MATH4050 Real Analysis
Assignment 2

There are 10 questions in this assignment. The page number and question number for each question correspond to that in Royden's Real Analysis, 3rd or 4th edition.

In this assignment, $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers. E is a subset of $\mathbb{R}$. $x \in \mathbb{R}$ is called a point of closure of E if each neighbourhood of x intersects E.

1. * (3rd: P.39, Q12)
   Show that $x = \lim x_n$ if and only if every subsequence of $\{x_n\}$ has in turn a subsequence that converges to x.
   Q: How about $x \in \{-\infty, \infty\}$?

2. (3rd: P.39, Q13)
   Show that the real number l is the limit superior of the sequence $\{x_n\}$ if and only if (i) given $\varepsilon > 0$, $\exists n$ such that $x_k < l + \varepsilon$ for all $k \geq n$, and (ii) given $\varepsilon > 0$ and n, $\exists k \geq n$ such that $x_k > l - \varepsilon$.

3. * (3rd: P.39, Q14)
   Show that $\limsup x_n = \infty$ if and only if given Δ and n, $\exists k \geq n$ such that $x_k > \Delta$.

4. (3rd: P.39, Q15)
   Show that $\liminf x_n \leq \limsup x_n$ and $\liminf x_n = \limsup x_n = l$ if and only if $l = \lim x_n$.

5. * (3rd: P.39, Q16)
   Prove that
   $$\limsup x_n + \liminf y_n \leq \limsup (x_n + y_n) \leq \limsup x_n + \limsup y_n.$$
   provided the right and left sides are not of the form $\infty - \infty$.

6. (3rd: P.39, Q17)
   Prove that if $x_n > 0$ and $y_n \geq 0$, then
   $$\limsup (x_n y_n) \leq (\limsup x_n)(\limsup y_n)$$
   provided the product on the right is not of the form $0 \cdot \infty$.

7. (3rd: P.46, Q27)
   Show that x is a point of closure of E if and only if there is a sequence $\{y_n\}$ with $y_n \in E$ and $x = \lim y_n$.

8. (3rd: P.46, Q28; 4th: P.20, Q30(i))
   A number x is called an **accumulation point** of a set E if it is a point of closure of $E \setminus \{x\}$. Show that the set E' of accumulation points of E is a closed set.

9. (3rd: P.46, Q29; 4th: P.20, Q30(ii))
   Show that $\overline{E} = E \cup E'$.

10. (3rd: P.46, Q30; 4th: P.20, Q31)
    A set E is called **isolated** if $E \cap E' = \phi$. Show that every isolated set of real numbers is countable.