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An open mapping theorem

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1. *Introduction.* Let (E, τ) be a complete, semi-metrizable topological vector space. Let p be a pseudo-norm (not to be confused with a semi-norm, cf. (8), p. 18) inducing the topology τ. For each positive real number r, let

$$V_r = \{x \in E: p(x) \leqslant r\}.$$

Let f be a continuous linear function from E into a topological vector space F. The open mapping theorem of Banach may be stated as follows: If f is nearly open, that is, if the closure $\overline{f(V_r)}$ of each $f(V_r)$ is a neighbourhood of O in F then $f(V_\beta) \supseteq \overline{f(V_\alpha)}$ whenever $\beta > \alpha > O$; in particular, each $f(V_r)$ is a neighbourhood of O. We note that f, identifying with its graph, is a closed linear subspace of the product space $E \times F$. In this paper, we shall employ techniques developed by Kelley (6) and Baker (1) to extend the theorem to the case where f is taken to be a closed cone in $E \times F$. The generalized theorem throws some light onto the duality theory of ordered spaces. In particular, the theorem of Andô–Ellis is generalized to (not assumed, *a priori* to be complete) normed vector spaces.

2. *Main results.* Recall that any subset S of $E \times F$ is called a *relation*. If $A \subseteq E$, we write

$$S(A) = \{y \in F: (a, y) \in S \quad \text{for some} \quad a \in A\}.$$

The inverse relation of S is the set $S^{-1} = \{(y, x) \in F \times E: (x, y) \in S\}$.

THEOREM 1. *Let (E, τ) be a complete, semi-metrizable topological vector space and let p, V_r be as in section* 1. *Let F be a topological vector space. Let S be a closed cone in $E \times F$, and suppose that $\overline{S(V_r)}$ is a neighbourhood of O in F, for each $r > O$. Then, whenever $\beta > \alpha > O$, we have that $S(V_\beta) \supseteq \overline{S(V_\alpha)}$; consequently each $S(V_r)$ is a neighbourhood of O in F.*

Proof. Following Kelley ((6), p. 203), we first show that if $y \in \overline{S(A)}$ for some $A \subseteq E$ then there exists a set B of arbitrary small diameter such that $A \cap B \neq \phi$ and $y \in \overline{S(B)}$. To see this, let $\epsilon > O$. Then $\overline{S(V_\epsilon)}$ is a neighbourhood of O in F hence contains a symmetric neighbourhood W of O. Notice that the neighbourhood $y + W$ of y must intersect $S(A)$. Hence there exist $a \in A$ and $y' \in y + W$ such that $(a, y') \in S$. Let $B = a + V_\epsilon$. Then $A \cap B \neq \phi$, the diameter of B is less than (or equal to) 2ϵ and $y \in \overline{S(B)}$. In fact, since W is symmetric, $y \in y' + W$, so $y = y' + w$ for some $w \in W \subseteq \overline{S(V_\epsilon)}$. There exists a net $\{w_\alpha\}$ in $S(V_\epsilon)$ and hence a net $\{v_\alpha\}$ in V_ϵ such that

$$(v_\alpha, w_\alpha) \in S \quad \text{and} \quad w_\alpha \to w.$$

Since $(a, y') \in S$ and S is a cone, it follows that $w_\alpha + y' \in S(a+v_\alpha) \subseteq S(B)$; passing to the limit, we have $y = w + y' \in \overline{S(B)}$.

We now apply the above observation to complete proof as follows. Let $\beta > \alpha > O$ and let $\epsilon = \beta - \alpha$. Let $y_0 \in \overline{S(V_\alpha)}$. Write A_0 for V_α. By the first part of our proof, there exists a set A_1 of diameter less than $\frac{1}{2}\epsilon$ such that $A_0 \cap A_1 \neq \phi$ and $y_0 \in \overline{S(A_1)}$. Inductively, we can construct a sequence $\{A_n\}$ of sets such that the diameter of A_n is less than $\epsilon/2^n$, $A_n \cap A_{n+1} \neq \phi$ and $y_0 \in \overline{S(A_n)}$. For each $n = O, 1, 2, \ldots$, take $a_n \in A_n \cap A_{n+1}$. Then $\{a_n\}$ is a Cauchy sequence in the complete space E so converges to, say, a. It is easy to see that $a \in V_{\alpha+\epsilon} \equiv V_\beta$. Since the sets A_n are 'small', it is also easy to see that each neighbourhood of a contains A_n for large enough n. Next we show that $(a, y_0) \in S$. In fact let $G \times H$ be a neighbourhood of (a, y_0) in $E \times F$, where G, H are neighbourhoods of a and y_0 respectively. Then, for large n, $A_n \subseteq G$ so $y_0 \in \overline{S(A_n)} \subseteq \overline{S(G)}$; hence the neighbourhood H intersects $S(G)$. Then there exist $h \in H$ and $g \in G$ such that $(g, h) \in S$. This shows that the arbitrary neighbourhood $G \times H$ of (a, y_0) intersects S. Since S is closed, it follows that $(a, y_0) \in \overline{S} = S$. Therefore $y_0 \in S(a) \subseteq S(V_\beta)$, and so $\overline{S(V_\alpha)} \subseteq S(V_\beta)$. The theorem is thus proved.

By considering the inverse relation, we have the following variant of Theorem 1:

THEOREM 1′. *Let E, V_r and F be as in Theorem* 1. *Let T be a closed cone in $F \times E$, and suppose that $\overline{T^{-1}(V_r)}$ is a neighbourhood of O in F for each $r > O$. Then, whenever*

$$\beta > \alpha > O, \quad T^{-1}(V_\beta) \supseteq \overline{T^{-1}(V_\alpha)};$$

consequently each $T^{-1}(V_r)$ is a neighbourhood of O in F.

The above theorem is a numerical version of a theorem of Baker ((1), Theorem 12), and the numerical character is essential for its application to ordered normed spaces (see Theorem 3 below).

COROLLARY 1. *Let E, V_r, and F be as in Theorem* 1, *and suppose further that F is Hausdorff. Let f be a continuous linear function from E into F. If $\overline{f(V_r)}$ is a neighbourhood of O in F for each $r > O$ then $\overline{f(V_\beta)} \supseteq \overline{f(V_\alpha)}$ whenever $\beta > a > O$. Consequently f is an open mapping if* (*and only if*) *f is nearly open.*

Proof. Since f is linear and continuous, f (identifying with its graph) must be a closed cone in $E \times F$.

The corollary is slightly more general than that given in ((8), p. 76) and in ((4), p. 36); Schaefer considers the case when F is metrizable and while Husain considers a non-numerical version in the case when E is Hausdorff.

3. *Applications*. Let $(F, \mathscr{T})$ be a topological vector space with a cone C. Recall that ((5), p. 94) $(F, C, \mathscr{T})$ has the *open decomposition property* (resp. *semi-open decomposition property*) if for each neighbourhood U of O, the set

$$U \cap C - U \cap C \text{ (resp. } \overline{U \cap C - U \cup C})$$

is also a neighbourhood of O. The first stated decomposition property implies that $E = C - C$, and the second only implies $E = \overline{C - C}$. Duhoux (2) has recently shown that if E' is order-convex in the algebraic dual then the two decomposition properties are in fact equivalent.

Suppose $(E, \mathscr{T})$ is semi-metrizable and let p be a pseudo-norm inducing $\mathscr{T}$. For $r > 0$, let $U_r = \{x \in F : p(x) \leqslant r\}$, and

$$U_r^* = U_r \cap C - U_r \cap C.$$

Then the sets U_r^* form a neighbourhood basis at O for a uniquely determined semi-metrizable vector topology $\mathscr{T}_1$ in the linear subspace F_1 spanned by C. For each $x \in F_1$ we define

$$p^*(x) = \inf\{r > O : x \in U_r^*\},$$

and let

$$V_r = \{x \in F_1 : p^*(x) \leqslant r\}.$$

Then $V_\alpha \subseteq U_\beta^* \subseteq V_\beta$ whenever $O < \alpha < \beta$. It follows that p^* is a pseudo-norm inducing the topology $\mathscr{T}_1$ in F_1.

THEOREM 2 (*Jameson* (5), p. 105). *Let $(F, \mathscr{T})$ be a metrizable topological vector space and let C be a $\mathscr{T}$-complete cone in F. Suppose for each $r > 0$, $\overline{U_r \cap C - U_r \cap C}$ is a neighbourhood of O. Then $U_\beta \cap C - U_\beta \cap C$ contains $\overline{U_\alpha \cap C - U_\alpha \cap C}$ and hence is a neighbourhood of O, whenever $\beta > \alpha > O$. In particular, if $(E, C, \mathscr{T})$ has the semi-open decomposition property then it has the open decomposition property.*

Proof. By a theorem of Klee (cf. (8), p. 221), $(F_1, \mathscr{T}_1)$ is a complete metrizable space. Let i be the identity map from F_1 into F. Since $\mathscr{T}_1$ is obviously finer than $\mathscr{T}$, i is continuous (and linear), so its graph is a closed cone in $F_1 \times F$. By assumption each $\overline{i(V_r)}$ is a neighbourhood. Hence we can apply Theorem 1 to conclude that $i(V_\beta) \supseteq \overline{i(V_\alpha)}$ whenever $\beta > \alpha > 0$. Hence, if $\beta > \gamma > \alpha$.

$$i(U_\beta^*) \supseteq i(V_\gamma) \supseteq \overline{i(V_\alpha)} \supseteq \overline{i(U_\alpha^*)},$$

i.e. $U_\beta \cap C - U_\beta \cap C \supseteq \overline{U_\alpha \cap C - U_\alpha \cap C}$ in the space $(F, \mathscr{T})$.

The next application of Theorem 1 is concerned with ordered normed spaces. Let $(X, \|.\|)$ be a normed space ordered by a cone C. Let Σ denote the closed unit ball in X and let α be a positive real number. Following (3), we say that C is

(i) *α-normal* if $(\Sigma + C) \cap (\Sigma - C) \subseteq \alpha\Sigma$,

(ii) *α-generating* if $\Sigma \subseteq \alpha . \operatorname{co}(\Sigma^+ \cup -\Sigma^+)$, where $\Sigma^+ = \Sigma \cap C$.

A theorem of Krein–Grosberg states that if C is α-generating then the dual cone C' is α-normal in the Banach dual X'. Conversely, Ellis (3) has shown that if X and C are complete and if C' is α-normal then C is $(\alpha + \epsilon)$-generating for each $\epsilon > O$.

THEOREM 3. *Let $(X, \| \|)$ be a normed space and let C be a norm-complete cone in X. Suppose the dual cone C' is α-normal in the Banach dual space $(X', \| \|)$ for some $\alpha > O$. Then C is $(\alpha + \epsilon)$-generating for each $\epsilon > O$, and $(X, \|.\|)$ is complete.*

Proof. Let Σ' denote the closed unit ball in X'. Then, since C is closed in X, it is easy to verify that (via the Hahn–Banach theorem),

$$(\Sigma' + C')^\pi = \Sigma'^\pi \cap C'^\pi = \Sigma \cap (-C) = -\Sigma^+$$

where the polars are taken in X. By the Alaoglu theorem, $\Sigma' + C'$ and $\Sigma' - C'$ are $\sigma(X', X)$-closed convex sets. Also since C' is α-normal,

$$(\Sigma' + C') \cap (\Sigma' - C') \subseteq \alpha\Sigma',$$

it follows from the bipolar theorem that

$$(1/\alpha)\Sigma \subseteq [(\Sigma'+C')\cap(\Sigma'-C')]^{\pi} = \overline{\mathrm{co}}((\Sigma'+C')^{\pi}\cup(\Sigma'-C')^{\pi}) = \overline{\mathrm{co}}\,(-\Sigma^{+}\cup\Sigma^{+}). \quad (1)$$

Now let us consider the subspace $X_1 = C-C$ endowed with the semi-norm p, the Minkowski functional of $\mathrm{co}\,(\Sigma^{+}\cup -\Sigma^{+})$. Since C is $\|.\|$-complete, (X_1, p) is complete by Klee's theorem (cf. (8), p. 221). Let $V_r = \{x\in X_1: p(x) \leqslant r\}$, and let i be the identity map from (X_1, p) into $(X, \|.\|)$. By (1), $(1/\alpha)\Sigma \subseteq \overline{i(V_1)}$. Hence each $\overline{i(V_r)}$ is a neighbourhood of O in $(X, \|.\|)$. Applying Theorem 1, we conclude that

$$i(V_\beta) \supseteq \overline{i(V_\alpha)} \quad \text{whenever} \quad \beta > \alpha > O.$$

Write D for $\mathrm{co}\,(\Sigma^{+}\cup -\Sigma^{+})$ then $D \subseteq V_1$ and $V_\alpha \subseteq \beta D$ whenever $\beta > \alpha > O$. Hence it follows that

$$\overline{\mathrm{co}}\,(-\Sigma^{+}\cup\Sigma^{+}) = \overline{i(D)} \subseteq \overline{i(V_1)} \subseteq i(V_{1+\epsilon}) \subseteq i((1+2\epsilon)\,D) \quad (\epsilon > O),$$

and from (1) we conclude that

$$\frac{1}{1+2\epsilon}\cdot\frac{1}{\alpha}\Sigma \subseteq D = \mathrm{co}\,(\Sigma^{+}\cup -\Sigma^{+}).$$

Therefore C is $\alpha(1+2\epsilon)$-generating for each $\epsilon > O$, equivalently C is $(\alpha+\epsilon)$-generating for each $\epsilon > O$. Furthermore, i is continuous from (X_1, p) onto $(X, \|.\|)$, and is open by Corollary of Theorem 1. Therefore $(X, \|.\|)$ is homeomorphic to (X_1, p) and hence complete.

4. *Continuous maps from order-infrabarrelled spaces.* We first recall that an ordered locally convex topological space E is called an *order-infrabarrelled space* if each barrel in E which absorbs all order-intervals is a neighbourhood of the origin. A barrelled space with a cone must be order-infrabarrelled but the converse is not true as shown in (7).

Now let E, F be two ordered topological vector spaces and suppose the positive cone F^{+} in F is closed. Let t be a sublinear function from E into F and let T be the 'subgraph' of t:

$$T = \{(x,y)\in E\times F: t(x) \leqslant y\}.$$

Since t is sublinear, it is easily seen that T is a cone in $E\times F$. Further, if t is continuous, then T is closed. In fact, suppose $\{(x_\alpha, y_\alpha)\}$ is a net in T convergent to (x_0, y_0) in $E\times F$. Then $y_\alpha - t(x_\alpha) \leqslant O$ in F. Passing to the limit, since F^{+} is closed and t is continuous, it follows that $y_0 - t(x_0) \geqslant O$, i.e. $(x_0, y_0)\in T$. A partial converse of this observation is given in the following theorem:

THEOREM 4. *Let E be an order-infrabarrelled space with the open decomposition property, and let F be a complete, metrizable locally convex topological vector space with a normal cone. Let t be a monotonic and sublinear function from E into F, and let*

$$T = \{(x,y)\in E\times F: t(x) \leqslant y\}.$$

If T is closed then t is continuous.

Proof. When we say the cone in F is normal we mean the order-convex and convex neighbourhoods of O in F form a local basis. In this case, to show that the sublinear map t is continuous it is sufficient to show that t is continuous at the origin O (1). Take an *order-convex* and *circled* convex neighbourhood V of O in F. We have to show that $t^{-1}(V)$ is a neighbourhood of O in E. Let E^+ denote the positive cone in E and consider the set $t^{-1}(V) \cap E^+$. If x, y are in $t^{-1}(V) \cap E^+$ and if $O \leqslant \lambda$, $\mu \leqslant 1$ with $\lambda + \mu = 1$ then $O \leqslant t(\lambda x + \mu y) \leqslant \lambda t(x) + \mu t(y) \in V$; so $t(\lambda x + \mu y) \in V$ and $\lambda x + \mu y \in t^{-1}(V) \cap E^+$. This shows that $t^{-1}(V) \cap E^+$ is convex. Notice also that it absorbs each positive element in E. Next, let

$$U = t^{-1}(V) \cap E^+ - t^{-1}(V) \cap E^+.$$

Then U is a symmetric convex set in E. Also t sends U into V. In fact, if $u = x_1 - x_2 \in U$ and $x_1, x_2 \in t^{-1}(V) \cap E^+$ then since t is monotonic and sublinear, we have

$$-t(x_2) \leqslant t(x_1) - t(x_2) \leqslant t(x_1 - x_2) \leqslant t(x_1).$$

Since V is symmetric and order-convex and since $t(x_1), t(x_2) \in V$, it follows that

$$t(u) = t(x_1 - x_2) \in V.$$

This shows that $u \in t^{-1}(V)$ and so that $U \subseteq t^{-1}(V)$. Further, since E has the open decompositon property, $E = E^+ - E^+$, so U absorbs every element of E. Finally, we show that U absorbs all intervals in E. To this end, let $I = [O, x]$ be an order-interval in E, where $x \geqslant O$. Then $t(x) \geqslant O$. Since V is a neighbourhood of O, we can find $M > O$ such that $t(x) \in MV$. Since t is monotonic and V is order-convex, we have

$$t(I) \subseteq [O, t(x)] \subseteq MV.$$

Hence $I \subseteq M(t^{-1}(V) \cap E^+) \subseteq MU$. This shows that U absorbs I. Consequently, since E^+ is generating in E, U absorbs all order-intervals (cf. (5), p. 132). We have shown that the closure $\overline{U}$ is a barrel in the order-infrabarrelled space E, and absorbs all intervals; hence $\overline{U}$ is a neighbourhood of O in E. Since $T^{-1}(V) \supseteq t^{-1}(V) \supseteq U$, it follows that $\overline{T^{-1}(V)}$ is a neighbourhood of O in E. Since V is arbitrary, it follows from Theorem 1 (or Theorem 1′) that $T^{-1}(V)$ is a neighbourhood of O in E. Let

$$W = T^{-1}(V) \cap E^+ - T^{-1}(V) \cap E^+.$$

Since E has the open decomposition property, W is also a neighbourhood of O in E. To complete our proof, we show that $W \subseteq t^{-1}(V)$. To see this, let $w = w_1 - w_2 \in W$, where w_1, w_2 are in $T^{-1}(V) \cap E^+$. Then, for $i = 1, 2$, there exists $v_i \in V$ such that $(w_i, v_i) \in T$. Hence $O \leqslant t(w_i) \leqslant v_i$ for each i, and

$$-v_2 \leqslant -t(w_2) \leqslant t(w_1) - t(w_2) \leqslant t(w) = t(w_1 - w_2) \leqslant t(w_1) \leqslant v_1.$$

Since V is symmetric and order-convex, it follows that $t(w) \in V$. Therefore $W \subseteq t^{-1}(V)$.

Remark. The theorem was proved by Baker (1) in the special case when E is a barrelled space and when the monotonic sublinear function t satisfies some additional properties. Our proof is modified from that given by Baker. As pointed out in ((1), p. 244), theorems of this kind are essentially generalizations of the following well-known theorem: if F is a Frechet space (i.e. a complete and metrizable locally convex space) and a vector lattice with a closed and normal cone then the lattice operations in F are continuous.

Finally, following suggestions of ((9), p. 203) and ((1), p. 243), we conclude with an application of Theorem 4 to a study of FK spaces. Let F be an FK space (cf. (9), p. 202). Then F is a subspace of the sequence space s and so is naturally ordered by the relative cone $F^+ = F \cap s^+$.

THEOREM 5. *Let E be as in Theorem 4, F an FK space, and suppose that the cone F^+ is normal in F. Let t be a monotonic and sublinear function from E into F. Then t is continuous if and only if it is continuous as a map into s.*

Proof. Suppose that t is continuous as a map from E into s. Then it is not difficult to verify that the subgraph T of t is closed in $E \times F$, hence t is continuous into F by Theorem 4. This proves one half of the theorem; the other half is obvious since the topology in F is finer than that of s.

COROLLARY. *Let E, F be FK spaces, and suppose that the cone E^+ is generating in E and F^+ is normal in F. Then any monotonic and sublinear function from E into F is continuous.*

Proof. It is well known that a closed and generating cone in a Fréchet space gives the open decomposition (cf. (5), p. 105). Thus E satisfies the conditions in Theorem 4. Since the topology in E is always finer than that of s, t is continuous from E into s. Thus Theorem 5 can be applied.

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