Series

Let $\sum_{n=1}^{\infty} a_n$ be a "formal series" : thus we are to concern with two sequences $(a_n), (s_n)$ of real numbers with

$$s_n := a_1 + \cdots + a_n = \sum_{i=1}^{n} a_i . \qquad (1)$$

This series is said to be convergent if

$$\lim_n s_n \text{ exists in } \mathbb{R};$$

in this case $\sum_{i=1}^{\infty} a_n$ will also be used to denote the $\lim_n s_n = \lim_n \sum_{i=1}^{n} a_i$ (the "sum" of the series $\sum_{n=1}^{\infty} a_n$)

s_n is called nth partial sum of the series.

Th 1. Suppose the series $\sum_{n=1}^{\infty} a_n$ converges : $s := \lim_n s_n$ exists in $\mathbb{R}$ (where s_n as in (1)). Then $\lim_n a_n = 0$.

proof. By assumption $s = \lim_n s_n \left(= \lim_n (s_{n+1})\right)$ and hence $\lim_n (s_{n+1} - s_n) = s - s = 0$. By (1), $s_{n+1} - s_n = a_{n+1}$ & so $\lim_n a_n = 0$.

Th 2. Suppose $\sum_{n=1}^{\infty} |a_n|$ converges. Then $\sum_{n=1}^{\infty} a_n$ converges ("Absolutely convergent series converges").

proof. Let $t_n = \sum_{i=1}^{\infty} |a_i| \ \forall n$. Then, by assumption

(t_n) converges and so is Cauchy. Let $\varepsilon > 0$.
Then $\exists\, N \in \mathbb{N}$ s.t

$$|t_m - t_n| < \varepsilon \quad \forall\, m > n \geq N$$

that is
$$\sum_{i=n+1}^{m} |a_i| < \varepsilon \quad \forall\, m > n \geq N$$

and consequently
$$|s_m - s_n| = \left|\sum_{i=n+1}^{m} a_i\right| < \varepsilon \quad \forall\, m > n \geq N$$

and
$$|s_m - s_n| < \varepsilon \quad \forall\, m, n \geq N$$

by symmetry. This shows that (s_n) is Cauchy and so converges in $\mathbb{R}$. This means that $\sum_{n=1}^{\infty} a_n$ converges.

(Non-Negative Series: Usually simply called Positive Series)

Theorem 3 Let $a_n \geq 0$, $b_n \geq 0$. Then

(i) The series $\sum_{n=1}^{\infty} a_n$ converges (in notation $\sum_{n=1}^{\infty} a_n < +\infty$) iff $\exists\, M > 0$ s.t. $\sum_{i=1}^{n} a_i \leq M \;\forall n$. (MCT)

(convergence criterion for "positive" (or nonnegative) series.)

(ii) If $0 \leq a_n \leq b_n \;\forall n$ & $\sum_{n=1}^{\infty} b_n$ converges then $\sum_{n=1}^{\infty} a_n$ converges (Comparison Test for positive series)

Proof. Ex. (Use the Monotone Conv. Th)

Ex. Let $r \in (0,1)$. Then $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ & $\sum_{r=1}^{\infty} r^n = \frac{r}{1-r}$.

Theorem 4 (Ratio/Root test). Let $0 \leq a_n \; \forall n$. Then the series $\sum a_n$ converges in each of the following cases:

(i) $$r := \lim_n \frac{a_{n+1}}{a_n} < 1 \; ;$$

(ii) $$r := \lim_n \sqrt{a_n} < 1 \, .$$

Proof. (i). Let $c \in (r,1)$. Then $\exists N \in \mathbb{N}$ s.t. $\frac{a_{n+1}}{a_n} < c \;\; \forall n \geq N$ (pl. supply reasons). Hence $a_{N+1} < c a_N$, $a_{N+2} < c a_{N+1} < c^2 a_N$ and inductively $a_{N+j} < c^j a_N \; \forall j \in \mathbb{N}$. Since $\sum_{j=1}^{\infty} c^j < +\infty$ (pl. supply reasons) it follows from Th 3 that $\sum_{j=1}^{\infty} a_{N+j} < +\infty$ & so $\sum_{n=1}^{\infty} a_n < +\infty$.

(ii) Similar (pl. supply details).

Application to Contractive Sequences ((x_n) is called a contractive seq if $\exists \; c \in (0,1)$ s.t. $|x_{n+2} - x_{n+1}| \leq c |x_{n+1} - x_n| \;\; \forall n \in \mathbb{N}$..)

Th 5. Any contractive seq (x_n) converges.

Proof. By assumption

$$|x_{n+2} - x_{n+1}| \leq c |x_{n+1} - x_n| \leq c^2 |x_n - x_{n-1}|$$
$$\leq c^3 |x_{n-1} - x_{n-2}| \leq \cdots \leq c^n |x_2 - x_1|$$

By Th 3 (ii) (applied to $|x_{n+2} - x_{n+1}|$, $c^n |x_2 - x_1|$ in place of a_n, b_n) $\sum_{n=1}^{\infty} (x_{n+2} - x_{n+1}) = \lim_N \sum_{n=1}^{N} (x_{n+2} - x_{n+1}) = \lim_N (x_{N+2} - x_2)$ exists in $\mathbb{R}$

$\therefore \lim_n x_n$ exists in $\mathbb{R}$

Alternatively, one can use the Cauchy Criterion as

$$|x_n - x_{n+j}| \leqslant |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{n+j-1} - x_{n+j}|$$

$$\leqslant c^{n-1}|x_2 - x_1| + c^n |x_2 - x_1| + \cdots + c^{n+j-2}|x_2 - x_1|$$

$$\leqslant \frac{c^{n-1}}{1-c}|x_2 - x_1| \leqslant \frac{c^{N-1}}{1-c} \quad (\forall\, m > n \geqslant N).$$

This implies that (x_n) is Cauchy (why?)

Examples

1. $\sum_{n=1}^{\infty} \frac{1}{n} = +\infty$.

$$S_{2^n} = 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right)$$

$$\geqslant 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) + \left(\frac{1}{8}\right) \times 4 + \left(\frac{1}{2^4}\right) \times 2^3 + \cdots \left(\frac{1}{2^n}\right) \times 2^{n-1}$$

$$= 1 + \left(\frac{1}{2}\right) + n \cdot \left(\frac{1}{2}\right), \quad \forall\, n \in \mathbb{N}$$

This shows that $\{S_n : n \in \mathbb{N}\}$ is not bounded and so $\lim_n S_n = +\infty$ (as $S_n > 0\ \forall n$).

2. $\sum_{n=1}^{\infty} \frac{1}{n^2} < +\infty$

$$S_{2^n - 1} = 1 + \left(\frac{1}{2^2} + \frac{1}{3^2}\right) + \left(\frac{1}{4^2} + \cdots + \frac{1}{7^2}\right) + \left(\frac{1}{8^2} + \cdots + \frac{1}{15^2}\right) + \cdots + \left(\frac{1}{(2^{n-1})^2} + \cdots + \frac{1}{(2^n - 1)^2}\right)$$

$$\le 1 + \left(\frac{1}{2^2}\right)\times 2 + \left(\frac{1}{4^2}\right)\times 2^2 + \cdots + \left(\frac{1}{(2^{n-1})^2}\times 2^{n-1}\right)$$

$$= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2^2}\right) + \cdots + \left(\frac{1}{2^{n-1}}\right) \to \frac{1}{1-\frac{1}{2}} = 2$$

$\therefore$ $\{S_n : n \in \mathbb{N}\}$ is bounded & the series converges to a finite limit.

3. $\sum_{n=1}^{\infty} \frac{1}{n^p} < +\infty$ ($1 < p < +\infty$; we assume basic knowledge on n^p with positive irrational p)

$$S_{2^n-1} \le 1 + \left(\frac{1}{2^p}\right)\times 2 + \left(\frac{1}{4^p}\right)\times 2^2 + \cdots + \left(\frac{1}{(2^{n-1})^p}\times 2^{n-1}\right)$$

$$= 1 + \frac{1}{2^{p-1}} + \left(\frac{1}{2^{p-1}}\right)^2 + \cdots + \left(\frac{1}{2^{p-1}}\right)^{n-1}$$

$$\to \frac{1}{1-c} \quad \text{with } c := \frac{1}{2^{p-1}} \in (0,1)$$

4. $\sum_{n=1}^{\infty} \frac{1}{n!} < +\infty$

$$S_n = 1 + \frac{1}{2\times 1} + \frac{1}{3\times 2} + \cdots + \frac{1}{n(n-1)\cdots 2\cdot 1}$$

$$\le 1 + \frac{1}{2} + \frac{1}{2\times 2} + \cdots + \frac{1}{2\times 2\times\cdots\times 2\times 1}$$

$= (n-1)$th partial sum of $1 + \sum_{k=1}^{n-1}\left(\frac{1}{2}\right)^k \to 2$

5. $$e_n = \left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \cdots + \frac{n(n-1)\cdots 2}{(n-1)!}\left(\frac{1}{n}\right)^{n-1} + \left(\frac{1}{n}\right)^n \frac{n\cdots 2\cdot 1}{n!}$$

$$< 1 + \sum_{k=1}^{n}\frac{1}{k!} \to 3$$ so $e_n < 3 \ \forall n$

Note can show $e_n \uparrow_n$ (so $e := \lim e_n$ exists in $\mathbb{R}$). Indeed, writing

$$e_n = 2 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{(n-1)!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-2}{n}\right)$$

$$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$

(n many terms)

$$e_{n+1} = 2 + \frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+2}\right) + \frac{1}{(n-1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-2}{n+1}\right)$$

$$+ \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)$$

$$+ \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right)\left(1-\frac{n}{n+1}\right)$$

($n+1$ many terms with each of the first n terms dominating that in the expansion of e_n)

$\therefore\ e_n < e_{n+1}$.