Tutorial2

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1. **Exercise 4 on P27**

   Consider the Neumann problem

$$\Delta u = f(x, y, z) \quad \text{in } D$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on bdy } D$$

   (a) What can we surely add to any solution to get another solution? So we don't have uniqueness.

   (b) Use the divergence theorem and the PDE to show that

$$\iiint_D f(x, y, z) dx\ dy\ dz = 0$$

   is a necessary condition for the Neumann problem to have a solution.

   (c) Can you give a physical interpretation of part (a) and/or (b) for either heat flow or diffusion?

   Solution:

   (a) Adding a constant C to a solution will give another solution, so we do not have uniqueness if there is a solution;

   (b) Integrating $f(x, y, z)$ on D and using the divergence theorem, we obtain

$$\iiint_D f(x, y, z) dxdydz = \iiint_D \Delta u dxdydz = \iiint_D \nabla \cdot \nabla u dxdydz = \iint_{\partial D} \nabla u \cdot n dS = 0$$

   (c) For the heat flow, the equation which is independent of time t shows that the temperature u of the object reaches a steady state when there is an heat source or sink $f(x, y, z)$. At the same time, the Nuemann boundary condition means that the object is insulated,thus there is no heat flows in or out across the boundary. The part (b) shows that in order to make the PDE and the boundary condition hold simultaneously, we need $\iiint_D f(x, y, z)dxdydz = 0$, that is, the total heat source or sink on the domain D should be 0 since the heat is steady and no heat flows in and out across the boundary.(otherwise it won't be steady if $\iiint_D f(x, y, z)dxdydz \neq 0$). The part (a) means that if a given heat distribution is a steady state then rising or lowering the heat uniformly at every point is also a possible steady heat distribution. The difference between the steady state u and $u + C$ is that they have the different total heat energy.

   For diffusion, the eqution means that the concentration u of the substance reaches a steady state when there is an external source or sink of the substance $f(x, y, z)$. The Nuemann condition means that the container is impermeable, i.e, no substance escape or enter across the boundary. The part (b) shows that in order to make the PDE and the boundary condition hold simultaneously, we need $\iiint_D f(x,y,z)dxdydz = 0$, that is, the total external source or sink of the substance should be 0 since the state of the substance is steady and no substance escape or enter across the boundary.(otherwise it won't be steady if $\iiint_D f(x,y,z)dxdydz \neq 0$). The part (a) means that if a given concentration distribution is a steady state then rising or lowering the concentration uniformly at every point is also a possible steady concentration distribution. The difference between the steady state u and $u+C$ is that they have the different total mass.

2. **Example 2 on P35:The Plucked String with $c = 1$**
   For a vibrating string with the speed $c = 1$,consider an infinitely long string which satisfies the wave eqution:
   $$\partial_t^2 u - \partial_x^2 u = 0 \qquad -\infty < x < \infty$$
   Soppose that the the initial position is
   $$\phi(x) = \begin{cases} b - \dfrac{b|x|}{a} & |x| < a \\ 0 & |x| > a \end{cases}$$
   and the initial velocity $\psi(x) = 0$ for all x.

   The solution of this initial condition problem by d'Alembert Formula is
   $$u(x,t) = \frac{1}{2}[\phi(x+t) + \phi(x-t)]$$
   See the Figure 2 on Page 36.

   The effect of the initial position $\phi(x)$ is a pair of waves travelling in either direction, one to the left $\frac{1}{2}\phi(x+t)$ and another to the right $\frac{1}{2}\phi(x-t)$, at the speed $c = 1$ and at half the original amplitude $\frac{d}{2}$. You can see this phenomenen by the graphs on page 36 clearly.

   Remark: The above example shows the effect of the initial position $\phi(x)$, and the effect of the initial velocity can be observed by Exercise 5 in Section 2.1 on page 37.

3. **Discussion of Huygens's Principle**
   Causality Principle: The solution of the wave equation travels no more than the speed c.
   Huygens's Principle: The wave travels at the speed c exactly.

   $N = 2$,for instance,when you drop a pebble onto a clam pond, surface waves are created which satisfy 2-dimentional wave equation with speed c. A water bug whose distance from the point of impact is δ experiences a wave first at time $t = \delta/c$ and then continues to feel ripples.Thus some wave travels with speed c, some travels slower.

   $N = 3$, for example, the piano just plays a note at $t = 0$ and the listener hear the note at $t = d/c$ where d is the distance between the piano and the listener,and c is the sound speed in air.And then,$t > d/c$, the listener can not hear any sound. Thus the sound wave travels at speed c exactly.

   From the above physical phenomenen, Huygens's Principle is valid in 3 dimention, but not 2 dimention. And next we show that this fact can be derived by the solution of the wave equation directly.

   Consider the following wave equation:
   $$\partial_t^2 u = c^2 \Delta u$$
   $$u(\mathbf{x}, t) = \phi(\mathbf{x}), \quad \partial_t u(\mathbf{x}, t) = \psi(\mathbf{x})$$

When $n = 1$, the solution is given by d'Alembert formula.
When $n = 2$, the solution is given by Poisson's fromula:(Chaper9,P227)

$$u(x_0, y_0, t_0) = \iint_D \frac{\psi(x, y)}{[c^2t_0^2 - (x - x_0)^2 - (y - y_0)^2]^{1/2}} \frac{dxdy}{2\pi c}$$

$$+\frac{\partial}{\partial t_0} \iint_D \frac{\phi(x, y)}{[c^2t_0^2 - (x - x_0)^2 - (y - y_0)^2]^{1/2}} \frac{dxdy}{2\pi c}$$

where D is the disk $\{(x - x_0)^2 + (y - y_0)^2 \leq c^2t_0^2\}$.
When $n = 3$, the solution is given by Kirchhoff's formula:(Chapter9.P222)

$$u(x_0, y_0, z_0, t_0) = \frac{1}{4\pi c^2 t_0} \iint_S \psi(x, y, z)dS + \frac{\partial}{\partial t_0}[\frac{1}{4\pi c^2 t_0} \iint_S \phi(x, y, z)dS]$$

where S is the sphere $\{(x - x)^2 + (y - y_0)^2 + (z - z_0)^2 = c^2t_0^2\}$.
According to above formulas, if $n = 2$, the value $u(x_0, y_0, t_0)$ depends on the value of ψ and ϕ on the disk D, thus the domain of the infulence of the point (x_1, y_1, t) is a disk $\{(x-x_1)^2+(y-y_1)^2 \leq c^2t^2\}$, which means that some part of the wave travels at speed c and some part lags behind.If $n = 3$, the value $u(x_0, y_0, z_0, t_0)$ depends only on the value of ψ and ϕ on the sphere S, thus the domain of the infulence of the point (x_1, y_1, z_1, t) is a sphere $\{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2 = c^2t^2\}$, which means that the wave travels at speed c exactly. Thus Huygens's Principle holds in 3 dimention, but not 2 dimention.

Remark: Here just provides a understanding of the Huygens's Principle and its relationship with the wave equation.You can find more information in chapter 9 or any other reference book where you can also find the proof of the Poisson's formula and Kirchhoff's formula.