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Differentiability of Convex Functions on a Locally Convex Topological Vector Space

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We introduce the notion of a smooth set in a locally convex topological vector space and extend Asplund's result on the strong differentiability space. We also establish Gateaux differentiability of a continuous convex function in a locally convex topological vector space. In particular, we extend Mazur's classical theorem on Gateaux differentiability from a separable Banach space to a separable locally convex topological vector space.

Keywords: Topological vector space, smooth set, uniform differentiability, Gateaux differentiability.

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1. Introduction

In 1968, Asplund in his pioneering paper [1] introduced and studied a strong differentiability space: a Banach space X is said to be a *strong differentiability space* if every continuous convex function on an open convex subset D of X is Frechet differentiable at each point of a dense G_δ subset of D; in particular he proved that if a Banach space X can be given an equivalent norm such that the corresponding dual norm in X^* is locally uniformly rotund then X is a strong differentiability space. In 1975, Namioka and Phelps [8] renamed a strong differentiability space as an *Asplund space*. Afterwards Asplund spaces have been extensively studied with many significant results (cf. [5, 9, 7] and the references therein). Recall that a Banach space X is *Fréchet smooth* if its norm is Fréchet differentiable at each point of $X \setminus \{0\}$. It is known that X is Fréchet smooth if the dual space X^* is locally uniformly rotund. Hence the following theorem by Ekeland and Lebourg is a valuable improvement over Asplund's theorem (cf. [3, 10]).

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Theorem A. *Every Fréchet smooth Banach space is an Asplund space.*

The Mazur theorem is a classical result on differentiability theory on Banach spaces, which was proved by Mazur [7] in 1933 and can be stated as follows:

Theorem M. *If D is a nonempty open convex subset of a separable Banach space and $f\colon D \to \mathbb{R}$ is a continuous convex function then there exists a sequence $\{G_n\}$ of dense open subsets of D such that f is Gâteaux differentiable at each point of $\bigcap_{n=1}^{\infty} G_n$.*

In this paper, we introduce the notion of *smoothness* of a bounded closed convex set Ω in a locally convex topological vector space X. In the special case when X is a Banach space, X is Fréchet smooth if and only if its unit ball B_X is smooth. In terms of a smooth set, we extend Theorem A to a locally convex topological vector space. Moreover, we establish Gateaux differentiability of continuous convex functions on a locally convex topological vector space, which extends Mazur's theorem to the locally convex topological vector space setting.

2. Preliminaries

In what follows, X is always assumed to be a locally convex topological vector space with the topological dual X^*. Let $\varphi\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. For $x \in \mathrm{dom}(\varphi)$, let $\partial\varphi(x)$ denote the *subdifferential* of φ at x, that is,

$$\partial\varphi(x) := \{x^* \in X^* : \ \langle x^*, u - x\rangle \leq \varphi(u) - \varphi(x) \quad \forall u \in X\}.$$

It is known that if φ is continuous at $x \in \mathrm{dom}(\varphi)$ then $\partial\varphi(x) \neq \emptyset$.

For a nonempty set A in X, we say that φ is *A-differentiable* at $x \in \mathrm{dom}(\varphi)$ if there exists $u^* \in X^*$ such that the limit

$$\lim_{t\to 0} \frac{\varphi(x + th) - \varphi(x)}{t} = \langle u^*, h\rangle$$

holds uniformly with respect to h in A.

Recall that a nonempty set Ω in X is *balanced* if $\lambda\Omega \subset \Omega$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \leq 1$ (so, in particular, a balanced set is symmetric and contains the origin). Let Ω be a balanced convex closed subset of X and let $\mu_\Omega\colon X \to \mathbb{R}_+ \cup \{+\infty\}$ be the *Minkowski functional* of Ω defined by

$$\mu_\Omega(x) := \inf\{t > 0 : \ x \in t\Omega\} \quad \forall x \in X,$$

where $\inf\emptyset$ is understood as $+\infty$. Then, $\mathrm{dom}(\mu_\Omega) = \mathrm{span}(\Omega) := \bigcup_{t>0} t\Omega$,

$$\mu_\Omega(x + y) \leq \mu_\Omega(x) + \mu_\Omega(y), \ \ \mu_\Omega(tx) = |t|\mu_\Omega(x) \quad \forall (x, y, t) \in X \times X \times \mathbb{R} \tag{1}$$

and

$$\Omega = \{x \in X : \ \mu_\Omega(x) \leq 1\}. \tag{2}$$

Hence μ_Ω is lower semicontinuous.

The following lemma is a straightforward consequence of [4, Proposition 2.5] and useful for our later analysis.

Lemma 2.1. *Let $f\colon X \to \mathbb{R} \cup \{+\infty\}$ be a convex function. Let $a \in X$ and V be a balanced convex neighborhood V of 0 such that*

$$\sup_{x\in V} |f(x)| < +\infty \quad \forall x \in a + 2V. \tag{3}$$

Then there exists $L \in (0,\ +\infty)$ such that

$$|f(x_1) - f(x_2)| \le L\mu_V(x_1 - x_2) \quad \forall x_1, x_2 \in a + V, \tag{4}$$

where μ_V is the Minkowski functional of V.

The following lemma provides a characterization for the A-differentiability of a continuous convex function on a locally convex topological vector space, which is known in the case when X is a normed space and $A = B_X$ (cf. [9]). For completeness, we provide its proof which follows the idea of the proof of [9, Proposition 1.25].

Lemma 2.2. *Let G be an open convex subset of a locally convex topological vector space X and $\varphi\colon G \to \mathbb{R}$ be a continuous convex function. Let A be a bounded subset of X. For each $n \in \mathbb{N}$, let*

$$G_n := \left\{ x \in G :\ \exists t > 0 \text{ s.t. } \sup_{h\in A} \frac{\varphi(x+th) + \varphi(x-th) - 2\varphi(x)}{t} < \frac{1}{n} \right\}.$$

Then, each G_n is an open subset of G, and φ is A-differentiable at $x \in G$ if and only if $x \in \bigcap_{n=1}^{\infty} G_n$.

Proof. First we prove that each G_n is open. To do this, let $u \in G_n$, and we have to show that u is an interior point of G_n. By the continuity of φ, there exists a balanced convex neighborhood V of 0 such that $u + 2V \subset G$ and

$$\sup_{x\in u+2V} |\varphi(x)| < +\infty.$$

This and Lemma 2.1 imply that there exists $L \in (0,\ +\infty)$ such that

$$|\varphi(x_1) - \varphi(x_2)| \le L\mu_V(x_1 - x_2) \quad \forall x_1, x_2 \in u + V, \tag{5}$$

where μ_V is the Minkowski functional of V. Since $u \in G_n$ and A is bounded, the definition of G_n and the convexity of φ imply that there exists $r > 0$ such that $rA \subset \frac{1}{2}V$ and

$$\varepsilon := \frac{1}{n} - \sup_{h\in A} \frac{\varphi(u+rh) + \varphi(u-rh) - 2\varphi(u)}{r} > 0. \tag{6}$$

Take $\eta \in (0,\ \frac{1}{2})$ such that $\frac{4L\eta}{r} < \varepsilon$, and let

$$\Gamma_{(x,h)} := \frac{\varphi(x+rh)+\varphi(x-rh)-2\varphi(x)}{r} - \frac{\varphi(u+rh)+\varphi(u-rh)-2\varphi(u)}{r}$$

for all $(x,h) \in G \times A$. Then, by (5), one has

$$|\Gamma_{(x,h)}| \leq \frac{4L\mu_V(x-u)}{r} \leq \frac{4L\eta}{r} < \varepsilon \quad \forall (x,h) \in (u+\eta V) \times A.$$

It follows from (6) that

$$\sup_{h\in A} \frac{\varphi(x+rh)+\varphi(x-rh)-2\varphi(x)}{r} < \frac{1}{n} \quad \forall x \in u+\eta V.$$

This implies that $u+\eta V \subset G_n$. Hence, u is an interior point of G_n. It is clear that $x \in \bigcap_{n=1}^{\infty} G_n$ if φ is A-differentiable at $x \in G$. Next suppose that $x \in \bigcap_{n=1}^{\infty} G_n$. Then, from the convexity of φ, it is easy to verify that the limit

$$\lim_{t\to 0^+} \frac{\varphi(x+th)+\varphi(x-th)-2\varphi(x)}{t} = 0$$

uniformly holds with respect to h in A. Noting that $\partial\varphi(x) \neq \emptyset$ and

$$\frac{\varphi(x)-\varphi(x-th)}{t} \leq \langle x^*, h\rangle \leq \frac{\varphi(x+th)-\varphi(x)}{t}$$

for all $(x^*, h, t) \in \partial\varphi(x) \times X \times (0,\ +\infty)$, it follows that

$$\lim_{t\to 0} \frac{\varphi(x+th)-\varphi(x)}{t} = \langle x^*, h\rangle$$

and holds uniformly with respect to h in A for each $x^* \in \partial\varphi(x)$. This show that φ is A-differentiable at x. The proof is complete. □

Definition 2.3. We say that a balanced convex set Ω in X is *smooth* if for each $x \in \Omega$ with $\mu_\Omega(x) = 1$, the limit

$$\lim_{t\to 0} \frac{\mu_\Omega(x+th)+\mu_\Omega(x-th)-2\mu_\Omega(x)}{t} = 0$$

holds uniformly with respect to h in Ω.

In the case when X is a Banach space, it is easy to verify that the norm of X is Fréchet differentiable at each $x \in X \setminus \{0\}$ if and only if the unit ball B_X of X is a smooth set in X.

Let Ω be a smooth set and x an arbitrary element in $\mathrm{span}(\Omega)$ such that $\mu_\Omega(x) > 0$. Then, Ω is balanced and convex, the Minkowski functional μ_Ω is positively homogenous, and the limit

$$\lim_{t\to 0} \frac{\mu_\Omega(x+th) + \mu_\Omega(x-th) - 2\mu_\Omega(x)}{t} = 0$$

holds uniformly with respect to h in Ω. It follows from (1) and (2) that

$$\mu'_\Omega(x,h) := \lim_{t\to 0} \frac{\mu_\Omega(x+th) - \mu_\Omega(x)}{t} \le 1$$

exists uniformly with respect to h in Ω, and so

$$\lim_{t\to 0} \frac{\mu_\Omega(x+th)^2 - \mu_\Omega(x)^2}{t} = 2\mu_\Omega(x)\mu'_\Omega(x,h) \quad \text{for all} \quad h \in \Omega.$$

This and the convexity of μ_Ω imply that

$$\begin{aligned} 0 &\le \frac{\mu_\Omega(x+th)^2 - \mu_\Omega(x)^2}{t} - 2\mu_\Omega(x)\mu'_\Omega(x,h) \\ &= \frac{(\mu_\Omega(x+th) - \mu_\Omega(x))^2 + 2\mu_\Omega(x)(\mu_\Omega(x+th) - \mu_\Omega(x) - \mu'_\Omega(x,h)t)}{t} \\ &\le \mu_\Omega(h)^2 t + 2\mu_\Omega(x)\left(\frac{\mu_\Omega(x+th) - \mu_\Omega(x)}{t} - \mu'_\Omega(x,h)\right) \end{aligned}$$

for all $(h,t) \in \Omega \times (0,\ +\infty)$, and so

$$\lim_{t\to 0^+} \frac{\mu_\Omega(x+th)^2 - \mu_\Omega(x)^2}{t} = 2\mu_\Omega(x)\mu'_\Omega(x,h)$$

holds uniformly with respect to h in Ω. Noting that Ω is balanced, this implies

$$\lim_{t\to 0} \frac{\mu_\Omega(x+th)^2 + \mu_\Omega(x-th)^2 - 2\mu_\Omega(x)^2}{t} = 0$$

uniformly with respect to h in Ω, that is, for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \frac{\mu_\Omega(x+th)^2 + \mu_\Omega(x-th)^2 - 2\mu_\Omega(x)^2}{t} \right| < \varepsilon \quad \forall (h,t) \in \Omega \times (-\delta,\ \delta). \tag{7}$$

Proposition 2.4. *Let Ω be a smooth set in a locally convex topological vector space X, $\{v_n\} \subset X$ and let $\{\alpha_n\} \subset (0,\ +\infty)$ be such that*

$$\sum_{n=1}^{\infty} \alpha_n < +\infty. \tag{8}$$

Define $f(x) := \sum_{n=1}^{\infty} \alpha_n \mu_\Omega(x - v_n)^2$ *for all* $x \in X$*. Then, for any* $L \in (0,\ +\infty)$ *and* $x \in \bigcap_{n=1}^{\infty} (v_n + L\Omega)$ *the limit*

$$\lim_{t \to 0^+} \frac{f(x+th) + f(x-th) - 2f(x)}{t} = 0$$

holds uniformly with respect to h *in* Ω*.*

Proof. Let $L \in (0,\ +\infty)$ and $x \in \bigcap_{n=1}^{\infty} (v_n + L\Omega)$. Then

$$\mu_\Omega(x - v_n + th) \le \mu_\Omega(x - v_n) + \mu_\Omega(th) \le L + |t| \quad \forall (t,h) \in \mathbb{R} \times \Omega. \tag{9}$$

Hence

$$\begin{aligned} |\triangle_\Omega(n,t,h)| &= (\mu_\Omega(x - v_n + th) + \mu_\Omega(x - v_n))|\mu_\Omega(x - v_n + th) - \mu_\Omega(x - v_n)| \\ &\le (\mu_\Omega(x - v_n + th) + \mu_\Omega(x - v_n))\mu_\Omega(th) \\ &\le (2L+1)|t| \end{aligned}$$

for all $(t,h) \in [-1,\ 1] \times \Omega$, where $\triangle_\Omega(n,t,h) := \mu_\Omega(x - v_n + th)^2 - \mu_\Omega(x - v_n)^2$. It follows that

$$\begin{aligned} 0 &\le \frac{\mu_\Omega(x - v_n + th)^2 + \mu_\Omega(x - v_n - th)^2 - 2\mu_\Omega(x - v_n)^2}{t} \\ &= \frac{\triangle_\Omega(n,t,h) + \triangle_\Omega(n,-t,h)}{t} \\ &\le 4L + 2 \end{aligned}$$

for all $(n,t,h) \in \mathbb{N} \times (0,\ 1] \times \Omega$. Let $\varepsilon \in (0,\ +\infty)$. Then, by (8), there exists $N \in \mathbb{N}$ such that

$$0 \le \sum_{n=N}^{\infty} \alpha_n \frac{\mu_\Omega(x - v_n + th)^2 + \mu_\Omega(x - v_n - th)^2 - 2\mu_\Omega(x - v_n)^2}{t} < \frac{\varepsilon}{2}$$

for all $(t,h) \in (0,\ 1] \times \Omega$. On the other hand, by (7), there exists $\delta \in (0,\ 1)$ such that

$$0 \le \sum_{n=1}^{N} \alpha_n \frac{\mu_\Omega(x - v_n + th)^2 + \mu_\Omega(x - v_n - th)^2 - 2\mu_\Omega(x - v_n)^2}{t} < \frac{\varepsilon}{2}$$

for all $(t,h) \in (0,\ \delta) \times \Omega$. Therefore, by the definition of f,

$$0 \le \frac{f(x+th) + f(x-th) - 2f(x)}{t} < \varepsilon \quad \forall (t,h) \in (0,\ \delta) \times \Omega.$$

The proof is complete. □

Proposition 2.5. *Let X be a locally convex topological vector space and assume that $\{v_1, \cdots, v_n\} \subset X$ is linearly independent. Then*

$$\Omega := \left\{ \sum_{k=1}^{n} t_k v_k : \sum_{k=1}^{n} t_k^2 \le 1 \right\}$$

is a compact smooth subset of X.

Proof. Clearly, Ω is a compact balanced convex subset of X, and it is easy to verify that

$$\mu_\Omega(x) = \big(\sum_{k=1}^{n} t_k^2\big)^{\frac{1}{2}} \quad \forall x = \sum_{k=1}^{n} t_k v_n \in \text{span}\{v_1, \cdots, v_n\}.$$

Since $(t_1, \ldots, t_n) \mapsto \big(\sum_{k=1}^{n} t_k^2\big)^{\frac{1}{2}}$ is continuously differentiable on $\mathbb{R}^n \setminus \{0\}$, it follows that μ_Ω is uniformly differentiable at each $x \in (\text{span}\{v_1, \cdots, v_n\}) \setminus \{0\}$ in Ω. Hence Ω is a smooth subset of X.

3. Main results

In this section, we extend the two classical theorems on the differentiability of convex functions mentioned in Section 1 to the general case of locally convex topological vector spaces.

Theorem 3.1. *Let X be a locally convex topological vector space and let Ω be a bounded balanced convex closed subset of X such that the following conditions hold:*

(i) *Ω is smooth, and* (ii) *Ω is either compact or sequentially complete.*

Then, for every open convex subset G of X and every continuous convex function $\varphi\colon G \to \mathbb{R}$ there exists a sequence $\{G_n\}$ of open subsets of G such that $\bigcap_{n=1}^{\infty} G_n$ is dense in G and φ is Ω-differentiable at each point of $\bigcap_{n=1}^{\infty} G_n$.

Proof. For each $n \in \mathbb{N}$, let

$$G_n := \left\{ x \in G : \exists t > 0 \text{ s.t. } \sup_{h \in \Omega} \frac{\varphi(x+th) + \varphi(x-th) - 2\varphi(x)}{t} < \frac{1}{n} \right\}.$$

By Lemma 2.2, each G_n is open, and φ is Ω-differentiable at $x \in G$ if and only if $x \in \bigcap_{n=1}^{\infty} G_n$. Hence we only need to show that $\bigcap_{n=1}^{\infty} G_n$ is dense in G. Let $v_0 \in G$ and U be a neighborhood of v_0 such that $U \subset G$. It suffices to show that there exists $v \in U$ such that φ is Ω-differentiable at v. By the continuity of φ, there exists a balanced convex neighborhood V of 0 such that φ is bounded on

$$X_0 := v_0 + 2\text{cl}(V) \subset U \tag{10}$$

and $\varphi(v_0) + 1 > \sup\{\varphi(u) : u \in X_0\}$, that is,

$$f_0(v_0) - 1 < \inf\{f_0(u) : u \in X_0\}, \tag{11}$$

where $f_0 := -\varphi$. Since Ω is bounded, there exists $\lambda > 0$ such that

$$\lambda\Omega \subset V. \tag{12}$$

Now we construct inductively sequences $\{f_n\}$ and $\{v_n\}$ as follows. Define the function $f_1 \colon X_0 \to \mathbb{R} \cup \{+\infty\}$ by

$$f_1(x) := f_0(x) + \frac{\mu_\Omega(x - v_0)^2}{2\lambda^2} \quad \forall x \in X_0. \tag{13}$$

Then, since f_0 is bounded on X_0, one has

$$-\infty < \inf_{x \in X_0} f_1(x) \leq f_1(v_0) = f_0(v_0). \tag{14}$$

Thus, we can choose $v_1 \in X_0$ such that

$$f_1(v_1) \leq \frac{1}{4} f_1(v_0) + (1 - \frac{1}{4}) \inf_{x \in X_0} f_1(x) \leq f_0(v_0). \tag{15}$$

We also note that

$$f_1(v_1) - \inf_{x \in X_0} f_1(x) \leq \frac{1}{4}(f_1(v_0) - \inf_{x \in X_0} f_1(x)) \leq \frac{1}{4} \tag{16}$$

(thanks to (11)). Given $n \in \mathbb{N} := \{1, 2, \cdots\}$, suppose that f_n and $v_n \in X_0$ have been chosen. We define f_{n+1} as follows:

$$f_{n+1}(x) := f_n(x) + \frac{\mu_\Omega(x - v_n)^2}{2^{n+1}\lambda^2} = f_0(x) + \sum_{k=0}^{n} \frac{\mu_\Omega(x - v_k)^2}{2^{k+1}\lambda^2} \quad \forall x \in X_0. \tag{17}$$

Then

$$-\infty < \inf_{x \in X_0} f_{n+1}(x) \leq f_{n+1}(v_n) = f_n(v_n) \tag{18}$$

and hence there exists $v_{n+1} \in X_0$ such that

$$f_{n+1}(v_{n+1}) \leq \frac{1}{4^{n+1}} f_{n+1}(v_n) + (1 - \frac{1}{4^{n+1}}) \inf_{x \in X_0} f_{n+1}(x) \leq f_n(v_n). \tag{19}$$

As a consequence of (17) $\inf_{x \in X_0} f_{n+1}(x) \geq \inf_{x \in X_0} f_n(x)$, and it follows that the sequence $\{f_n(v_n) - \inf_{x \in X_0} f_n(x)\}$ is decreasing. Further, $\inf_{x \in X_0} f_1(x) \geq \inf_{x \in X_0} f_0(x)$ by the definition of f_1 in (13), and one also has $f_1(v_1) \leq f_0(v_0)$. Consequently

$$f_n(v_n) - \inf_{x \in X_0} f_n(x) \leq f_1(v_1) - \inf_{x \in X_0} f_1(x) \leq f_0(v_0) - \inf_{x \in X_0} f_0(x) < 1$$

thanks to (11). Therefore, by (19) and (18), one has

$$\begin{aligned} 0 &\leq f_{n+1}(v_{n+1}) - \inf_{x \in X_0} f_{n+1}(x) \leq \frac{1}{4^{n+1}}(f_{n+1}(v_n) - \inf_{x \in X_0} f_{n+1}(x)) \\ &\leq \frac{1}{4^{n+1}}(f_n(v_n) - \inf_{x \in X_0} f_n(x)) < \frac{1}{4^{n+1}}. \end{aligned} \tag{20}$$

This combined with (16) implies that

$$0 \leq f_n(v_n) - \inf_{x \in X_0} f_n(x) < \frac{1}{4^n} \quad \forall n \in \mathbb{N}. \tag{21}$$

For each $n \in \mathbb{N}$, we define a closed set D_n containing v_n by

$$D_n := \{x \in X_0 : f_n(x) \leq f_n(v_n) + \frac{1}{4^n}\}. \tag{22}$$

Then

$$D_{n+1} \subset D_n \quad \forall n \in \mathbb{N} \tag{23}$$

because of the inequalities $f_{n+1}(v_{n+1}) \leq f_n(v_n)$ and $f_n(x) \leq f_{n+1}(x)$ for all $x \in X_0$ (see (19) and (17)). Next we show that

$$D_n \subset v_{n-1} + \frac{\sqrt{5}\lambda}{2^{\frac{n}{2}}}\Omega \quad \forall n \in \mathbb{N}. \tag{24}$$

For $n = 1$, let $u \in D_1$. Then, by (13) and (15),

$$f_0(u) + \frac{\mu_\Omega(u - v_0)^2}{2\lambda^2} = f_1(u) \leq f_1(v_1) + \frac{1}{4} \leq f_0(v_0) + \frac{1}{4}.$$

(11) implies $\frac{\mu_\Omega(u-v_0)^2}{2\lambda^2} < 1 + \frac{1}{4}$ and therefore $\mu_\Omega(u - v_0) < \frac{\sqrt{5}\lambda}{\sqrt{2}}$; thus $u - v_0 \in \frac{\sqrt{5}\lambda}{\sqrt{2}}\Omega$. This shows that (24) holds when $n = 1$. Let $n \in \mathbb{N}$ and $u \in D_{n+1}$. Then

$$f_{n+1}(u) = f_n(u) + \frac{\mu_\Omega(u - v_n)^2}{2^{n+1}\lambda^2} \leq f_{n+1}(v_{n+1}) + \frac{1}{4^{n+1}}.$$

It follows from (19) and (21) that

$$\begin{aligned} \frac{\mu_\Omega(u - v_n)^2}{2^{n+1}\lambda^2} &\leq f_{n+1}(v_{n+1}) - f_n(u) + \frac{1}{4^{n+1}} \\ &\leq f_n(v_n) - \inf_{x \in X_0} f_n(x) + \frac{1}{4^{n+1}} < \frac{1}{4^n} + \frac{1}{4^{n+1}}. \end{aligned}$$

Therefore,

$$\mu_\Omega(u - v_n) < \frac{\sqrt{5}\lambda}{2^{\frac{n+1}{2}}} \quad \forall n \in \mathbb{N} \text{ and } \forall u \in D_{n+1}. \tag{25}$$

This shows that (24) also holds for all $n \in \mathbb{N} \setminus \{1\}$. Since each D_n is a closed set containing v_n and Ω is bounded and either compact or sequentially complete, it is easy to verify from (23) and (24) that there exists $v \in v_0 + \frac{\sqrt{5}\lambda}{\sqrt{2}}\Omega$ such that

$$v = \lim_{n \to \infty} v_n \text{ and } \bigcap_{n=1}^{\infty} D_n = \{v\}. \tag{26}$$

By (10) and (12), one has $v \in v_0 + \frac{\sqrt{5}}{\sqrt{2}}V \subset U$. Therefore, it suffices to show that φ is Ω-differentiable at v. To do this, let

$$\rho_\infty(x) := \frac{1}{\lambda^2} \sum_{n=1}^{\infty} \frac{\mu_\Omega(x - v_{n-1})^2}{2^n} \quad \forall x \in X.$$

Since Ω is balanced, it is easy from (24) and (26) to verify that $v \in \bigcap_{n=0}^{\infty}(v_n+\sqrt{5}\lambda\Omega)$. Thus, by Proposition 2.4,

$$\lim_{t\to\infty}\frac{\rho_\infty(v+th)+\rho_\infty(v-th)-2\rho_\infty(v)}{t}=0$$

uniformly holds with respect to h in Ω. In consequence, for any $\varepsilon>0$ there exists $\delta\in(0,\ (2-\frac{\sqrt{5}}{\sqrt{2}})\lambda)$ such that

$$\frac{\rho_\infty(v+th)+\rho_\infty(v-th)-2\rho_\infty(v)}{t}<\varepsilon \quad \forall (t,h)\in(0,\ \delta)\times\Omega. \tag{27}$$

We claim that

$$\rho_\infty(v)-\varphi(v)=\inf_{x\in X_0}(\rho_\infty(x)-\varphi(x)). \tag{28}$$

Granting this, one has

$$\frac{\varphi(v+th)+\varphi(v-th)-2\varphi(v)}{t}\le\frac{\rho_\infty(v+th)+\rho_\infty(v-th)-2\rho_\infty(v)}{t} \tag{29}$$

for all $(t,h)\in(0,\ \delta)\times\Omega$ because

$$v\pm th\in v_0+\frac{\sqrt{5}\lambda}{\sqrt{2}}\Omega\pm t\Omega\subset v_0+2\lambda\Omega\subset v_0+2V\subset X_0$$

for all $(t,h)\in(0,\ \delta)\times\Omega$ (thanks to (12), (24) and (26)). Let $x^*\in\partial\varphi(v)$. Then, by the convexity of φ,

$$0\le\frac{\varphi(v+th)-\varphi(v)}{t}-\langle x^*,h\rangle\le\frac{\varphi(v+th)+\varphi(v-th)-2\varphi(v)}{t}$$

for all $(t,h)\in(0,\ \delta)\times\Omega$. It follows from (27) and (29) that

$$0\le\frac{\varphi(v+th)-\varphi(v)}{t}-\langle x^*,h\rangle<\varepsilon \quad \forall (t,h)\in(0,\ \delta)\times\Omega.$$

This shows that φ is Ω-differentiable at v. It remains to show that (28) holds. To show this, suppose to the contrary that there exists $z\in X_0$ such that

$$\rho_\infty(z)-\varphi(z)<\rho_\infty(v)-\varphi(v). \tag{30}$$

By (17) and recalling $\varphi=-f_0$,

$$f_n(z)=-\varphi(z)+\frac{1}{\lambda^2}\sum_{k=1}^{n}\frac{\mu_\Omega(z-v_{k-1})^2}{2^k}<\rho_\infty(v)-\varphi(v) \quad \forall n\in\mathbb{N}. \tag{31}$$

On the other hand, by (26), (22) and (19), one has

$$f_{n+k}(v)\le f_{n+k}(v_{n+k})+\frac{1}{4^{n+k}}\le f_n(v_n)+\frac{1}{4^n} \ \ \forall n,k\in\mathbb{N}.$$

Letting $k \to \infty$, it follows that

$$-\varphi(v) + \rho_\infty(v) = \lim_{k\to\infty} f_{n+k}(v) \le f_n(v_n) + \frac{1}{4^n} \quad \forall n \in \mathbb{N}.$$

Thus, by (22) and (31), we see that $z \in \bigcap_{n=1}^{\infty} D_n = \{v\}$, contradicting (30). This shows that (28) holds. The proof is complete. □

Remark 3.2. The Borwein-Preiss smooth variational principle is fundamental in variational analysis and can be stated as follows (cf. [2, Theorem 2.6]): *Let X be a Banach space, $g\colon X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function, and let the constants $\varepsilon > 0$, $\lambda > 0$ and $p \ge 1$ be given. Suppose that $x_0 \in X$ satisfies $g(x_0) < \inf_{x\in X} g(x) + \varepsilon$. Then there exist $v \in X$ and sequences $\{v_n\}$ in X and $\{\mu_n\}$ in $(0, +\infty)$ with $\sum_{n=1}^{\infty} \mu_n = 1$ such that $\|x_0 - v\| < \lambda$ and*

$$g(x) + \frac{\varepsilon}{\lambda^p} \sum_{n=1}^{\infty} \mu_n \|x - v_n\|^p \ge g(v) + \frac{\varepsilon}{\lambda^p} \sum_{n=1}^{\infty} \mu_n \|v - v_n\|^p \quad \forall x \in X.$$

The proof of Theorem 3.1 uses the main ideas of the proof of [2, Theorem 2.6]. In particular, our proof uses the auxiliary function

$$f_{n+1}(x) := f_n(x) + \frac{\mu_\Omega(x - v_n)^2}{2^{n+1}\lambda^2} = f_0(x) + \sum_{k=0}^{n} \frac{\mu_\Omega(x - v_k)^2}{2^{k+1}\lambda^2}$$

which is only a slight modification of the corresponding one in the proof of [2, Theorem 2.6] (with $\mu_\Omega(x - v_n)$ replacing the norm $\|x - v_n\|$ in [2]). □

In the special case when X is a Fréchet smooth Banach space and Ω is the unit ball of X, Theorem 3.1 reduces to Theorem A in Section 1. With the help of Theorem 3.1, we can extend Mazur's classical theorem on the Gateaux differentiability to general separable locally convex topological vector spaces in place of separable Banach spaces.

Theorem 3.3. *Let X be a separable locally convex topological vector space, G be a nonempty open convex subset of X, and let $\varphi\colon G \to \mathbb{R}$ be a continuous convex function. Then there exists a sequence $\{G_n\}$ of open dense subsets of G such that φ is Gâteaux differentiable at each point of $\bigcap_{n=1}^{\infty} G_n$.*

Proof. By the separability of X, take a sequence $\{v_n\}$ of X such that $\{v_1, \cdots, v_n\}$ is linearly independent for each $n \in \mathbb{N}$ and

$$X = \mathrm{cl}(\mathrm{span}(\{v_n : n \in \mathbb{N}\})). \tag{32}$$

Let

$$\Omega_n := \left\{ \sum_{k=1}^{n} t_k v_k : \sum_{k=1}^{n} t_k^2 \le 1 \right\} \quad \forall n \in \mathbb{N}. \tag{33}$$

Then, by Proposition 2.5, each Ω_n is a compact smooth subset of X. It follows from Theorem 3.1 that for each $n \in \mathbb{N}$ there exists a sequence $\{G_k^n\}$ of open subsets of G such that φ is Ω_n-differentiable at each point of $\bigcap_{k=1}^{\infty} G_k^n$ and G_k^n is dense in G for each $k \in \mathbb{N}$. Therefore, it suffices to show that φ is Gateaux differentiable at each point of $\bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty} G_k^n$. To do this, let $x \in \bigcap_{n=1}^{\infty}\bigcap_{k=1}^{\infty} G_k^n$ and $x_1^*, x_2^* \in \partial\varphi(x)$. Then, φ is Ω_n-differentiable at x, and so

$$\langle x_1^*, h\rangle = \langle x_2^*, h\rangle \quad \forall h \in \Omega_n \text{ and } \forall n \in \mathbb{N}.$$

It follows from (32) and (33) that $x_1^* = x_2^*$, and so $\partial\varphi(x)$ is a singleton. This shows that φ is Gâteaux differentiable at x. The proof is complete.

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