1. Let $p > 0$. Define $x \sim y$ if $x - y$ is divisible by p: $x - y = mp$ for some $m \in \mathbb{Z}$ (this relation is an "equivalence relation": reflexive, transitive and symmetric.)

Show that, $\forall x \in \mathbb{R}$, there exists only and only one $\tilde{x} \in (0, p]$ s.t. $x \sim \tilde{x}$ [Hint: Let m be the largest integer such that $x + mp \leq 0$; set $\tilde{x} := x + (m+1)p$.]

2. Let $f : \mathbb{R} \to \mathbb{R}$ be cts and be periodic with period $p > 0$: $f(x+p) = f(x)$ $\forall x \in \mathbb{R}$. Show that f is uniformly cts.

3 Let $f : \mathbb{R} \to \mathbb{R}$ be cts and suppose that there exist $a \in \mathbb{R}$ s.t.

(i) f is Lipschitz on $(-\infty, a]$ with Lip-constant K:
$|f(x) - f(y)| \leq K|x - y|$ $\forall x, y \in (-\infty, a]$;

(ii) $\lim_{x \to +\infty} f(x)$ exists in $\mathbb{R}$.

Show that f is unif cts on $\mathbb{R}$.

Hint for Q2: Let $\varepsilon > 0$. By Unif. Continuity Th (applied to $[-p, 2p]$) to get $\delta > 0$. Then, if $x, y \in \mathbb{R}$ with $|x - y| < \delta \wedge p$ one has $|f(x) - f(y)| < \varepsilon$: by Q1, $\tilde{x} = x + mp \in (0, p]$ with some $m \in \mathbb{Z}$. Set $\tilde{y} := y + mp$, and note $f(\tilde{x}) - f(\tilde{y}) = f(x) - f(y)$ & $\tilde{y} \in (-p, 2p]$