligned}&\int_0^\pi \int_0^\pi (\sin my \sin nz)(\sin m'y \sin n'z) dydz \\ =&(\int_0^\pi \sin my \sin m'y dy)(\int_0^\pi \sin nz \sin n'z dz) = 0,\end{aligned}$$

so the eigenfunctions $\{\sin my \sin nz\}$ are orthogonal on the squre $\{0 < y < \pi,\ 0 < z < \pi\}$. $\square$

3. Let $u(x,y) = X(x)Y(y)$, then

$$\frac{X''}{X} = -\frac{Y''}{Y} = \lambda, \qquad X(0) = Y'(0) = Y'(\pi) = 0.$$

Hence,

$$\lambda_n = n^2,\ Y_n(y) = \cos(ny),\ X_0 = x,\ X_{n+1} = \sinh[(n+1)x],\ n = 0,1,2,\ldots$$

Therefore,

$$u(x,y) = A_0 x + \sum_{n=1}^\infty A_n \sinh(nx)\cos(ny).$$

By the inhomogeneous boundary condition, we get

$$A_0\pi + \sum_{n=1}^{\infty} A_n \sinh(n\pi)\cos(ny) = \frac{1}{2}(1+\cos 2y),$$

which implies

$$A_0 = \frac{1}{2\pi},\ A_2 = \frac{1}{2\sinh(2\pi)},\ A_n = 0,\ \text{if } n \neq 0, 2.$$

Therefore,

$$u(x,y) = \frac{x}{2\pi} + \frac{1}{2\sinh(2\pi)}\sinh(2x)\cos(2y). \qquad \square$$

4. Let u_1 satisfies

$$\Delta u_1 = 0,\ \text{in the squre } \{0 < x < 1,\ 0 < y < 1\},$$
$$u_1(x,0) = x,\ u_1(x,1) = u_{1,x}(0,y) = u_{1,x}(1,y) = 0,$$

and u_2 satisfies

$$\Delta u_2 = 0,\ \text{in the squre } \{0 < x < 1,\ 0 < y < 1\},$$
$$u_2(x,0) = u_2(x,1) = u_{2,x}(0,y) = 0,\ u_{2,x}(1,y) = y^2,$$

then $u = u_1 + u_2$ is a harmonic function which we want to find.

By the method of separate variables,

$$u_1 = -\frac{A_0}{2}(y-1) + \sum_{n=1}^{\infty} A_n \cos(n\pi x)[\cosh(n\pi y) - \coth(n\pi)\sinh(n\pi y)],$$

where

$$A_0 = 1,\ A_n = 2\int_0^1 x\cos(n\pi x)dx = \frac{2}{n^2\pi^2}[(-1)^n - 1],\ n = 1, 2, \ldots.$$

And

$$u_2 = \sum_{n=1}^{\infty} B_n \cosh(n\pi x)\sin(n\pi y),$$

where

$$\begin{aligned} B_n &= \frac{2}{n\pi\sinh(n\pi)}\int_0^1 y^2\sin(n\pi y)dy \\ &= \frac{2}{\sinh(n\pi)}\left\{\frac{(-1)^{n+1}}{n^2\pi^2} + \frac{2}{n^4\pi^4}[(-1)^n - 1]\right\},\ n = 1, 2, \ldots. \end{aligned}$$

Therefore,

$$\begin{aligned} u = -\frac{1}{2}(y-1) &+ \sum_{n=1}^{\infty} A_n \cos(n\pi x)[\cosh(n\pi y) - \coth(n\pi)\sinh(n\pi y)] \\ &+ \sum_{n=1}^{\infty} B_n \cosh(n\pi x)\sin(n\pi y), \end{aligned}$$

where

$$A_n = \frac{2}{n^2\pi^2}[(-1)^n - 1],\ B_n = \frac{2}{\sinh(n\pi)}\left\{\frac{(-1)^{n+1}}{n^2\pi^2} + \frac{2}{n^4\pi^4}[(-1)^n - 1]\right\},\ n = 1, 2, \ldots. \qquad \square$$

6. Let $u(x,y,z) = X(x)Y(y)Z(z)$, then

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0, \quad X'(0) = X'(1) = Y'(0) = Y'(1) = Z'(0) = 0.$$

Hence,

$$X_m(x) = \cos(m\pi x),\ Y_n(y) = \cos(n\pi y),\ m,n = 0,1,2,\ldots,$$

and

$$Z'' = (m^2+n^2)\pi^2 Z,\ Z'(0) = 0.$$

Therefore,

$$\begin{aligned} u(x,y,z) =& \frac{1}{4}A_{00} + \frac{1}{2}\sum_{m=0}^{\infty} A_{m0}\cos(m\pi x)\cosh(m\pi z) + \frac{1}{2}\sum_{n=0}^{\infty} A_{0n}\cos(n\pi y)\cosh(n\pi z) \\ &+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\cos(m\pi x)\cos(n\pi y)\cosh(\sqrt{m^2+n^2}\pi z). \end{aligned}$$

Finally, by the inhomogeneous boundary condition, we get

$$\begin{aligned} g(x,y) =& \frac{1}{2}\sum_{m=0}^{\infty} A_{m0} m\pi\sinh(m\pi)\cos(m\pi x)\cosh(m\pi z) + \frac{1}{2}\sum_{n=0}^{\infty} A_{0n} n\pi\sinh(n\pi)\cos(n\pi y)\cosh(n\pi z) \\ &+ \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} A_{mn}\sqrt{m^2+n^2}\pi\sinh(\sqrt{m^2+n^2}\pi)\cos(m\pi x)\cos(n\pi y)\cosh(\sqrt{m^2+n^2}\pi z), \end{aligned}$$

which implies

$$A_{mn} = \frac{4}{\sqrt{m^2+n^2)\pi\sinh(\sqrt{m^2+n^2}\pi)}}\int_0^1\int_0^1 g(x,y)\cos(m\pi x)\cos(n\pi y)dxdy, \qquad m^2+n^2 \neq 0,$$

and A_{00} is any constant. We can prove that they are all solutions by the Hopf maximum. □

7(a). Let $u(x,y) = X(x)Y(y)$, then

$$\frac{X''}{X} + \frac{Y''}{Y} = 0, \quad X(0) = X(\pi) = 0.$$

Hence,

$$X_n(x) = \sin(n\pi),\ n = 1,2,\ldots,\ \text{and}\ Y'' = n^2 Y,\ \lim_{y\to 0} Y(y) = 0.$$

Thus,

$$u(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi)e^{-ny}.$$

Finally, by the inhomogeneous condition $h(x) = \sum_{n=1}^{\infty} A_n\sin(nx)$, we have

$$A_n = \frac{2}{\pi}\int_0^{\pi} h(x)\sin(nx)dx.$$

And the solution is

$$u(x,y) = \sum_{n=1}^{\infty}\frac{2}{\pi}\left(\int_0^{\pi} h(x)\sin(nx)dx\right)\sin(nx)e^{-ny}. \qquad \square$$

Exercise 6.3

1. (a) By the Maximum Principle,

$$\max_{\bar{D}} u = \max_{\partial D} u = \max_{\theta}(3\sin 2\theta + 1) = 4.$$

(b) By the Mean Value property,

$$u(0,0) = \frac{1}{2\pi}\int_0^{2\pi}(3\sin 2\theta + 1)d\theta = 1. \qquad \square$$

2. By the formula (10)-(12) in the textbook,

$$u(x,y) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} r^n(A_n \cos n\theta + B_n \sin n\theta),$$

where

$$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\theta)\cos n\theta d\theta, \quad B_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\theta)\sin n\theta d\theta.$$

Since $h(\theta) = 1 + 3\sin\theta$, we get

$$A_0 = 2,\ A_n = 0\ (n > 0),\ B_1 = \frac{3}{a},\ B_m = 0\ (m > 1).$$

Hence,

$$u(r,\theta) = 1 + \frac{3r}{a}\sin\theta.$$

3. As before, since

$$h(\theta) = \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta,$$

we get

$$A_n = 0\ (n \in \mathrm{N}),\ B_1 = \frac{3}{4a},\ B_3 = -\frac{1}{4a^3},\ B_m = 0\ (m \neq 1, 3).$$

Use the same way, the solution should be

$$u(r,\theta) = \frac{3}{4a}r\sin\theta - \frac{r^3}{4a^3}\sin 3\theta. \qquad \square$$

Problem 4. By Poisson's formula,

$$u(x,y) = u(r,\theta) = (1-r^2)\int_0^{2\pi}\frac{u(1,\phi)}{1-2r\cos(\theta-\phi)+r^2}\frac{d\phi}{2\pi} \leq \frac{1-r^2}{(1-r)^2}\int_0^{2\pi}u(1,\phi)\frac{d\phi}{2\pi} = \frac{1+r}{1-r}u(0,0),$$

since $u \geq 0, \cos(\theta-\phi) \leq 1$ and u has the Mean-Value Property. Similarly,

$$u(x,y) = u(r,\theta) = (1-r^2)\int_0^{2\pi}\frac{u(1,\phi)}{1-2r\cos(\theta-\phi)+r^2}\frac{d\phi}{2\pi} \geq \frac{1-r^2}{(1+r)^2}\int_0^{2\pi}u(1,\phi)\frac{d\phi}{2\pi} = \frac{1-r}{1+r}u(0,0),$$

since $u \geq 0, \cos(\theta-\phi) \geq -1$ and u has the Mean-Value Property.

Problem 5. (a)Use the Strong Maximum Principle.

(b)By Problem 4($r = 1/2$), $\frac{1}{3} = \frac{1-1/2}{1+1/2} \leq u(x,y) \leq \frac{1+1/2}{1-1/2} = 3$

Problem 6. Since u is a harmonic function in $B_1(0) \setminus (0,0)$, $u(x,y)$ is smooth in $B_1(0) \setminus (0,0)$. Define

$$v(x,y) = v(r,\theta) = \frac{1/4 - r^2}{2\pi} \int_0^{2\pi} \frac{h(\phi)}{1/4 - 2\cos(\theta - \phi) + r^2} d\phi, for\ r < 1/2,$$

where $h(\phi) = u(1/2, \phi)$, and $w := u - v$, then $v(x,y)$ is a harmonic function in $B_{1/2}(0)$, w is a harmonic function in $B_{1/2}(0) \backslash (0,0)$, $w = 0$ on $\partial B_{1/2}(0)$ and w is bounded in $B_{1/2}(0)$. Now, it suffices to show that $w \equiv 0$ in $B_{1/2}(0) \setminus (0,0)$:

For any fixed $(x_0, y_0) \in B_{1/2}(0) \setminus (0,0)$, $r_0 := \sqrt{x_0^2 + y_0^2}$. $\forall \varepsilon > 0$, define $v_\varepsilon(r) := -\varepsilon \log(2r)$, which is harmonic in $B_{1/2}(0) \setminus (0,0)$. Since $v_\varepsilon = 0$ on $\partial B_{1/2}(0)$ and $\lim_{r \to 0^+} v_\varepsilon = +\infty$, we can choose r_1 small enough such that $0 < r_1 < r_0$ and $v_\varepsilon(r) > \sup_{B_{1/2} \backslash (0,0)} w$ on $r = r_1$. Thus, by the Maximum Principle on $A := \{(x,y) : r_1 < \sqrt{x^2 + y^2} < 1/2\}$, we get $w(x_0, y_0) \leq -\varepsilon \log(2r_0)$. Let $\varepsilon \to 0^+$, we get $w(x_0, y_0) \leq 0$. Similarly, for $-w$ we get $-w(x_0, y_0) \leq 0$. Therefore, $w(x_0, y_0) = 0$.

Exercise 6.4

1. Since the only difference between the formulas of harmonic function in the interior and exterior of a disk is that r and a are replaced by r^{-1} and a^{-1}. Therefore, by the result in the exercise 6.4.2

$$u(r,\theta) = 1 + \frac{3a}{r} \sin\theta. \qquad \square$$

6. Using the separation of variables technique, we have

$$\Theta'' + \lambda\Theta = 0, \quad r^2 R'' + rR' - \lambda R = 0.$$

So the homogenous conditions lead to

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(0) = \Theta(\pi) = 0.$$

Hence,

$$\lambda_n = n^2,\ \Theta(\theta) = \sin n\theta,\ n = 1, 2, \ldots,$$

and then

$$R_n(r) = r^n,\ n = 1, 2, \ldots.$$

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^n \sin n\theta.$$

Finally, the inhomogeneous boundary condition requires that

$$\pi \sin\theta - \sin 2\theta = \sum_{n=1}^{\infty} A_n \sin n\theta,$$

which implies

$$A_1 = \pi,\ A_2 = -1,\ A_n = 0\ (n \neq 1, 2).$$

So the solution is

$$u(r,\theta) = \pi r \sin\theta - r^2 \sin 2\theta. \qquad \square$$

9. It is obvious that $u(r,\theta) = \theta$ is a solution. Hence, by the uniqueness theorem, $u(r,\theta) = \theta$ is the unique solution. $\square$

10. By the example 1 in the textbook Section 6.4 (Please do it again) and letting $\beta = \pi/2$, $h(\theta) = 1$, we have

$$u(r,\theta) = \sum_{n=1}^{\infty} A_n r^{2n} \sin 2n\theta,$$

where

$$A_n = a^{1-2n} \frac{2}{n\pi} \int_0^{\pi/2} \sin(2n\theta) d\theta = a^{1-2n} \frac{1}{n^2\pi}[1-(-1)^n].$$

The first two nonzero terms are

$$\frac{2}{a\pi} r^2 \sin 2\theta, \ \frac{2}{9a^5\pi} r^6 \sin 6\theta. \qquad \square$$

11. Multiplying u in the both sides of equation and using the divergence theorem,

$$\int_{\partial D} u \frac{\partial u}{\partial n} - \int_D |\nabla u|^2 = 0.$$

Using Robin boundary condition,

$$-a \int_{\partial D} u^2 - \int_D |\nabla u|^2 = 0,$$

which implies $\nabla u = 0$ in the D and $u = 0$ on ∂D since $a > 0$. So $u \equiv 0$ in D.

13. It is similiar to the Example 1 in Section 6.4 in the textbook. Here we only give the result and leave the details to you.

For the eigenvalue problem of $\Theta(\theta)$, we have

$$\lambda_n = (\frac{n\pi}{\beta-\alpha})^2, \quad \Theta_n(\theta) = \sin \frac{n\pi(\theta-\alpha)}{\beta-\alpha}, \ n = 1, 2, \ldots.$$

For the eigenvalue problem of $R(r)$, we have

$$R_n(r) = A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}, \ n = 1, 2, \ldots.$$

So the solution is

$$u(r,\theta) = \sum_{n=1}^{\infty} (A_n r^{\frac{n\pi}{\beta-\alpha}} + B_n r^{-\frac{n\pi}{\beta-\alpha}}) \sin \frac{n\pi(\theta-\alpha)}{\beta-\alpha}.$$

By setting $r = a$ and $r = b$ the coefficients A_n and B_n should satisfy

$$\begin{cases} A_n a^{\frac{n\pi}{\beta-\alpha}} + B_n a^{-\frac{n\pi}{\beta-\alpha}} = \dfrac{2}{\beta-\alpha} \displaystyle\int_\alpha^\beta g(\theta) \sin \frac{n\pi(\theta-\alpha)}{\beta-\alpha} d\theta \\ A_n b^{\frac{n\pi}{\beta-\alpha}} + B_n b^{-\frac{n\pi}{\beta-\alpha}} = \dfrac{2}{\beta-\alpha} \displaystyle\int_\alpha^\beta h(\theta) \sin \frac{n\pi(\theta-\alpha)}{\beta-\alpha} d\theta \end{cases},$$

then

$$\begin{cases} A_n = \dfrac{Aa^{\frac{n\pi}{\beta-\alpha}} - Bb^{\frac{n\pi}{\beta-\alpha}}}{a^{2\frac{n\pi}{\beta-\alpha}} - b^{2\frac{n\pi}{\beta-\alpha}}} \\ B_n = \dfrac{Aa^{-\frac{n\pi}{\beta-\alpha}} - Bb^{-\frac{n\pi}{\beta-\alpha}}}{a^{-2\frac{n\pi}{\beta-\alpha}} - b^{-2\frac{n\pi}{\beta-\alpha}}} \end{cases},$$

where

$$A = \frac{2}{\beta-\alpha} \int_\alpha^\beta g(\theta) \sin \frac{n\pi(\theta-\alpha)}{\beta-\alpha} d\theta, \ B = \frac{2}{\beta-\alpha} \int_\alpha^\beta h(\theta) \sin \frac{n\pi(\theta-\alpha)}{\beta-\alpha} d\theta. \qquad \square$$

Problem 7. Write $u(r,\theta) = R(r)\Theta(\theta)$, then $r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta'}{\Theta} =: \lambda$ and $\Theta'(0) = 0 = \Theta(\pi)$,$\Rightarrow \lambda = \frac{\int_0^\pi |\Theta'|^2}{\int_0^\pi \Theta^2} > 0$ since u is nontrivial. $\Rightarrow$ We can write $\lambda = \beta^2,\ \beta > 0$ and $\Theta'' + \beta^2\Theta = 0$,

$\Rightarrow \beta_n = 1/2 + n$, $\Theta_n = \cos(\beta_n\theta)$, $n = 0, 1, 2, ...$

$\Rightarrow u(r,\theta) = \sum_{n=0}^{\infty}(c_n r^{\beta_n} + d_n r^{-\beta_n})\cos(\beta_n\theta)$

By the boundary conditions, $u(1,\theta) = \cos^3(\theta/2) = 1/4\cos(3\theta/2) + 3/4\cos(\theta/2)$ and $u(2,\theta) = 4\cos(5\theta/2)$

$\Rightarrow c_1 + d_1 = 1/4;\ c_0 + d_0 = 3/4;\ c_n + d_n = 0\ if\ n \neq 0, 1;\ c_2 2^{\beta_2} + d_2 2^{-\beta_2} = 4;\ c_n 2^{\beta_n} + d_n 2^{-\beta_n} = 0\ if\ n \neq 2$

$\Rightarrow c_n = d_n = 0\ if n \neq 0, 1, 2;\ c_0 = -3/4,\ d_0 = 3/2,\ c_1 = -1/28,\ d_1 = 2/7,\ c_2 = \frac{16\sqrt{2}}{31} = -d_2.$