

Tutorial 0: Prerequisite

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1. Derivatives, partial derivatives, directional derivatives

Suppose $f(x)$ is a function of one variable x , we say that $f(x)$ is differentiable at a point $x = a$ provided that the limit of $\frac{f(x)-f(a)}{x-a}$ exists as $x \rightarrow a$. The value of the limit is denoted by $f'(a)$ or $\frac{df}{dx}(a)$.

For the two variables function or the multiple variables function, we introduce the partial derivatives and directional derivatives.

For example, suppose $f(x, y)$ is a function of two variables x and y . We define the partial derivatives $\frac{\partial f}{\partial x}$ by $\lim_{x \rightarrow a} \frac{f(x, y) - f(a, y)}{x - a}$.

Suppose $f(x)$ is a function, $x \in \mathbb{R}^n$, the directional derivatives of $f(x)$ at a point $a \in \mathbb{R}^n$ in the direction of the vector v is $\lim_{t \rightarrow 0} \frac{f(a+tv) - f(a)}{t} = v \cdot \nabla f(a)$.

Remark: Derivative is local, since we define the derivative by limit.

2. Mixed derivatives are equal: If a function $f(x, y)$ is of class C^2 , then $\partial_{xy}u = \partial_{yx}u$.

Remark: The same true for derivatives of any order provided these derivatives are continuous. Throughout this course, we assume, unless stated otherwise, that all the derivatives exist and are continuous.

3. Chain rule.

The Chain Rule deals with functions of functions.

For example, consider the chain $s, t \mapsto x, y \mapsto u$. Suppose u is a function of x, y of class C^1 , and x, y are differential functions of s, t , then

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial t}$$

$$\frac{\partial u}{\partial s} = \frac{\partial u}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial u}{\partial y} \frac{\partial y}{\partial s}$$

4. Integrals of Derivatives.

For one variable, we have fundamental theorem of calculus. Then if f is differential in $[a, b]$, we have

$$f(b) - f(a) = \int_a^b f'(x) dx$$

For two variables, we introduce the Green's Formula. For high variables, Gauss Formula or Divergence Theorem.

Green's Formula.

Let D be a bounded plane domain with a piecewise C^1 boundary curve $C = \text{bdy}D$. Consider C to be parametrized so that it is traversed once with D on the left. Let $p(x, y)$ and $q(x, y)$ be any C^1 functions defined on $\bar{D} = D \cup C$. Then

$$\iint_D (q_x - p_y) dx dy = \int_C p dx + q dy.$$

Divergence Theorem:

Let D be a bounded spatial domain with a piecewise C^1 boundary surface S . Let \vec{n} be the unit outward normal vector on S . Let $f(x)$ be any C^1 vector field on $\bar{D} = D \cup S$. Then

$$\iiint_D \nabla \cdot f dx = \iint_S f \cdot \vec{n}.$$

5. Derivatives of integrals.

Thm 1 Suppose that a and b are constants. If both $f(x, t)$ and $\partial f / \partial t$ are continuous in the rectangle $[a, b] \times [c, d]$, then

$$\frac{d}{dt} \int_a^b f(x, t) dx = \int_a^b \frac{\partial f}{\partial t}(x, t) dx$$

for $t \in [c, d]$.

Thm 2 Let $f(x, t)$ and $\partial f / \partial t(x, t)$ be continuous functions in $(-\infty, \infty) \times (c, d)$. Assume that the integrals $\int_{-\infty}^{\infty} |f(x, t)| dx$ and $\int_{-\infty}^{\infty} |\partial f / \partial t| dx$ converge uniformly (as improper integrals) for $t \in (c, d)$. Then

$$\frac{d}{dt} \int_{-\infty}^{\infty} f(x, t) dx = \int_{-\infty}^{\infty} \frac{\partial f}{\partial t}(x, t) dx$$

for $t \in (c, d)$.

Thm 3 If $I(t)$ is defined by $I(t) = \int_{a(t)}^{b(t)} f(x, t) dx$, where $f(x, t)$ and $\partial f / \partial t$ are continuous on the rectangle $[A, B] \times [c, d]$, where $[A, B]$ contains the unions of all intervals $[a(t), b(t)]$, and if $a(t)$ and $b(t)$ is differentiable on $[c, d]$, then

$$\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} f(x, t) dx + f(b(t), t)b'(t) - f(a(t), t)a'(t)$$

Remark: For the two or three variables we have the similar theorems.

6. Change of coordinates formula.

If the transformation $x = g(x', y')$ and $y = h(x', y')$ carries the domain D' onto the domain D in a one-to-one manner and is of class C^1 , and if $f(x, y)$ is a continuous function defined on \overline{D} , then

$$\iint_D f(x, y) dx dy = \iint_{D'} f(g(x', y'), h(x', y')) \cdot |J(x', y')| dx' dy'$$

where J is the Jacobian determinant. The size of the jacobian factor $|J|$ expresses the amount that areas are stretched or shrunk by the transformation.

7. Infinite series of functions and their differentiation.

Consider $\sum_{n=1}^{\infty} f_n(x)$, where $f_n(x)$ could be any functions.

We say that this infinite series converges to $f(x)$ pointwise in an interval (a, b) if it converges to $f(x)$ (as a series of numbers) for each of $a < x < b$.

We say that the series converges uniformly to $f(x)$ in $[a, b]$ if

$$\max_{a \leq x \leq b} |f(x) - \sum_{n=1}^N f_n(x)| \rightarrow 0, \quad \text{as } N \rightarrow \infty.$$

Comparison Test:

If $|f_n(x)| \leq c_n$ for all n and for all $a \leq x \leq b$, where c_n are constants, and if $\sum_{n=1}^{\infty} c_n$ converges, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly in the interval $[a, b]$, as well as absolutely.

Convergence Theorem: (term by term integration)

If $\sum_{n=1}^{\infty} f_n(x) = f(x)$ is uniformly in $[a, b]$ and $f_n(x)$ is continuous in $[a, b]$, then the sum $f(x)$ is also continuous in $[a, b]$, and

$$\sum_{n=1}^{\infty} \int_a^b f_n(x) dx = \int_a^b f(x) dx$$

Convergence of Derivatives.

If all the functions $f_n(x)$ is differentiable in $[a, b]$ and if the series $\sum_{n=1}^{\infty} f_n(c)$ converges for some c , and if the series of derivatives $\sum_{n=1}^{\infty} f'_n(x)$ converges uniformly in $[a, b]$, then $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly to a function $f(x)$ and

$$\sum_{n=1}^{\infty} f'_n(x) = f'(x)$$

8. ODE

First order ODE: $\frac{dy}{dx} = f(t, y)$.

First order linear equation:

$$\frac{dy}{dt} + p(t)y = q(t)$$

where $p(t)$ and $q(t)$ are given functions. By multiplying both sides of the equation with an integrating factor $\mu(x) = e^{\int p(t)dt}$, we arrive

$$\frac{d}{dt}[\mu(t)y] = q(t)\mu(t)$$

thus the general solution is

$$y = e^{-\int p(t)dt} \left\{ \int q(t)e^{\int p(t)dt} dt + C \right\}.$$

where C is an arbitrary constant. f Seperable Equations:

$$M(x)dx + N(y)dy = 0$$

where $M(x)$ and $N(y)$ are given function. Let H_1 and H_2 are the antiderivatives of M and N respectively. Rewrite the equation as

$$H_1'(x) + H_2'(y) \frac{dy}{dx} = 0$$

Thus the general solution is

$$H_1(x) + H_2(y) = C$$

where C is an arbitrary constant.

Exact Equations:

$$M(x, y) + N(x, y)y' = 0$$

where $M(x, y)$ and $N(x, y)$ are given functions.

If the equation is exact, $M_y = N_x$, that is, there exists a function $\psi(x, y)$ such that

$$\frac{\partial \psi}{\partial x}(x, y) = M(x, y), \frac{\partial \psi}{\partial y}(x, y) = N(x, y)$$

and such that $\psi(x, y) = C$ defines $y = \phi(x)$ implicitly as a differentiation function of x , thus the above ODE turns to $\frac{d}{dx}\psi(x, \phi(x)) = 0$, hence the general solution is $\psi(x, y) = C$ where C is an arbitrary constant.

If the equation is not exact, multiply the equation by an undetermined integrating factors $\mu(x, y)$ such that $\mu(x, y)M(x, y) + \mu(x, y)N(x, y)y' = 0$ is exact, i.e., $(\mu M)_y = (\mu N)_x$, and then solve the exact equation to get the general solution.

Second order ODE (To be continued next time)

Reference book of ODE: Elementary Differential Equations and Boundary Value Problems (William E. Boyce, Richard C. DiPrima)