MATH2050B 1920 HW6

TA's solutions[1] to selected problems

Q1. Let (x_n) be a C-contractive sequence ($0 < C < 1$):

$$|x_{n+1} - x_n| \leq C|x_n - x_{n-1}|, \qquad \forall n \geq 2.$$

Show by MI that $|x_{n+1} - x_n| \leq C^{n-1}|x_2 - x_1|$ and that $|x_m - x_n| \leq (C^{m-2} + \cdots + C^{n-1})|x_2 - x_1|$, $\forall m > n$. Using ϵ-N definition and $\lim_n C^n = 0$ show hence that (x_n) is Cauchy.

Solution. Claim. $|x_{n+1} - x_n| \leq C^{n-1}|x_2 - x_1|$.

It is clear that the inequality holds for $n = 1$. Suppose that $|x_{k+1} - x_k| \leq C^{k-1}|x_2 - x_1|$ for some k. By assumption that (x_n) is C-Cauchy, $|x_{k+2} - x_{k+1}| \leq C|x_{k+1} - x_k|$. By induction hypothesis, $|x_{k+2} - x_{k+1}| \leq C^k|x_{k+1} - x_k|$. So the claim is proved by MI.

Claim. $|x_m - x_n| \leq (C^{m-2} + \cdots + C^{n-1})|x_2 - x_1|$ for all $m > n$.

Let m, n, $m > n$. Then by triangle inequality and the previous claim,

$$\begin{aligned} |x_m - x_n| &= |(x_m - x_{m-1}) + (x_{m-1} - x_{m-2}) + \cdots + (x_{n+1} - x_n)| \\ &\leq (C^{m-2} + C^{m-3} + \cdots + C^{n-1})|x_2 - x_1| \end{aligned}$$

Claim. (x_n) is Cauchy.

We use the fact that $\sum_{k=1}^{\infty} C^k$ is convergent.

Let $\epsilon > 0$. Since the series $\sum_{k=1}^{\infty} C^k$ is convergent, therefore there is $N \in \mathbb{N}$ so that for all $m, n > N$, $m > n$, we have

$$\sum_{k=n-1}^{m-2} C^k < \epsilon.$$

Therefore for all $m, n > N$, $m > n$, by previous claim we have

$$|x_m - x_n| < \epsilon|x_2 - x_1|.$$

Because $|x_2 - x_1|$ is fixed and ϵ can be arbitrarily small. So (x_n) is Cauchy.

Q2. Respectively by MCT and by **Q1**, show the sequence (x_n) converges, where $x_1 = 99$ and

$$x_{n+1} = \frac{1}{3}(x_n + 10), \qquad \forall n$$

Find the limit.

Solution. (MCT method) **Claim.** (x_n) is decreasing.

$x_2 = 109/3 < x_1$. Suppose that for some k, $x_k < x_{k-1}$. Then $+10$ on both sides, multiply $\frac{1}{3}$ on both sides:

$$x_{k+1} = \frac{1}{3}(x_k + 10) < \frac{1}{3}(x_{k-1} + 10) = x_k$$

Thus (x_n) is decreasing by MI.

[1] please kindly send an email to `nclliu@math.cuhk.edu.hk` if you have spotted any typo/error/mistake.

By definition of (x_n), x_n is always positive, so it is bounded below by 0. So MCT applies.

("**Q1**" Method) For all $n \geq 3$,

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{3}(x_n + 10) - \frac{1}{3}(x_{n-1} + 10) \\ &= \frac{1}{3}(x_n - x_{n-1}) \end{aligned}$$

So (x_n) is C-contractive with $C = \frac{1}{3}$.

To find the limit, since the sequence converges, suppose $L = \lim_n x_n$, then

$$L = \frac{1}{3}(L + 10),$$

and hence $L = 5$.

Q3. Use MCT to show that (y_n) converges; find its limit:

$$y_1 := 81 \qquad \text{and} \qquad y_{n+1} = \sqrt{y_n}\ \forall n$$

Solution. Claim. (y_n) is decreasing and bounded below by 1.

We have $y_1 = 81$, $y_2 = 9$, so $y_1 > y_2 > 1$. If for some k, $y_{k-1} > y_k > 1$, then $y_k > y_{k+1} > 1$ by taking square roots. The claim follows by MI.

By MCT (y_n) converges, say $L = \lim_n y_n$, then $L \geq 1$ because $y_n > 1$ for all n. Now since $y_{n+1} = \sqrt{y_n}$,

$$L = \sqrt{L}.$$

This gives $L^2 = L$. So $L = 0$ or $L = 1$. But $L \geq 1$, so $L = 1$.

Q4. Let (x_n) be a bounded sequence and recall that

$$\limsup_n x_n := \lim_n y_n (= l \in \mathbb{R}, \text{ say}),$$

where $y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}$ for all n. Let α, β be real numbers such that

$$\alpha < l < \beta$$

Show that

(i) $\exists N \in \mathbb{N}$ s.t.
$$x_n < \beta,\ \forall n \geq N$$

(ii) $\forall N \in \mathbb{N}$, $\exists n \geq N$ s.t.
$$\alpha < x_n$$

Remark. For a sequence (x_n), (y_n) defined by $y_n = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}$ may not be a subsequence of (x_n)(many of you think it is so). It can happen that for any n, k, $y_n \neq x_k$.

Solution. For (i), Since $l < \beta$, so there is $N \in \mathbb{N}$ s.t.

$$y_m < \beta,\ \forall m \geq N.$$

In particular $y_N < \beta$. Note $y_N = \sup\{x_N, x_{N+1}, x_{N+2}, \dots\}$, so

$$x_n < \beta, \forall n \geq N$$

For (ii), suppose not. Then there is $N \in \mathbb{N}$ so that for all $n \geq N$, $x_n \leq \alpha$. So $y_n \leq \alpha$ for all $n \geq N$. In this case y_n cannot converge to l. Contradiction.

Q5. With $\alpha = l - \frac{1}{k}$ and $\beta = l + \frac{1}{k}$ in **Q4**, show that $\exists$ a strictly increasing sequence (n_k) of natural numbers such that

$$l - \frac{1}{k} < x_{n_k} < l + \frac{1}{k} \ \ \forall k \in \mathbb{N}.$$

Show that $\lim_k x_{n_k} = \limsup_n x_n$.

Solution. For $k = 1$, by **Q4** (i) there is N_1 so that $x_n < l + \frac{1}{1}$, for all $n \geq N_1$. For this N_1, by **Q4** (ii), there is $n_1 \geq N$ s.t. $l - \frac{1}{1} < x_{n_1}$.

For $k = 2$, by **Q4** (i), there is N_2 so that $x_n < l + \frac{1}{2}$ for all $n \geq N_2$. For the number $\max(N_2, n_1)$, there is $n_2 \geq \max(N_2, n_1)$ s.t. $l - \frac{1}{2} < x_{n_2}$.

Inductively, if we have constructed $x_{n_1}, x_{n_2}, \dots, x_{n_k}$, we can apply the same step to construct $x_{n_{k+1}}$. (****Please fill in the details****)

Since

$$l - \frac{1}{k} < x_{n_k} < l + \frac{1}{k}, \forall k \in \mathbb{N},$$

taking $k \to \infty$ gives $\lim_k x_{n_k} = \limsup_n x_n$.

Q6. Show conversely that if (x_{m_k}) is a convergent subsequence of (x_n) then

$$\lim_k x_{m_k} \leq \limsup_n x_n.$$

Solution. Note $x_{m_k} \leq y_{m_k}$. The sequence (y_{m_k}) is a subsequence of (y_k), so is convergent to $\limsup_n x_n$. So

$$\lim_k x_{m_k} \leq \lim_k y_{m_k} = \limsup_n x_n.$$

Q7. Let X consist of all real numbers expressible as the limit of a convergent subsequence of (x_n). Show that $\max X = \limsup_n x_n$. Show further that $\min X = \liminf_n x_n$, i.e. $\min X = \lim_n z_n$, where $z_n = \inf\{x_n, x_{n+1}, \dots\}$.

Solution. By **Q5**, $\limsup_n x_n \in X$. By **Q6**, for all $x \in X$, $x \leq \limsup_n x_n$. So $\max X = \limsup_n x_n$.

Consider $-X := \{-x : x \in X\}$. Then $-X$ consists of all real numbers expressible as the limit of a convergent subsequence of $(-x_n)$. So by the above,

$$\max(-X) = \limsup_n (-x_n).$$

Note $\max(-X) = -\min X$, and $\limsup_n (-x_n) = -\liminf_n x_n$, so $\min X = \liminf_n x_n$.

Q8. Let $0 < x_n$ and $\limsup_n \frac{x_{n+1}}{x_n} = \gamma \in (0, 1)$. Show that $\sum_{n=1}^{\infty} x_n < +\infty$.

Solution. Let η be a number such that $\gamma < \eta < 1$. Since $\limsup \frac{x_{n+1}}{x_n} = \gamma < \eta$, so (by **Q4**) there is $N \in \mathbb{N}$ such that

$$\frac{x_{n+1}}{x_n} < \eta, \ \ \forall n \geq N.$$

This shows that $x_{N+k} < \eta^k x_N$ for all $k \geq 1$.

For all large m, $m > N$, we have

$$\begin{aligned}\sum_{n=1}^{m} x_n &= \sum_{n=1}^{N} x_n + \sum_{n=1}^{m-N} x_{N+n} \\ &\leq \sum_{n=1}^{N} x_n + x_N \sum_{n=1}^{m-N} \eta^n \\ &\leq \sum_{n=1}^{N} x_n + x_N \sum_{n=1}^{\infty} \eta^n\end{aligned}$$

Note that the R.H.S. is independent of m. (R.H.S. depends on N, and N depends on η only). Hence $\sum_{n=1}^{\infty} x_n < +\infty$.