THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050B Mathematical Analysis I

Tutorial 1 (September 11, 13)

The following were discussed in the tutorial this week:

1 Negation and Quantifiers

Example 1. Negate the following statements.

(a) If n^2 is divisible by 4, then n is divisible by 2.

(b) For any real number x, $x^2 \geq 0$.

(c) For any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $1/N < \varepsilon$.

2 Algebraic Properties of $\mathbb{R}$

The Field Axioms of $\mathbb{R}$. $(\mathbb{R}, +, \cdot)$ *satisfies the following properties:*

(A1) "+" *commutative*	**(M1)** "$\cdot$" *commutative*	**(D)** *distributive law*
(A2) "+" *associative*	**(M2)** "$\cdot$" *associative*	
(A3) "0"	**(M3)** "1", $1 \neq 0$	
(A4) "+" *inverse*	**(M1)** "$\cdot$" *inverse*	

Example 2. Let $a, b \in \mathbb{R}$. Prove the following statements. State clearly any axioms or theorems used in each step.

(a) $a \cdot 0 = 0$,

(b) If $a + b = 0$, then $b = -a$,

(c) $(-1)a = -a$

(d) $(-a)(-b) = ab$

Solution. We only prove (d) here. Note that

$$\begin{aligned}
(-a)(-b) + (-ab) &= (-a)(-b) + (-a)b + (-(-a)b) + (-ab) && \text{(by A3, A4)} \\
&= (-a)(-b) + (-a)b + (-1)(-a)b + (-1)ab && \text{(by (c))} \\
&= (-a)(-b + b) + (-1)b(-a + a) && \text{(by M1, M2, D)} \\
&= (-a)(0) + (-1)b(0) && \text{(by A4)} \\
&= 0 + 0 && \text{(by (a))} \\
&= 0. && \text{(by A3)}
\end{aligned}$$

By (b), we have $(-a)(-b) = -(-ab) = ab$. ◀

3 Order Properties of $\mathbb{R}$

The Order Properties of $\mathbb{R}$. *There is a nonempty subset $\mathbb{P}$ of $\mathbb{R}$, called the set of positive real numbers, that satisfies the following properties:*

(I) $a, b \in \mathbb{P} \implies a + b \in \mathbb{P}$,

(II) $a, b \in \mathbb{P} \implies ab \in \mathbb{P}$,

(III) If $a \in \mathbb{R}$, then exactly one of the following holds:

$$a \in \mathbb{P}, \qquad a = 0, \qquad -a \in \mathbb{P}.$$

Write $a > 0$ if $a \in \mathbb{P}$; and write $a > b$ if $a - b \in \mathbb{P}$.

Example 3. Let $a, b \in \mathbb{R}$. Show that if $a > 0$, then $1/a > 0$.

Solution. Note that $1/a \neq 0$. By Theorem 2.18(a), $(1/a)^2 > 0$. Now $1/a = a \cdot (1/a)^2 \in \mathbb{P}$ since both $a, (1/a)^2 \in \mathbb{P}$. Hence $1/a > 0$. ◀

4 The Completeness Property of $\mathbb{R}$

Definition 4.1. Let S be a nonempty subset of $\mathbb{R}$.

(a) Suppose S is bounded above. Then $u \in \mathbb{R}$ is said to be a **supremum** of S if

  (i) u is an upper bound of S (that is, $s \le u$ for all $s \in S$);

  (ii) if v is any upper bound of S, then $u \le v$.

  Here (ii) is equivalent to

  (ii)' if $v < u$, then there exists $s_v \in S$ such that $v < s_v$.

(b) Suppose S is bounded below. Then $w \in \mathbb{R}$ is said to be an **infimum** of S if

  (i) w is a lower bound of S (that is, $w \le s$ for all $s \in S$);

  (ii) if v is any lower bound of S, then $v \le w$.

  Here (ii) is equivalent to

  (ii)" if $w < v$, then there exists $s_v \in S$ such that $s_v < v$.

Remark. 1. Supremum and infimum may not be elements of S.

2. u and w above are unique and we write $\sup S = u$, $\inf S = w$.

The Completeness Property of $\mathbb{R}$. *Every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$.*

Example 4. Let $A := \{x \in \mathbb{R} \setminus \{0\} : 1/x < x\}$. Find $\sup A$ and $\inf A$. Justify your answers.

Solution. Note that

$$\frac{1}{x} < x \iff \frac{x^2-1}{x} > 0 \iff \frac{(x-1)(x+1)}{x} > 0 \iff x \in (-1,0) \cup (1,\infty).$$

Thus $A = (-1,0) \cup (1,\infty)$.

It is easy to see that A is not bounded above, so $\sup A$ does not exist.

Next we want to show that $\inf A = -1$. Clearly

$$x > -1 \quad \text{for all } x \in A.$$

So -1 is a lower bound of A. Let $v > -1$.

Want: show that v is not a lower bound of A, that is $\exists\ s_v \in A$ s.t. $s_v < v$.

Take $s_v := \min\{(v-1)/2, -1/2\}$. Then

$$-1 < s_v \leq -1/2 < 0,$$

so that $s_v \in A$. Moreover,

$$s_v \leq (v-1)/2 < (v+v)/2 = v.$$

Hence $\inf A = -1$. ◀