# Tutorial 4

Feb 18, 2016

1. Using reflection method to solve the following problem

$$\partial_t^2 u - c^2 \partial_x^2 u = 0, \qquad x > 0, t > 0$$
$$u(x, t = 0) = \phi(x), \partial_t u(x, t = 0) = \psi(x), x > 0$$
$$\partial_x u(x = 0, t) = 0, t > 0$$

**Solution:** Use the reflection method, and first consider the following Cauchy Problem:

$$\partial_t^2 v - c^2 \partial_x^2 v = 0, \qquad -\infty < x < \infty, t > 0$$
$$v(x, t = 0) = \phi_{even}(x), \partial_t v(x, t = 0) = \psi_{even}(x), -\infty < x < \infty$$

where $\phi_{even}(x)$ and $\psi_{even}(x)$ are even extension of $\phi$ and $\psi$.Then the unique solution is given by d'Alembert formula:

$$v(x, t) = \frac{1}{2}[\phi_{even}(x + ct) + \phi_{even}(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)dy$$

And since $\phi_{even}(x)$ and $\psi_{even}(x)$ are even, so is $v(x, t)$ for $t > 0$,which implies

$$\partial_x v(x = 0, t) = 0, t > 0$$

Set $u(x, t) = v(x, t), x > 0$ ,then $u(x, t)$ is the unique solution of Neumann Problem on the half-line.More presicely, if $x > ct$,

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy$$

if $0 < x < ct$,

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(ct - x)] + \frac{1}{2c}\Big\{\int_0^{ct-x} \psi(y)dy + \int_0^{x+ct} \psi(y)dy\Big\}.$$

2. Example 1 on P57
Solve

$$\partial_t v - k\partial_x^2 v = 0, x > 0, t > 0$$
$$v(x, t = 0) = \phi(x) = 1, x > 0$$
$$v(x = 0, t) = 0, t > 0$$

**Solution:**By the solution fomula, we have

$$v(x, t) = \frac{1}{\sqrt{4\pi kt}}\int_0^{\infty} e^{-\frac{(x-y)^2}{4kt}} - e^{-\frac{(x+y)^2}{4kt}} dy$$
$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-p^2}dp - \frac{1}{\sqrt{\pi}}\int_{\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-q^2}dq$$
$$= [\frac{1}{2} + \frac{1}{2}\mathcal{E}rf(\frac{x}{\sqrt{4kt}})] - [\frac{1}{2} - \frac{1}{2}\mathcal{E}rf(\frac{x}{\sqrt{4kt}})]$$
$$= \mathcal{E}rf(\frac{x}{\sqrt{4kt}})$$

3. Example 2 on P58
   Solve

$$\partial_t v - k\partial_x^2 v = 0, x > 0, t > 0$$

$$v(x, t = 0) = \phi(x) = 1, x > 0$$

$$\partial_x v(x = 0, t) = 0, t > 0$$

**Solution:**By the solution fomula, we have

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty e^{-\frac{(x-y)^2}{4kt}} + e^{-\frac{(x+y)^2}{4kt}} dy$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\frac{x}{\sqrt{4kt}}} e^{-p^2} dp + \frac{1}{\sqrt{\pi}} \int_{\frac{x}{\sqrt{4kt}}}^{+\infty} e^{-q^2} dq$$

$$= [\frac{1}{2} + \frac{1}{2}\mathcal{E}rf(\frac{x}{\sqrt{4kt}})] + [\frac{1}{2} - \frac{1}{2}\mathcal{E}rf(\frac{x}{\sqrt{4kt}})] = 1$$