Chapter 2. General Scalar Conservation Laws

§2.1 Convex Conservation Laws

Consider the general conservation laws

$$\partial_t u + \partial_x f(u) = 0 \tag{2.1}$$

where $f \in C^2$. In chapter 1 we set $f(u) = \frac{1}{2} u^2$ in the case of Burgers equation. We say (2.1) is a convex conservation law if $f''(u) > 0$ for all u in the consideration.

Note that not all the stationary conservation laws is convex. For example, $\partial_t u + \partial_x \left(\frac{u^3}{3}\right) = 0$ with $f(u) = \frac{u^3}{3}$ not be convex.

For the general convex conservation laws, we have all similar results as in Burgers equation, such as existence and uniqueness of the entropy weak solution, L^1-contraction principle, large time asymptotic behaviour of solutions with decaying order in time (for periodic initial data, bounded and integrable initial data $u_0 \in L^1 \cap L^\infty$, and in the case $\lim_{x \to \pm\infty} u_0(x) = u_\pm$), and existence of profile (N-wave). We omit the proofs of these properties and leave them to readers as exercises since their proofs in convex conservation laws are as same as in the Burgers equation.

Theorem 2.1

All the theory we derived for Burgers equations goes to the scalar convex conservation laws.

§2.2 General Conservation Laws (Kruzkov's theory)

Consider the general conservation laws

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 \\ u(x, t = 0) = u_0(x) \end{cases} \tag{2.2}$$

where $f \in C^1$ needs not be convex, and initial data $u_0(x) \in L^\infty$. Before considering all properties of solutions, we need to impose stronger definition than in convex conservation laws which is in the sense of distribution because the solution may fail to satisfy the entropy condition in general. We give a definition of weak solution of general scalar conservation laws as below.

Definition 2.1

A bounded function $u(x,t) \epsilon L^\infty(R^1 \times [0,T])$ is called an entropy weak solution to (2.2) if

(a) for all smooth nonnegative test function $\varphi \,\epsilon\, C_0^\infty(R^1 \times [0,T]), \varphi \geq 0$, one has

$$\int\int |u(x,t)-k|\partial_t \varphi + sgn(u(x,t)-k)(f(u(x,t))-f(k))\partial_x \varphi \, dx\, dt \geq 0$$

for any constant k.

(b) there is a measure zero set $E_0 \subseteq [0, T]$ such that $\int_{|x| \leq R} |u(x,t)| dx$ is well defined for $t \,\epsilon\, [0, T] \backslash E_0$ and

$$\lim_{t \to 0, t \,\epsilon\, [0,T] \backslash E_0} \int_{|x| \leq R} |u(x,t) - u_0(x)| dx = 0$$

Remark:

1. We can see clearly that if u is an entropy weak solution, then u must be a weak solution in the sense of distribution, by taking k to be large or small.

2. This formulation comes from entropy - entropy flux consideration stated as below.

Definition 2.2 We say $(\eta(u), q(u))$ is an entropy - entropy flux pairs if

$$\nabla q(u) = \nabla\, \eta(u) \cdot \nabla\, f(u).$$

This definition does naturally come from the transformation. Suppose $u(x, t)$ is a smooth solution of general scalar conservation law of (2.2), then $\partial_t\, u + \nabla\, f(u) \cdot \partial_x\, u = 0$. Multiply $\nabla\, \eta(u)$ on both sides and from the identity in Definition 2.2, we get $\partial_t\, \eta(u) + \partial_x\, q(u) = 0$.

In particular, if η is a convex function, then we call (η, q) a convex entropy - entropy flux pair.

For weak solutions of (2.2), we claim that $\partial_t \eta(u) + \partial_x q(u) \leq 0$. Assuming the claim, let $\eta(u)$ be a regularization of $|u-k|$ and k be any fixed constant, one deduces that u satisfies (a) in Definition 2.1. To prove the claim, consider the viscous conservation laws $\partial_t u + \partial_x f(u) = \varepsilon \partial_x^2 u, \quad \varepsilon > 0$. Let (η, q) be a convex entropy - entropy flux pairs, $\nabla^2 \eta(u) \geq 0$. Multiply $\nabla \eta(u)$ on both sides, we have

$$\begin{aligned}
\partial_t \eta(u) + \partial_x q(u) &= \varepsilon \nabla \eta(u) \partial_x^2 u \\
&= \varepsilon \partial_x(\nabla \eta(u) \partial_x u) - \varepsilon \nabla^2 \eta(u)(\partial_x u)^2 \\
&\leq \varepsilon \partial_x^2 \eta(u)
\end{aligned}$$

let $\varepsilon \to 0$, the viscous limit solution gives our claim.

For the general scalar conservation laws, we will adapt Kruzkov's result on the well-posedness of the Cauchy problem.

Theorem 2.2 Assume that $f \in C^1(R^1)$, $u_0 \in L^\infty(R^1)$, then there exists a unique entropy weak solution to the problem (2.2). Furthermore, if $u(x,t), v(x,t)$ are entropy weak solutions to (2.2) with initial data $u_0(x), v_0(x) \in L^\infty(R^1)$, respectively, then

$$\int_{|x|\leq R} |u(x,t) - v(x,t)| dx \leq \int_{|x|\leq R+Nt} |u_0(x) - v_0(x)| dx \quad (2.3)$$

for $t \in [0,T] \backslash E_0$. Here
$M = \max_{x \in R^1} \{|u_0(x)|, |v_0(x)|\}, \quad N = \max_{|u|\leq M} |f'(u)|$, and E_0 is the same as in Definition 2.1.

Remark: In the proof, we will show the existence and the validity of (2.3) for smooth, compactly supported initial data. We can see that (2.3) is much more than the uniqueness. It also allows us to approximate the solution with "bad" initial data by solutions with "good" initial data. Actually, the proof also works for scalar equations

$$\partial_t u + \sum_{1 \le \alpha \le d} \partial_{x_\alpha} f^\alpha(x, t, u) = 0, \quad x \epsilon R^d,$$

with $d \geq 2$.

Proof: Without loss of generality, we can assume $u_0 \epsilon C_0^\infty(R^1)$. To make the proof easy to follow, we will separate it into several steps.

Step 1: Approximate solutions

We consider

$$\begin{cases} \partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \, \partial_x^2 u^\varepsilon, \\ u^\varepsilon(x, t=0) = u_0(x). \end{cases} \qquad (2.4)$$

For fixed $\varepsilon > 0$, the maximum principle implies $||u^\varepsilon(x,t)||_{L^\infty(R^1)} \leq ||u_0(x)||_{L^\infty(R^1)} \leq M$, which is enough to ensure the global (for t) existence of the smooth solution, i.e.

$$u^\varepsilon \in C^\infty([0, T] \times R^1), \quad \forall \, T > 0.$$

Now we assume that there is a subsequence $\{\varepsilon_j\}_{j=1}^{\infty}, \varepsilon_j \to 0$ as $j \to +\infty$, such that $u^{\varepsilon_j}(x,t) \to u^0(x,t)$, a.e., which will be proved in Step 2. Then here we show u^0 is an entropy weak solution to (2.2).

Let $(\eta(u), q(u))$ be any convex entropy - entropy flux pair, then multiply the equation (2.4) by $\nabla\eta(u^\varepsilon)$. By the definition of entropy - entropy flux, i.e. $\nabla q = \nabla\eta \cdot \nabla f$, we obtain

$$\begin{aligned}
& \partial_t\,\eta(u^\varepsilon) + \partial_x\,q(u^\varepsilon) \\
= \;& \varepsilon\,\nabla\eta(u^\varepsilon)u^\varepsilon_{xx} \\
= \;& \varepsilon\,\partial_x(\nabla\eta(u^\varepsilon)\cdot\partial_x u^\varepsilon) - \varepsilon\,\nabla^2\eta(u^\varepsilon)\cdot(\partial_x u^\varepsilon)^2 \\
\le \;& \varepsilon\,\partial_x(\nabla\eta(u^\varepsilon)\cdot\partial_x u^\varepsilon).
\end{aligned}$$

Then multiply the above inequality by any $\varphi \in C_0^\infty([0,T]\times R^1), \varphi \geq 0$, and integrate by parts to give

$$\begin{aligned} & -\iint_Q (\eta(u^\varepsilon)\cdot\partial_t\varphi + q(u^\varepsilon)\partial_x\varphi)dx\,dt \\ \leq\ & -\varepsilon\iint_Q \nabla\eta(u^\varepsilon)\cdot\partial_x u^\varepsilon\cdot\partial_x\varphi\,dx\,dt \\ =\ & R.H.S., \end{aligned}$$

where $Q = R^1 \times (0,T)$.

Now we prove $R.H.S. \to 0$ as $\varepsilon \to 0$. Multiply (2.4) by u^ε and then integrate over Q to give

$$\frac{1}{2}\cdot\frac{d}{dt}\int_{R^1}|u^\varepsilon(x,t)|^2dx + \int_{R^1} u^\varepsilon(x,t)\partial_x f(u^\varepsilon)dx = -\varepsilon\int_R |\partial_x u^\varepsilon(x,t)|^2dx,$$

The second term on the left hand side is zero. Then integrate the above equation over $(0, T)$ to give

$$\begin{aligned} & \int_{R^1} |u^\varepsilon(x,t)|^2 dx + \varepsilon \iint_Q |\partial_x u^\varepsilon(x,t)|^2 dx\, dt \\ = & \int_{R^1} |u^\varepsilon(x, t=0)|^2 dx = \int_{R^1} |u_0(x)|^2 dx. \end{aligned}$$

Therefore the standard energy estimate shows that

$$\varepsilon \iint_Q |\partial_x u^\varepsilon(x,t)|^2 dx\, dt \le M_1 < +\infty, \quad M_1 = \int_{R^1} |u_0(x)|^2 dx.$$

Then

$$\begin{aligned}
&\varepsilon \left| \iint_Q \nabla \eta(u^\varepsilon) \cdot \partial_x u^\varepsilon \cdot \partial_x \phi \, dx \, dt \right| \\
\leq \quad & C \sqrt{\varepsilon} \cdot \left(\varepsilon \iint_Q |\partial_x u^\varepsilon|^2 dx \, dt \right)^{\frac{1}{2}} \left(\iint_Q |\partial_x \phi|^2 dx \, dt \right)^{\frac{1}{2}} \\
\leq \quad & C \sqrt{\varepsilon} \cdot M_1{}^{\frac{1}{2}} \left(\iint_Q |\partial_x \phi|^2 dx \, dt \right)^{\frac{1}{2}} \\
& \to 0, \quad \text{as} \quad \varepsilon \to 0^+.
\end{aligned}$$

Hence, if we let $\varepsilon_j \to 0$, then $R.H.S. \to 0$, and by Dominated Convergence Theorem,

$$\iint_Q (\eta(u^0)\partial_t \phi + q(u^0)\partial_x \phi) dx \, dt \geq 0, \tag{2.5}$$

here we have used that $|\partial_t \phi|, |\partial_x \phi|$ are bounded and compactly supported.

In the above, we have assumed $\eta \epsilon C^2$. When $(\eta(u), q(u))$ is only in $W^{1,\infty}(R^1)$, we can approximate $(\eta(u), q(u))$ by C^2 convex entropy - entropy flux pairs. Then (2.5) holds true for any convex $(\eta, q) \epsilon W^{1,\infty}(R^1)$. In particular, we take $\eta(u) = |u - k|$, $q(u) = sign(u - k) \cdot (f(u) - f(k))$. This verifies that u_0 is the entropy weak solution.

Step 2: To prove $u^{\varepsilon_j}(x,t) \to u^0(x,t) \quad a.e.$

Actually, we cannot expect $|\partial_x u^\varepsilon(x,t)|_{L^\infty(R^1)}$ to be finite. But we can obtain the boundedness of $u^\varepsilon(x,t)$ in $BV(R^1)$, i.e. the L^1-estimate of $\partial_x u^\varepsilon(x,t)$.

Set $P = \partial_x u^\varepsilon(x,t)$, then

$$\begin{cases} \partial_t P + \partial_x (f'(u^\varepsilon) P) = \varepsilon \partial_x^2 P \\ P(x, t=0) = P_0(x) = \partial_x u_0(x) \end{cases}$$

Claim: $\partial_t |P| + \partial_x (f'(u^\varepsilon)|P|) \leq \varepsilon \partial_x^2 |P|$.

The proof of the Claim is just the same as what we have done in §1.8 for Burgers equation. We will omit the proof here.

From the Claim,

$$\int_{R^1} |P| dx \leq \int_{R^1} |P_0| dx \leq M_2 < +\infty,$$

i.e. $\quad TV\, u^\varepsilon(\cdot, t) = \int_{R^1} |\partial_x u^\varepsilon(\cdot, t)| dx \leq M_2 < +\infty.$

By Helley principle, there exists a subsequence $\{\varepsilon_j\}_{j=1}^{\infty}, \varepsilon_j \to 0$ as $j \to +\infty$, such that $u^{\varepsilon_j} \to u^0$, a.e. (by a similar argument and for the Burger's equation).

Then we go to the next step to give the important L^1- contraction property.

Step 3: Kruzkov's stability estimate (doubling variable argument)

Proposition 2.1 Let $u(x,t), v(x,t)$ be two entropy weak solution to (2.2) with initial data $u_0, v_0 \in L^{\infty}(R^1)$; respectively. Assume further that $|u(x,t)| \leq M, |v(x,t)| \leq M$, for some $M < +\infty$, then

$$\int_{S_t} |u(x,t) - v(x,t)| dx\, dt \leq \int_{S_\tau} |u(x,\tau) - v(x,\tau)| dx,$$

where

$$S_t = \{(x,t)|\, |x| \le R\},$$
$$S_\tau = \{(x,\tau)|\, |x| \le R + N(t-\tau)\},$$
$$N = \max_{|u|\le M} |f^{'}(u)|,$$

and $t, \tau \notin E_0^u \cup E_0^v, \quad t > \tau.$

Remark: The key idea of the proof of the Proposition is based on the symmetry of the entropy $\eta_*(u) = |u-k|$.

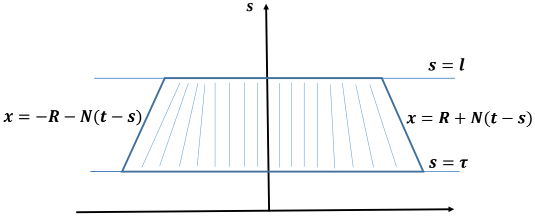

Figure 2.1

Proof of the Proposition: (doubling variable argument)

Step 3.1 By definition, $u(x,t)$ is an entropy weak solution, then $\forall\, \varphi \,\epsilon\, C_0^\infty(Q), \quad \varphi \geq 0,$

$$\iint_Q (|u(x,t) - k|\, \partial_t \phi(x,t) + \text{sign}(u(x,t) - k)(f(u(x,t)) - f(k))\, \partial_x \phi(x,t)) dx\, dt \geq 0,$$

for all $k \,\epsilon\, R^1$.

Now for any positive C^∞ function $\psi(x,t,y,\tau)$, we choose

$$\phi(x,t) = \psi(x,t,y,\tau), \quad k = k(y,\tau) = v(y,\tau),$$

for any fixed $(y,\tau) \epsilon\, Q$. Here we view (y,τ) as parameters. Then integrate the above inequality with respect to (y,τ) over Q to get

$$\iiiint_{Q\times Q} \Big(|u(x,t)-v(y,\tau)|\partial_t\psi(x,t,y,\tau)+\mathrm{sign}(u(x,t)-v(y,\tau))\cdot (f(u(x,t))-f(v(y,\tau)))\,\partial_x\psi(x,t,y,\tau)\Big)\,dx\,dt\,dy\,d\tau \geq 0 \tag{2.6}$$

Similarly, $v(y,\tau)$ is also an entropy weak solution. Then for fixed $(x,t)\epsilon\, Q$, we choose

$$\phi(y,\tau)=\psi(x,t,y,\tau), \quad k=k(x,t)=u(x,t).$$

Then we obtain

$$\iiiint_{Q\times Q} \Big(|v(y,\tau)-u(x,t)|\,\partial_\tau\psi(x,t,y,\tau)+\mathrm{sign}(v(y,\tau)-u(x,t))\cdot \\ (f(v(y,\tau))-f(u(x,t)))\,\partial_y\psi(x,t,y,\tau)\Big)\,dy\,d\tau\,dx\,dt \\ \geq 0 \tag{2.7}$$

From (2.6) and (2.7), we immediately get a symmetric inequality

$$\iiiint_{Q\times Q} \Big(|u(x,t)-v(y,\tau)|(\partial_t+\partial_\tau)\psi(x,t,y,\tau)+\mathrm{sign}(u(x,t)-v(y,\tau))\cdot \\ (f(u(x,t))-f(v(y,\tau)))\,(\partial_x+\partial_y)\psi(x,t,y,\tau)\Big)\,dx\,dt\,dy\,d\tau \\ \geq 0 \tag{2.8}$$

Step 3.2 Choice of the test functions

Let $\phi(x,t) \,\epsilon\, C_0^\infty(Q), \quad \phi \geq 0$. For any $h > 0$, define

$$\psi(x,t,y,\tau) = \phi\left(\frac{x+y}{2}, \frac{t+\tau}{2}\right) \delta_h(x-y)\, \delta_h(t-\tau).$$

where $\delta_h(x) = \frac{1}{h}\delta(\frac{x}{h})$ is the positive approximation to the Dirac mass at the origin and $\delta \,\epsilon\, C_0^\infty(R^1), \quad \int_{R^1} \delta(x)dx = 1$, the support of δ is [-1,1].

Then

$$\begin{aligned} (\partial_t + \partial_\tau)\psi(x,t,y,\tau) &= \partial_t\, \phi(\cdot,\cdot)\delta_h(x-y)\delta_h(t-\tau) \\ (\partial_x + \partial_y)\psi(x,t,y,\tau) &= \partial_x\, \phi(\cdot,\cdot)\delta_h(x-y)\delta_h(t-\tau), \end{aligned}$$

where $(\cdot,\cdot) = \left(\frac{x+y}{2}, \frac{t+\tau}{2}\right)$. Substitute them into (2.8) to give

$$
\begin{aligned}
0 \quad &\leq \quad \iiiint_{Q\times Q} |u(x,t)-v(y,\tau)|\partial_t\phi(\cdot,\cdot)\delta_h(x-y)\delta_h(t-\tau)dx\,dt\,dy\,d\tau \\
&\quad + \iiiint_{Q\times Q} \text{sign}(u(x,t)-v(y,\tau))\cdot(f(u(x,t))-f(v(y,\tau)))\cdot \\
&\quad \partial_x\phi(\cdot,\cdot)\delta_h(x-y)\delta_h(t-\tau)dx\,dt\,dy\,d\tau \\
&\equiv \quad I_1 + I_2.
\end{aligned}
$$

Now we want to prove

$$
\begin{aligned}
I_1 \quad &\to \quad \iint_Q |u(x,t)-v(x,t)|\partial_t\phi(x,t)dx\,dt, \\
I_2 \quad &\to \quad \iint_Q \text{sign}(u(x,t)-v(x,t))\cdot(f(u(x,t))-f(v(x,t)))\partial_x\phi(x,t)dx\,dt, \\
&\quad \text{as } h\to 0^+.
\end{aligned}
$$

The two quantities are in the same form, so we only need to work with one of them, say I_1.

Note that $\iint_Q \delta_h(x-y)\delta_h(t-\tau)dy\,d\tau = 1$, then

$$\begin{aligned}
& I_1 - \iint_Q |u(x,t)-v(x,t)|\partial_t\phi(x,t)dx\,dt \\
= \; & I_1 - \iiiint_{Q\times Q} |u(x,t)-v(x,t)|\partial_t\phi(x,t)\delta_h(x-y)\delta_h(t-\tau)dx\,dt\,dy\,d\tau \\
= \; & \iiiint_{Q\times Q} [(|u(x,t)-v(y,\tau)| - |u(x,t)-v(x,t)|)\,\partial_t\phi(\cdot,\cdot) \\
& +|u(x,t)-v(x,t)|\cdot(\partial_t\phi(\cdot,\cdot)-\partial_t\phi(x,t))] \\
& \cdot\delta_h(x-y)\delta_h(t-\tau)dx\,dt\,dy\,d\tau \\
\equiv \; & J_1 + J_2.
\end{aligned}$$

Then

$$
\begin{aligned}
|J_1| &\leq \iiiint_{Q\times Q} |v(x,t)-v(y,\tau)|\,|\partial_t\phi(\cdot,\cdot)|\cdot\delta_h(x-y)\delta_h(t-\tau)dx\,dt\,dy\,d\tau \\
&= \iiiint_{Q\times[-1,1]\times[0,1]} |v(x,t)-v(x-hy,t-h\tau)| \\
&\quad \cdot\left|\partial_t\phi\left(x-\frac{1}{2}hy, t-\frac{h}{2}\tau\right)\right|\cdot\delta(y)\,\delta(\tau)\,dx\,dt\,dy\,d\tau \\
&\equiv J_0(v).
\end{aligned}
$$

Let U be a compact neighborhood of the support of ϕ. Then for h sufficiently small, the above integral is taken on a bounded set $U\times[-1,1]\times[0,1]$, and the integrant is also bounded by a bounded function

$$2M\cdot||\partial_t\,\phi(\cdot,\cdot)||_{L^\infty}\cdot\delta(y)\,\delta(\tau).$$

If v is continuous, then $|v(x,t) - v(x - hy, t - h\tau)| \to 0$ as $h \to 0^+$. Hence the dominated convergence theorem shows that $J_0(v) \to 0$ as $h \to 0^+$. If v is not continuous, then for any small positive constant β, we can choose a continuous function w, such that $||v - w||_{L^1(Q)} \leq \beta$.

Then

$$\begin{aligned} J_0(v) \leq J_0(v-w) + J_0(w) &\leq 2 \cdot ||\partial_t \phi(\cdot,\cdot)||_{L^\infty} \cdot \beta + J_0(w) \\ &\to 2 \cdot ||\partial_t \phi(\cdot,\cdot)||_{L^\infty(Q)} \beta \quad as \quad h \to 0^+ \quad . \end{aligned}$$

Since β is arbitrarily small, we get $J_0(v) \to 0$ as $h \to 0^+$, which implies $|J_1| \to 0$ as $h \to 0^+$.

Also note that $\partial_t \phi(x,t)$ is Lipschitz continuous in t and x and with compact support. Then as h small enough,

$$\begin{aligned}
|J_2| &\le 2M \iiiint_{U\times[-1,1]\times[0,1]} \left| \partial_t\phi(x-\frac{h}{2}y, t-\frac{h}{2}\tau) - \partial_t\phi(x,t) \right| \\
&\quad \cdot\delta(y)\delta(\tau)\,dx\,dt\,dy\,d\tau \\
&\le 2M\cdot C\cdot h^2 \iiiint_{U\times[-1,1]\times[0,1]} (|y|+|\tau|)\delta(y)\delta(\tau)\,dx\,dt\,dy\,d\tau \\
&\le 2M\cdot C\cdot h^2\cdot \mathrm{meas}(U) \to 0 \quad as \quad h\to 0^+.
\end{aligned}$$

Therefore,

$$I_1 \to \iint_Q |u(x,t)-v(x,t)|\,\partial_t\phi(x,t)dx\,dt \quad as \quad h\to 0^+ \quad .$$

Similarly,

$$I_2 \to \iint_Q \operatorname{sign}(u(x,t)-v(x,t))\cdot(f(u(x,t))-f(v(x,t)))\,\partial_x\phi(x,t)dx\,dt$$

In conclusion, we obtain

$$\iint_Q (|u(x,t)-v(x,t)|\,\partial_t\phi(x,t)dx\,dt + \operatorname{sign}(u(x,t)-v(x,t))\cdot (f(u(x,t))-f(v(x,t)))\,\partial_x\phi(x,t)dx\,dt \geq 0 \qquad (2.9)$$

Step 3.3 L^1- Contraction

Let $\delta_h(x)$ be the standard Friedrichs mollifier. Define

$$S_h(x) = \int_{-\infty}^{x} \delta_h(y)dy.$$

Then $S_h(x)$ satisfies

$$\begin{array}{lll} (1) & S_h(x) \equiv 0, & x \leq -h; \\ (2) & S_h(x) \equiv 1, & x \geq h; \\ (3) & S_h'(x) = \delta_h(x) \geq 0, & 0 \leq S_h(x) \leq 1. \end{array}$$

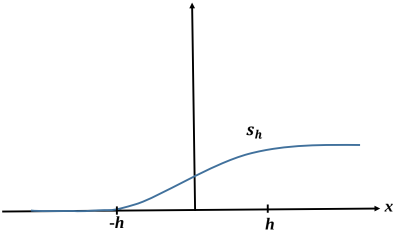

Figure 2.2

Now for any fixed $t, \tau \epsilon [0, T] \backslash (E_u^0 \cup E_v^0), t > \tau$, we define for $\tau < s < t$

$$\begin{aligned} \chi_\varepsilon(x, s) &= 1 - S_\varepsilon(|x| - R + N(s-t) + \varepsilon) \\ \Phi_h^\varepsilon(x, s) &= (S_h(s-\tau) - S_h(s-t))\chi_\varepsilon(x, s). \end{aligned}$$

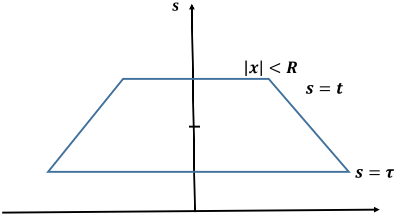

Figure 2.3

Then, it is easy to get

$$\begin{aligned} & \partial_s \Phi_h^{\varepsilon}(x,s) \\ = \; & -N(S_h(s-\tau) - S_h(s-t))\delta_{\varepsilon}(|x| - R + N(s-t) + \varepsilon) \\ & +(\delta_h(s-\tau) - \delta_h(s-t))\chi_{\varepsilon}(x,s), \end{aligned} \tag{2.10}$$

$$\begin{aligned} & \partial_x \Phi_h^{\varepsilon}(x,s) \\ = \; & (S_h(s-\tau) - S_h(s-t))(-\delta_{\varepsilon}(|x| - R + N(s-t) + \varepsilon)\,\mathrm{sign} x) \end{aligned} \tag{2.11}$$

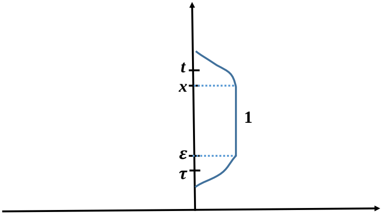

Figure 2.4

Substitute (2.10), (2.11) into (2.9) to give

$$\begin{aligned}
0 \leq & \iint |u(x,s)-v(x,s)|(\delta_h(s-\tau)-\delta_h(s-t))\chi_\varepsilon(x,s)dx\,ds \\
& + \iint [|u(x,s)-v(x,s)|(-N)(S_h(s-\tau)-S_h(s-t)) \\
& \delta_\varepsilon(|x|-R+N(s-t)+\varepsilon) \\
& +\mathrm{sign}(u(x,s)-v(x,s))\,(f(u(x,s))-f(v(x,s)))\cdot \\
& (S_h(s-\tau)-S_h(s-t))(-\mathrm{sign}x)\delta_\varepsilon(|x|-R+N(s-t)+\varepsilon)]dx\,ds \\
\equiv & \; I_1+I_2.
\end{aligned}$$

It is clear that

$$\begin{aligned} & I_2 \\ = & \iint |u(x,s)-v(x,s)|(S_h(s-\tau)-S_h(s-t))\delta_\varepsilon(|x|-R+N(s-t)+\varepsilon) \\ & \left[-N-(\text{sign}x)\frac{f(u(x,s))-f(v(x,s))}{u(x,s)-v(x,s)}\right] dx\, ds \\ \leq & \ 0. \end{aligned}$$

where we have used the fact $N = \max_{|u|\leq M} |f^{'}(u)|$.

Therefore, it yields

$$\begin{aligned} I_1 &= \iint |u(x,s)-v(x,s)|(\delta_h(s-\tau)-\delta_h(s-t))\chi_\varepsilon(x,s)dx\, ds \\ &\geq 0. \end{aligned}$$

That is

$$\begin{aligned} &\iint |u(x,s)-v(x,s)|\delta_h(s-t)\chi_\varepsilon(x,s)dx\,ds \\ \leq\ &\iint |u(x,s)-v(x,s)|\delta_h(s-\tau)\chi_\varepsilon(x,s)dx\,ds. \end{aligned}$$

Let $h,\varepsilon\to 0^+$ to reduce

$$\int_{|x|\leq R}|u(x,t)-v(x,t)|dx \leq \int_{|x|\leq R+N(t-\tau)}|u(x,\tau)-v(x,\tau)|dx. \tag{2.12}$$

After let $\tau \to 0^+$, we obtain

$$\int_{|x|\leq R} |u(x,t) - v(x,t)|dx \leq \int_{|x|\leq R+Nt} |u_0(x) - v_0(x)|dx. \quad (2.13)$$

This is L^1-Contraction of weak entropy solutions of (2.2). Certainly, uniqueness of the weak entropy solutions can be deduced from L^1-Contraction.

Remark: In the stability argument for (2.12), there was no requirement on the weak entropy solution except that u is bounded measurable. However, for the existence, we assume $u_0 \epsilon C_0^\infty(R^1)$. For general L^∞ initial data, one can use the L^1-contraction to approximate u_0 by a sequence $u_0^n \epsilon C_0^\infty(R^1)$.

§2.3 Existence by Weak Convergence Method

As shown in previous sections, when one deals with existence of weak solutions of conservation laws or other PDEs, the usual strategy is to construct approximate solutions for them. Then a priori estimates and compactness discussions for approximate solutions are crucial parts. Certainly, strong convergence is always good thing. However, strong convergence is not always easy to achieve. In most cases, we only have weak convergence or weak - $*$ convergence in L^p-space $(1 \leq p \leq \infty)$. What we are concerned with is whether weak or weak - $*$ convergence guarantees the global existence of weak solutions. The answer is decidedly negative in general. We give some examples to illustrate the point.

Consider a scalar equation:

$$\begin{cases} \partial_t u + \partial_x(a(x,t)u) = \varepsilon \partial_x^2 u, & x \in R^1, \quad t \in (0,T), \\ u(x,t=0) = u_0(x) \end{cases} \tag{2.14}$$

where $a(x,t) \in C^1(R^1 \times [0,T])$ for instance. Then maximum principle gives us an estimate on the approximate solution u^ε, that is,

$$|u^\varepsilon|_{L^\infty(R^1 \times (0,T))} \leq C(T, u_0).$$

We have

$$u^\varepsilon \overset{*}{\rightharpoonup} u^0 \quad \text{in} \quad L^\infty(R^1 \times (0,T)).$$

Then it is easy to see that $u^0(x,t)$ is a weak solution of the following equation:

$$\begin{cases} \partial_t u + \partial_x(a(x,t)u) = 0, \\ u(x,t=0) = u_0(x). \end{cases}$$

Indeed, since

$$\iint (u^\varepsilon\,\partial_t\varphi + a(x,t)u^\varepsilon\,\partial_x\varphi)dx\,dt = -\varepsilon\iint u^\varepsilon\,\partial_x^2\varphi\,dx\,dt,$$
$$\forall\,\varphi\,\epsilon\,C_0^\infty(R^1\times(0,T)),$$

as $\varepsilon \to 0^+$, we get it. This argument works just because $a(x,t)u$ is linear. Now, for the equation

$$\begin{cases} \partial_t\,u + \partial_x\,f(u) = \varepsilon\,\partial_x^2\,u, \\ u(x,t=0) = u_0(x), \end{cases} \tag{2.15}$$

We still have

$$|u^\varepsilon|_{L^\infty} \leq C$$

due to the maximum principle. Then

$$u^\varepsilon \overset{*}{\rightharpoonup} u^0 \quad \text{in} \quad L^\infty(R^1 \times (0, T)).$$

The question is whether u^0 be a solution to (2.15) with $\varepsilon = 0$? Or, under what condition, we can obtain that there exists a subsequence $\{u^{\varepsilon_j}\}$ of $\{u^\varepsilon\}$ such that

$$f(u^{\varepsilon_j}) \to f(u^0) \tag{2.16}$$

in the sense of distribution?

Some weak convergence methods are introduced to deal with this problem. The basic idea of weak convergence method is to get (2.16) without touching hard estimates. Compensated compactness is one kind of weak convergence methods in essence, which was introduced by L. Tartar and F. Murat in the end of 1970's. As we shall see, it is an efficient tool provided that the system has a sufficiently large number of entropies. This is the case for a scalar conservation law and also for 2×2 systems. On the other hand, since it needs more entropies, compensated compactness method has its limitation for more extensive applications for general systems.

In this section, we first give some preliminaries in §2.3.1, including some well-known inequalities and compactness theorems. Subsection §2.3.2 is about Young measures. Then we prove div-curl Lemma in §2.3.3 and finally, in §2.3.4 we give applications of the compensated compactness method to general conservation laws.

§2.3.1 Preliminaries

In this subsection, we first give some well-known facts, then we give a proof of two theorems on compactness of measures.

Fact 1. Gargliardo - Nirenberg - Sobolev inequality

Let $1 \leq q < n, \quad q^* = \frac{nq}{n-q}$. Then

$$||f||_{L^{q^*}(R^n)} \leq C||\nabla f||_{L^q(R^n)}.$$

Suppose $\Omega < R^n$ be a bounded open domain. Then we have

Fact 2. If $f \epsilon W^{1,q}(\Omega)$, then

$$||f||_{L^p(\Omega)} \leq C(\Omega, n, p)||f||_{W^{1,q}(\Omega)}, \quad 1 \leq p \leq q^*.$$

Fact 3. If $q > n$, then $W^{1,q}(\Omega)$ imbeds into $C(\Omega)$ compactly.

Fact 4. (Rellich's compactness theorem)

Let $\{f_k\}_{k=1}^{\infty}$ be a bounded sequence in $W^{1,q}(\Omega)$, then $\{f_k\}_{k=1}^{\infty}$ is precompact in $L^p(\Omega), \quad 1 \leq p < q^*$.

Fact 5. (Weak compactness of Measures)

Let $\{\mu_k\}$ be a bounded sequence in $\mathcal{M}(\Omega)$ (space of bounded measures). Then there exists a subsequence $\{\mu_{k_j}\}$ such that

$$\mu_{k_j} \rightharpoonup \mu \quad \text{in} \quad \mathcal{M}(\Omega),$$

i.e.

$$\int_\Omega \varphi \, d\mu_k \to \int_\Omega \varphi \, d\mu, \quad \forall \, \varphi \in C_0(\Omega).$$

Theorem 2.3 (Compactness of Measures)

Assume that $\{\mu_k\}_{k=1}^{\infty}$ is bounded in $\mathcal{M}(\Omega)$. Then $\{\mu_k\}_{k=1}^{\infty}$ is compact in $W^{-1,q}(\Omega), 1 \leq q < 1^* = \frac{n}{n-1}$.

($W^{-1,q}(\Omega)$ is the dual space of $W_0^{1,q'}, \frac{1}{q} + \frac{1}{q'} = 1$).

Proof:

Step 1. By the weak compactness of measures (fact 5), there must be a subsequence of $\{\mu_k\}_{k=1}^{\infty}$, which we still denote by itself, such that

$$\mu_k \rightharpoonup \mu \quad \text{in} \quad \mathcal{M}(\Omega),$$

i.e.

$$\langle \mu_k, \phi \rangle \;\to\; \langle \mu, \phi \rangle, \quad \forall\, \phi \in C_0(\Omega). \tag{2.17}$$

Step 2. We need to prove

$$||\mu_k - \mu||_{W^{-1,q}(\Omega)} \to 0 \quad \text{as} \quad k \to +\infty \tag{2.18}$$

By definition of the norm of $W^{-1,q}(\Omega)$, one has

$$||\mu_k - \mu||_{W^{-1,q}(\Omega)} = \sup_{\phi \,\epsilon\, B_1 \subset W_0^{1,q'}} |\langle \mu_k - \mu, \phi \rangle|,$$

where B_1 is the unit ball in $W_0^{1,q'}$.

Since $q < 1^* = \frac{n}{n-1}$, $\quad q' > n$, it concludes that $B_1 \subset W_0^{1,q'}$ is compact in $C_0(\Omega)$. Thus, for any $\phi \,\epsilon\, B_1$, one has from (2.17)

$$\langle \mu_k, \phi \rangle \to \langle \mu, \phi \rangle \quad \text{as} \quad k \to +\infty$$

However, this is not enough to get (2.18), which needs the convergence is uniform on $\phi \in B_1$. Note that $B_1 \subset W_0^{1,q'}(\Omega)$ is compact in $C_0(\Omega)$, so for any $\varepsilon > 0$, there exists a ε-net, that is, there exists a $N(\varepsilon)$ and a sequence $\{\Phi_k\}_{k=1}^{N(\varepsilon)} \subset C_0(\Omega)$ such that

$$\min_{1 \leq k \leq N(\varepsilon)} ||\phi - \phi_k||_{W^{1,q'}(\Omega)} \leq \varepsilon, \quad \forall \phi \in B_1 \subset W_0^{1,q'}(\Omega). \quad (2.19)$$

Then, for any $\phi \in B_1$, we first choose a function $\phi_i \in C_0(\Omega)(1 \leq i \leq N(\varepsilon))$ satisfying (2.19), then choose $K > 0$ large enough such that when $k > K$

$$|\langle \mu_k - \mu, \phi_i \rangle| \leq \varepsilon.$$

Consequently,

$$
\begin{aligned}
& |\langle \mu_k - \mu, \phi \rangle| \le |\langle \mu_k - \mu, \phi - \phi_i \rangle| + |\langle \mu_k - \mu, \phi_i \rangle| \\
\le \quad & 2\,\varepsilon\,\overline{\lim}_{k\to\infty} |\mu_k| + \varepsilon = (2M+1)\varepsilon,
\end{aligned}
$$

where M is the bound of $\{\mu_k\}$. Due to the arbitrary smallness of ε, we have

$$
\sup_{\phi \,\epsilon\, B_1 \subset W_0^{1,q'}} |\langle \mu_k - \mu, \phi \rangle| \to 0 \quad \text{as} \quad k \to +\infty.
$$

This proves (2.18) and the proof of the theorem is finished.

Theorem 2.4

Assume that

(1) $\{f_k\}_{k=1}^{\infty}$ is bounded in $W^{-1,p}(\Omega), p > 2$;
(2) $f_k = g_k + h_k$. $\{g_k\}_{k=1}^{\infty}$ is bounded in $\mathcal{M}(\Omega)$, and $\{h_k\}_{k=1}^{\infty}$ is precompact in $W^{-1,2}(\Omega)$.

Then $\{f_k\}_{k=1}^{\infty}$ is precompact in $W^{-1,2}(\Omega)$.

Proof: Consider

$$\begin{cases} -\Delta u_k = f_k, \quad x \epsilon \Omega, \\ u_k|_{\partial\Omega} = 0. \end{cases} \tag{2.20}$$

By standard elliptic regularity results, one has $u_k \in W^{1,p}(\Omega)$. Decompose $u_k = w_k + v_k$ such that

$$\begin{cases} -\Delta w_k = g_k, & x \in \Omega, \\ w_k|_{\partial\Omega} = 0. \end{cases}$$

$$\begin{cases} -\Delta v_k = h_k, & x \in \Omega, \\ v_k|_{\partial\Omega} = 0. \end{cases}$$

Since $\{g_k\}$ is precompact in $W^{-1,q}(\Omega)$ for $1 \leq q < 1^* = \frac{n}{n-1}, \{w_k\}$ is precompact in $W^{1,q}(\Omega)$ for $1 < q < 1^*$. It is also clear that $\{v_k\}$ is precompact in $W^{1,2}(\Omega)$. Therefore, one has that $\{u_k\}$ is precompact in $W^{1,q}(\Omega), \quad 1 < q < 1^* = \frac{n}{n-1}$.

From (2.20), $\{f_k\}$ is precompact in $W^{-1,q}(\Omega)$. Noting that $W^{-1,p}(\Omega) \subset W^{-1,2}(\Omega) \subset W^{-1,q}$ for $q < 2 < p$, and $\{f_k\}$ is uniformly bounded in $W^{-1,p}(\Omega)$ by assumption, one easily get $\{f_k\}$ is precompact in $W^{-1,2}(\Omega)$ through interpolation. Note that one has used the fact that $p' < 2 < q' \Rightarrow W^{1,q'} \subset W^{1,2} \subset W^{1,p'} \to W^{-1,p} \subset W^{-1,2} \subset W^{-1,q}$.

§**2.3.2** Young measure

In most of weak convergence methods in PDE, usually we take the approximate sequence and weak convergent subsequence argument to find a weak limit, and try to prove the weak limit is the solution to our problem. But the main difficulty which we deal with is the convergency of the nonlinear effects, that is , the weak convergency cannot apply to the nonlinear composition of the sequences in general. So we rise the following question:

If the sequence f_k is bounded in $L^\infty(\Omega; R^m)$, is there a subsequence f_{k_j} which is weak - $*$ convergent to f so that, for any continuous function $F \epsilon C(R^m)$, we have $F(f_{k_j}) \to F(f)$ in the sense of distribution?

One of the accessible way is to work with Young measure. It is rigorous for such a work because it gives an explicit form of the weak limit provided that we know the Young measure. Also it is an efficient tool to study concentration oscillation.

Theorem 2.5 (Existence of Young measure)

Assume that $\{f_k\}$ is uniformly bounded in $L^\infty(\Omega; R^m)$. Then there exists a subsequence $\{f_{k_j}\}$ of $\{f_k\}$ and for a.e. $x \epsilon \Omega$, there is a Borel probability measure ν_x on R^m such that for any continuous function $F \epsilon C(R^m)$, we have

$$F(f_{k_j}) \to \bar{F}(x) = \int_{R^m} F(y)\, d\,\nu_x(y) \quad \text{weak - } * \text{ convergent in } L^\infty(\Omega; R^m).$$

We call $d\,\nu_x(y)$ is a Young measure associated with $\{f_{k_j}\}$. First we give some remarks for Young measure.

Remark:

1. This theorem is nontrivial because it gives a representation of the weak convergence.

2. If $K \subset R^m$ is a compact set such that $f_k(x)\,\epsilon\,K$ for a.e. $x\,\epsilon\,\Omega$ and all k, then the support of the Young measure $supp\,\nu_x \subset K$ for a.e. $x\,\epsilon\,\Omega$.

3. One sufficient condition to give a positive answer to the question is that $\nu_x = \delta_{f(x)}(y) = \delta(y - f(x))$. Furthermore, the following proposition says that we can take limit for a.e. $x\,\epsilon\,\Omega$.

Proposition 2.2 Suppose the measure $d\nu_x$ is a unit point mass for a.e. $x \epsilon \Omega$, then $f_{k_j} \to f$ a.e. and $F(f_{k_j}) \to F(f)$ a.e.

Proof: Suppose $f_{k_j} \overset{*}{\rightharpoonup} f$ weak - $*$ in $L^\infty(\Omega; R^m)$. Since $d\nu_x$ is a unit point mass, $d\nu_x(y) = \delta(y - g(x))\, dy$. Take $F(y) = y$ and apply the Theorem 2.5, we get
$f_{k_j} \overset{*}{\rightharpoonup} \overline{y} = \int y\, \delta(y - g(x))dy = g(x)$, that is $g(x) = f(x)$ and $d\nu_x(y) = \delta(y - f(x))dy$.

Now let $F(y) = |y|^2$, then
$|f_{k_j}|^2 \overset{*}{\rightharpoonup} \overline{|y|^2} = \int |y|^2\, \delta(y - f(x))dy = |f(x)|^2$. Assume Ω is a bounded open set and χ_Ω is the corresponding characteristic function of Ω. Then $\chi_\Omega \epsilon\, L^1(\Omega)$ and

$$||f_{k_j}||^2_{L^2(\Omega)} = \int_\Omega |f_{k_j}|^2(x)\chi_\Omega(x)dx \to \int_\Omega |f|^2(x)\chi_\Omega(x)dx = ||f||^2_{L^2(\Omega)}$$

From this and the weak - $*$ convergence $f_{k_j} \stackrel{*}{\rightharpoonup} f$ in $L^\infty(\Omega; R^m)$, we deduce that $||f_{k_j} - f||_{L^2(\Omega)} \to 0$, which implies $f_{k_j} \to f$ a.e. $x \,\epsilon\, \Omega$.

Proof of Theorem 2.5: We follow the standard measure theory to find Young measure by taking projection of some product measure constructed by the sequence.

<u>Step 1:</u> Define a sequence of measure μ_k on $\Omega \times R^m$ by $\mu_k(E) = \int_\Omega \chi_E(x, f_k(x))dx$ for any measurable subset E of $\Omega \times R^m$.

Then $d\,\mu_k(y) = \delta(y - f_k(x))dx$ and $|\mu_k|(\Omega \times R^m) \leq L_n(\Omega)$ for all k, where $L_n(\Omega)$ is the n-dimensional Lebesgue measure.

By the weak convergence of measures, there is a measure $\mu \,\epsilon\, \mathcal{M}(\Omega \times R^m)$ such that $\mu_{k_j} \to \mu$ in $\mathcal{M}(\Omega \times R^m)$.

Step 2: Let σ be the projection of μ onto Ω, i.e., $\sigma(E) = \mu(E \times R^m)$ for any measurable set $E \subset \Omega$. We claim that σ is actually an n-dimensional Lebesgue measure, that is, $\sigma(E) = L_n(E)$ for all $E \subset \Omega$ measurable.

First, for any open set $V \subset \Omega$, by the weak convergence of measure, we get

$$\sigma(V) = \mu(V \times R^m) \leq \liminf_{k_j \to \infty} \mu_{k_j}(V \times R^m) \leq L_n(V)$$

hence $\sigma \leq L_n$.

Conversely, since $\{f_{k_j}\}$ is uniformly bounded in $L^\infty(\Omega; R^m)$, there is an $R > 0$ such that $supp\, \mu_k = \{(x, t) \epsilon\, \Omega \times R^m | \frac{d\, \mu_k}{d(x,y)}(x, y) \neq 0\} \subset \Omega \times B(0, R)$. Then for any compact set $K \subset \Omega$,

$$\sigma(K) = \mu(K \times R^m) = \mu(K \times \overline{B(0,R)}) \geq \limsup_{k_j \to \infty} \mu_{k_j}(K \times \overline{B(0,R)}) = L_n(K)$$

hence $\sigma \geq L_n$. This proves the claim.

Step 3: By slice measure theorem, we can find a Borel probability measure ν_x on R^m such that

$$\int G(x,y)d\,\mu(x,y) = \int_\Omega \int_{R^m} G(x,y)d\nu_x(y)d\sigma \qquad (2.21)$$

for all continuous bounded function G on $\Omega \times R^m$. We state the slice measure theorem as follows.

Theorem 2.6 (Slice measure theorem)

Let μ be a finite Radon measure on R^{m+n}. Let σ be the canonical projection of μ onto R^n. Then for a.e. $x \in R^n$, there exists Radon probability measure ν_x on R^m such that

(i) the function $x \mapsto \int_{R^m} G(x, y) d\nu_x(y)$ is measurable with respect to σ.

(ii) $\int_{R^{m+n}} G(x, y) d\mu = \int_{R^n} \int_{R^m} G(x, y) d\nu_x(y) d\sigma$.

This theorem is well-known in measure theory and we omit the proof.

Step 4: The only thing to do is that the Borel probability measure ν_x found in Step 3 is the measure we required. For any $\xi \epsilon C_0(\Omega), F \epsilon C_0(R^m)$, we want to prove $\int \xi(x) F(f_{k_j}) d\sigma \to \int_{R^n} \xi(x) \overline{F}(x) d\sigma$.

In (2.21), we choose $G(x,y) = \xi(x)F(y)$, then

$$\begin{aligned}
\lim_{k_j \to \infty} \int_\Omega \xi(x) F(f_{k_j}) d\sigma &= \lim_{k_j \to \infty} \int_\Omega \xi(x) F(y) d\mu_{k_j}(x,y) \\
&= \int_\Omega \xi(x) F(y) d\mu(x,y) \\
&= \int_\Omega \int_{R^m} \xi(x) F(y) d\nu_x(y)\, d\sigma \\
&= \int_\Omega \xi(x) \overline{F}(x) d\sigma
\end{aligned}$$

therefore $F(f_{k_j}) \overset{*}{\rightharpoonup} \overline{F}$ in $L^\infty(\Omega; R^m)$.

§2.3.3 Div-Curl Lemma

In this subsection, we consider the question as follows:

If $\{v_k\}, \{w_k\}$ are two bounded sequences in $L^2(\Omega; R^m)$. Assume that $v_k \rightharpoonup v$ in $L^2, w_k \rightharpoonup w$ in L^2. When $v_k \cdot w_k$ converges to $v \cdot w$ in the sense of distribution?

In most of previous analysis we may assume much stronger conditions on convergence, such as $D\, v_k$ converges weakly in L^2. Div-Curl lemma gives less conditions on the derivatives of the sequence, which is convenient to checking the convergence in various problems such as conservation laws and fluid mechanics.

Theorem 2.7 Let Ω be a bounded set in R^n. $\{v_k\}, \{w_k\}$ are uniformly bounded sequences in $L^2(\Omega; R^n)$ and $v_k \rightharpoonup v$ in $L^2, w_k \rightharpoonup w$ in L^2. Assume further that

(i) div $v_k = \sum_{i=1}^n \partial_{x_i} v_k^{(i)}$ is precompact in $W^{-1,2}$

(ii) curl w_k is precompact in $W^{-1,2}, (\mathrm{curl}\, w_k)_{ij} = \partial_{x_i} w_k^{(j)} - \partial_{x_j} w_k^{(i)}$.

Then $v_k \cdot w_k \to v \cdot w$ in the sense of distribution.

Proof: The main idea is to separate w_k into divergence free part and exact in gradient part by Hodge decomposition so that we can apply (i) and (ii) to each term multiplying with v_k to show the convergence.

Step 1: Consider the Laplace equation

$$\begin{cases} -\Delta\, u_k = w_k \\ u_k|_{\partial\Omega} = 0 \end{cases}$$

Since w_k is bounded in $L^2(\Omega)$, from the elliptic regularity with assuming the boundary $\partial\,\Omega\,\epsilon\, C^2$, we obtain that u_k is bounded in $W^{2,2}\,(\Omega)$.

Step 2: Let $z_k = -\text{div}\, u_k$ be a bounded sequence in $W^{1,2}(\Omega)$, $y_k = w_k - \nabla z_k$ be bounded in $L^2(\Omega)$. We claim that y_k is compact in L^2 by (ii). Indeed,

$$
\begin{aligned}
y_k^{(i)} &= (-\Delta\, u_k)^{(i)} + \partial_{x_i}(\text{div}\, u_k) \\
&= -\partial_{x_j}^2\, u_k^{(i)} + \partial_{x_i}\partial_{x_j}\, u_k^{(j)} \\
&= \partial_{x_j}(\partial_{x_i}\, u_k^{(j)} - \partial_{x_j}\, u_k^{(i)}) \\
&= \partial_{x_j}(\text{curl}\, u_k)_{ij}
\end{aligned}
$$

from the fact $-\Delta(\text{curl}\, u_k) = \text{curl}(-\Delta\, u_k) = \text{curl}\, w_k$ is precompact in $W^{-1,2}(\Omega)$, $\text{curl}\, u_k$ is compact in $W^{1,2}(\Omega)$ by the elliptic theory, hence y_k is compact in $L^2(\Omega)$. Also z_k is compact in $L^2(\Omega)$.

Step 3: Taking a subsequence of z_k, y_k, still denoted as z_k, y_k respectively, such that

$$z_k \rightharpoonup z \text{ weakly in } W^{1,2}(\Omega), \quad y_k \to y \text{ strongly in } L^2(\Omega),$$
$$u_k \rightharpoonup u \text{ weakly in } W^{2,2}(\Omega)$$

then $z = -\text{div}\, u$ and

$$\begin{cases} -\Delta u = w \\ u|_{\partial \Omega} = 0 \end{cases}$$

since all the involved terms are linear. Then we go to prove this proposition.

Step 4: For any $\varphi \epsilon C_0^\infty$, we need to show that

$$\int_\Omega (v_k \cdot w_k)\, \varphi \, dx \longrightarrow \int_\Omega (v \cdot w)\, \varphi \, dx$$

Now $w_k = y_k + \nabla z_k$, substitute into the integral on the left hand side and then take integration by parts, we get

$$\begin{aligned} & \int_\Omega v_k \cdot w_k \, \varphi \, dx \\ = & \int_\Omega v_k \cdot y_k \, \varphi \, dx + \int_\Omega v_k \cdot \nabla z_k \, \varphi \, dx \\ = & \int_\Omega v_k \cdot y_k \, \varphi \, dx - \int_\Omega \operatorname{div} v_k \, z_k \, \varphi \, dx - \int_\Omega v_k \cdot z_k \, \nabla \varphi \, dx \end{aligned}$$

The first integral $\int_\Omega v_k \cdot y_k \, \varphi \, dx$ converges to $\int_\Omega v \cdot y \, \varphi \, dx$ for $y_k \to y$ in $L^2(\Omega)$ and $v_k \rightharpoonup v$ in $L^2(\Omega)$. The third integral $-\int_\Omega v_k \cdot z_k \, \nabla\varphi \, dx \to -\int_\Omega v \cdot z \, \nabla\varphi \, dx$ for $z_k \to z$ in $L^2(\Omega)$. The second integral converges to $-\int_\Omega \operatorname{div} v \, z \, \varphi \, dx$ by (i) since

$$
\begin{aligned}
& -\int_\Omega \operatorname{div} v_k \cdot z_k \, \varphi \, dx + \int_\Omega \operatorname{div} v \cdot z \, \varphi \, dx \\
= \ & \int_\Omega (\operatorname{div} v - \operatorname{div} v_k) \, z_k \, \varphi \, dx + \int_\Omega \operatorname{div} v (z - z_k) \, \varphi \, dx \\
\to \ & 0
\end{aligned}
$$

and $z_k \rightharpoonup z$ weakly in $W^{1,2}(\Omega)$. This finishes the proof of theorem.

§2.3.4 Application to scalar conservation laws

Employing the div-curl lemma, we consider the following scalar conservation laws

$$\partial_t u^\varepsilon + \partial_x f(u^\varepsilon) = \varepsilon \partial_x^2 u^\varepsilon, \quad u^\varepsilon(x, t=0) = u_0^\varepsilon(x), \tag{2.22}$$

$$\partial_t u + \partial_x f(u) = 0, \quad u(x, t=0) = u_0(x). \tag{2.23}$$

Without loss of generality, we assume that

$$|u^\varepsilon|_{L^\infty(K)} \leq C_1(K), \quad \forall \varepsilon \ll \varepsilon_0. \tag{2.24}$$

$$\varepsilon \iint_K |\partial_x u^\varepsilon|^2 dx\, dt \leq C_2(K) \tag{2.25}$$

where $K = R \times (0, T)$. So we can extract a subsequence $\{u^{\varepsilon_k}\}_{k=1}^{\infty}$, such that there exists $u^0 \epsilon L^\infty(K)$ and

$$u^{\varepsilon_k} \to u^0 \quad \text{w-}* \text{ in } L^\infty(K) \text{ as } \varepsilon_k \to 0^+.$$

Now we want to know whether u^0 is a weak solution to (2.23).

Since u^ε is a classical solution to (2.22), multiply (2.22) by a test function and do integration by parts to deduce

$$\forall \varphi \,\epsilon\, C_0^\infty(R^1 \times [0, T]), \quad \text{supp}\, \varphi \subset K,$$
$$-\iint_K (\partial_t \varphi \, u^\varepsilon + \partial_x \varphi \, f(u^\varepsilon))\, dx\, dt = \varepsilon \iint_K \partial_x^2 \varphi \, u^\varepsilon dx\, dt.$$

Since $\partial_t\varphi,\ \partial_x\varphi,\ \partial_x^2\varphi$ are bounded and have compact support, it follows that the first term on the left hand side tends to $-\iint_K \partial_t\varphi\, u^0$, and the right hand side tends to 0 as $\varepsilon_k \to 0$. Hence u^0 is a weak solution iff $\iint \partial_x\varphi\, f(u^\varepsilon)dx\, dt \to \iint \partial_x\varphi\, f(u^0)dx\, dt$, i.e. u^0 is a weak solution iff $f(u^\varepsilon) \to f(u^0)$ in the sense of distribution. In most cases, the question is whether $u^\varepsilon \to u^0$ a.e. (x, t). Therefore, the question becomes whether the Young measure associated with the weak convergence is a Dirac mass.

Theorem 2.8 Under (2.24) and (2.25), and $f \epsilon C^1(R^1)$. Then u^0, the weak limit of u^{ε_k}, is a weak solution.

Proof:

Step 1: Let $\nu_{(x,t)}(y)$ be the Young measure associated with the weak convergence. Then

$$F(u^\varepsilon) \xrightarrow{w_*} \overline{F} = \int F(y) d\nu_{(x,t)}(y), \quad \forall\, F \in C(R^1).$$

In particular,

$$\begin{aligned} u^{\varepsilon_k} \quad &\xrightarrow{w_*} \quad u = \int y\, d\nu_{(x,t)}(y), \\ f(u^{\varepsilon_k}) \quad &\xrightarrow{w_*} \quad \overline{f} = \int f(y) d\nu_{(x,t)}(y). \end{aligned}$$

Step 2: Compensated compactness (Div-Curl lemma)

To apply the div-curl lemma, we will construct two sequences of functions, namely $\{v_k\}$ and $\{w_k\}$ using (2.22).

First choose $v_k = (u^{\varepsilon_k}, f(u^{\varepsilon_k}))$, then $\operatorname{div} v_k = \varepsilon_k\, \partial_x^2 u^{\varepsilon_k}$.

Let $(\eta(u), q(u))$ be a C^2- convex entropy - entropy flux pair, i.e. $\nabla\eta \cdot \nabla f = \nabla q$ and $\nabla^2\eta(U) \geq 0$, then
$\partial_t \eta(u^\varepsilon) + \partial_x q(u^\varepsilon) = \varepsilon\, \partial_x^2 \eta(u^\varepsilon) - \varepsilon\, \nabla^2\eta(u^\varepsilon) \cdot (\partial_x u^\varepsilon)^2$.

Now we choose $w_k = (-q(u^{\varepsilon_k}), \eta(u^{\varepsilon_k}))$, then

$$\begin{aligned} \operatorname{curl} w_k &= \partial_t \eta(u^{\varepsilon_k}) - \partial_x(-q(u^{\varepsilon_k})) \\ &= \partial_t \eta + \partial_x q \\ &= \varepsilon_k \, \partial_x^2 \eta(u^{\varepsilon_k}) - \varepsilon_k \, \nabla^2 \eta(u^{\varepsilon_k}) \cdot (\partial_x u^{\varepsilon_k})^2 \\ &\equiv f_k + h_k. \end{aligned}$$

It can be easily seen that $\{v_k\}$ and $\{w_k\}$ are bounded in $L^2(K)$.

Claim: $\operatorname{div} v_k$ is precompact in $W^{-1,2}(K)$.

Proof of the claim: $\forall \, \theta \in H_0^1(K) = W_0^{1,2}(K)$,

$$\langle \operatorname{div} v_k, \, \theta \rangle = \iint_K \operatorname{div} v_k \cdot \theta \, dx \, dt = \varepsilon_k \iint_K \partial_x^2 u^{\varepsilon_k} \, \theta \, dx \, dt.$$

Integrate by parts to give

$$
\begin{aligned}
|\langle \operatorname{div} v_k, \theta\rangle| &= \varepsilon_k \left| \iint_K \partial_x u^{\varepsilon_k} \, \partial_x \theta \, dx \, dt \right| \\
&\le \varepsilon_k \left(\iint_K (\partial_x u^{\varepsilon_k})^2 \, dx \, dt \right)^{\frac{1}{2}} \left(\iint_K (\partial_x \theta)^2 \, dx \, dt \right)^{\frac{1}{2}} \\
&\le \varepsilon_k \left(\iint_K |\partial_x u^{\varepsilon_k}|^2 \, dx \, dt \right)^{\frac{1}{2}} \cdot ||\theta||_{H_0^1(K)}.
\end{aligned}
$$

Since θ is arbitrary,

$$\begin{aligned} ||\mathrm{div}\, v_k||_{W^{-1,2}(K)} &\leq \varepsilon_k \left(\int\int_K |\partial_x u^{\varepsilon_k}|^2 \, dx\, dt \right)^{\frac{1}{2}} \\ &= (\varepsilon_k)^{\frac{1}{2}} \left(\varepsilon_k \int\int_K |\partial_x u^{\varepsilon_k}|^2 \, dx\, dt \right)^{\frac{1}{2}} \\ &\to 0 \text{ as } \varepsilon_k \to 0. \end{aligned}$$

by (2.25). This means $\mathrm{div}\, v_k$ is precompact in $W^{-1,2}(K)$.

Now we consider w_k with $\operatorname{curl} w_k = f_k + h_k$. A similar proof to that for $\operatorname{div} v_k$ shows that $f_k = \varepsilon_k \, \partial_x^2 \eta(u^{\varepsilon_k})$ is precompact in $W^{-1,2}(K)$. And $h_k = -\varepsilon_k \, \nabla^2 \eta(u^{\varepsilon_k}) |\partial_x u^{\varepsilon_k}|^2$ is a bounded sequence in $\mathcal{M}(K)$ since

$$\int\int_K |h_k| \, dx \, dt \leq \varepsilon_k \, C(K) \int\int_K |\partial_x u^{\varepsilon_k}|^2 dx \, dt \leq C(K) \, C_2(K) < +\infty.$$

So by the compactness theorem of measures, $\operatorname{curl} w_k$ is precompact in $W^{-1,2}(K)$ if $\operatorname{curl} w_k$ is bounded in $W^{-1,q}$, $q > 2$. This is true because $||u^{\varepsilon_k}||_{L^\infty(K)}$ is uniformly bounded which implies that $\partial_t \eta(u^{\varepsilon_k}) + \partial_x q(u^{\varepsilon_k})$ is uniformly bounded in $W^{-1,\infty}(K)$. Therefore, $\operatorname{curl} w_k$ is uniformly bounded in $W^{-1,q}(K)$ for any $q > 2$.

Then we can use the div-curl lemma and have

$$v_k \cdot w_k \rightharpoonup v \cdot w \quad \text{in the sense of distribution}$$

where $v = (u^0, \overline{f})$, $\quad w = (-\overline{q}, \overline{\eta})$. That is

$$-f(u^{\varepsilon_k})\eta(u^{\varepsilon_k}) + u^{\varepsilon_k}\, q(u^{\varepsilon_k}) \to -\overline{f}\,\overline{\eta} + u^0\,\overline{q} \quad \text{in the sense of distribution} \tag{2.26}$$

<u>Step 3:</u> Tartar's commutative relation

By Young measure theorem,

$$\begin{aligned} -f(u^{\varepsilon_k})\eta(u^{\varepsilon_k}) + u^{\varepsilon_k}\, q(u^{\varepsilon_k}) &\to \overline{-f(y)\,\eta(y) + y\,q(y)} \\ &= -\overline{f(y)\,\eta(y)} + \overline{y\,q(y)} \qquad (2.27) \\ &\quad \text{in the sense of distribution} \end{aligned}$$

Comparing (2.26) and (2.27), we have

$$-\overline{f(y)\,\eta(y)} + \overline{y\,q(y)} = -\overline{\eta(y)} \cdot \overline{f(y)} + u^0\,\overline{q(y)}.$$

By definition, it reads

$$\begin{aligned} & \int -f(y)\,\eta(y)\,d\nu_{(x,t)}(y) + \int y\,q(y)\,d\nu_{(x,t)}(y) \\ = & \; -\int f(y)\,d\nu_{(x,t)}(y) \cdot \int \eta(y)\,d\nu_{(x,t)}(y) + u^0 \int q(y)\,d\nu_{(x,t)}(y), \end{aligned}$$

then we rewrite it into a compact form

$$\begin{aligned} & \langle (\bar{f} - f)\eta, \nu \rangle + \langle (y - u^0) q, \nu \rangle = 0, \\ \text{or} \quad & \langle (\bar{f} - f)\eta + (y - u^0) q, \nu \rangle = 0. \end{aligned} \tag{2.28}$$

Although we have assumed $(\eta, q) \epsilon C^2$ in deducing (2.28), it actually holds for general convex (η, q) through approximation procedure.

The equation (2.28) shows that $(\bar{f} - f)\eta + (y - u^0)q$ must vanish on the support of $\nu_{(x,t)}$. But it is not enough to prove the strong convergence. We will analyze the support of $\nu_{(x,t)}$ in the following step.

Step 4: Reduction to the Dirac mass by choosing special entropy-entropy flux.

Case 1: Convex case, i.e. $f^{''} > 0$.
We fix any (x, t) and denote $u = u^0(x, t)$. Then choose

$$\eta_1(y) = f(y) - f(u) - f^{'}(u)(y - u),$$

so that $q_1(y) = \int_u^y (f^{'}(\xi))^2 \, d\xi - f^{'}(u)(f(y) - f(u))$, and
$\nabla^2 \eta = f^{''}(y)$.

Substitute (η_1, q_1) into (2.28) to give

$$\begin{aligned} 0 &= \langle (\overline{f} - f(y))(f(y) - f(u) - f^{'}(u)(y - u) \\ &\quad +(y-u)\int_u^y (f^{'}(\xi))^2 d\xi - (y-u)f^{'}(u)(f(y) - f(u)), \nu\rangle \\ &\equiv \langle I_1 + I_2 + I_3, \nu\rangle. \qquad (2.29) \end{aligned}$$

We rewrite I_1 as

$$
\begin{aligned}
I_1 &= (\bar{f} - f(u) + f(u) - f(y)) \cdot (f(y) - f(u) - f^{'}(u)(y-u)) \\
&= -(f(u) - f(y))^2 + (\bar{f} - f(u))(f(y) - f(u)) \\
&\quad -(f(u) - f(y))f^{'}(u)(y-u) - (\bar{f} - f(u))f^{'}(u)(y-u) \\
&= -(f(u) - f(y))^2 + (\bar{f} - f(u))\eta_1(y) + (y-u)f^{'}(u)(f(y) - f(u)) \\
&= -(f(u) - f(y))^2 + (\bar{f} - f(u))\eta_1(y) - I_3.
\end{aligned}
$$

Then by (2.29),

$$\langle -(f(u)-f(y))^2+(\bar{f}-f(u))\eta_1(y)+(y-u)\int_u^y (f'(\xi))^2 d\xi, \nu\rangle = 0.$$

That is

$$\langle -(f(u)-f(y))^2+(y-u)\int_u^y (f'(\xi))^2 d\xi, \nu\rangle + \langle(\bar{f}-f(u))\eta_1, \nu\rangle = 0.$$

Noting that

$$\langle -(f(u) - f(y))^2 + (y - u) \int_u^y (f^{'}(\xi))^2 d\xi, \nu \rangle \geq 0,$$

which follows from the Cauchy - Schwartz inequality, one has

$$\begin{aligned} &\langle\, (\overline{f} - f(u))\, \eta_1(y), \nu\, \rangle \leq 0, \qquad \text{i.e.} \\ &(\overline{f} - f(u))\, \langle \eta_1(y), \nu \rangle \leq 0. \end{aligned}$$

Since f is convex, we have $\overline{f} \geq f(u)$ and $\eta_1(y) \geq 0$. Thus either $\overline{f} - f(u) = 0$ or $\langle \eta_1(y), \nu \rangle = 0$. In the first possibility, together with the convexity of f, we must have $\nu_{(x,t)} = \delta(y - u)$. In the second possibility, we also have $\nu_{(x,t)} = \delta(y - u)$. Therefore we always have $\nu_{(x,t)} = \delta(y - u)$, and then u^0 is a weak solution.

Case 2: f is not convex. Here we will use Kruzkov's entropy.

Let $\eta(y) = |y - u|$, so that $q(y) = \text{sign}\,(y - u)\,(f(y) - f(u))$.

Tartar's commutative relation also holds for this (η, q). And (2.28) reads

$$\begin{aligned} 0 &= \langle (\bar{f} - f(y))|y - u| + |y - u|(f(y) - f(u)), \nu \rangle \\ &= \langle (\bar{f} - f(y))|y - u|, \nu \rangle \\ &= (\bar{f} - f(y))\langle |y - u|, \nu \rangle. \end{aligned}$$

So either $\bar{f} = f(u)$ or $\langle |y - u|, \nu \rangle = 0$. But in the second possibility, ν must be a Dirac mass, $\nu = \delta(y - u)$, and $\bar{f} = f(u)$ follows immediately. Therefore we always have $\bar{f} = f(u)$, and this implies u^0 is a weak solution.

The proof is complete.

We conclude this section and this chapter by the following remarks.

Remarks:

(1) One can see from the previous proof that when f is convex, the Young measure ν must be a Dirac mass. However, when f is not convex, we may have $\overline{f} = f(u)$ but ν is not be a Dirac mass.

(2) One can also see that the existence of entropy is crucial in the proof. Actually, this method is applicable to 2×2 systems but only for some special $n \times n \, (n > 2)$ systems. For general $n \times n$ systems, the existence of entropy is a very difficult problem because the systems determining the entropy and entropy flux is usually overdetermined and has no solution.