THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 4050 Real Analysis

Tutorial 2 (March 1)

The following were discussed in the tutorial this week.

1. In the following discussion, let $E \subseteq \mathbb{R}$ be a measurable set and let f_n, f be measurable functions on E.

2. Consider the following modes of convergence:

   (I) We say that $f_n \to f$ **pointwise almost everywhere** if there exists $A \subseteq E$ such that $m(E \setminus A) = 0$ and $f_n(x) \to f(x)$ for all $x \in A$.

   (II) We say that $f_n \to f$ **in measure** if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $m(\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}) < \varepsilon$.

   (III) We say that $f_n \to f$ in L^1 if $\|f_n - f\|_1 \to 0$, where $\|g\|_1 := \int_E |g|$.

3. (II) is equivalent to any of the following conditions:

   (i) For any $\varepsilon, \eta > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $m(\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}) < \eta$.

   (ii) For any $\varepsilon > 0$, $\lim_n m(\{x \in E : |f_n(x) - f(x)| \geq \varepsilon\}) = 0$.

4. The three modes of convergence do not imply the other, except that (III) $\implies$ (II).

5. If $f_n \to f$ in measure, then (f_n) has a subsequence (f_{n_k}) such that $f_{n_k} \to f$ almost everywhere.

6. Suppose that $m(E) < \infty$. If $f_n \to f$ almost everywhere, then $f_n \to f$ in measure.

7. (I) $\implies$ (III) and (II) $\implies$ (III) if (f_n) is "dominated" by an integrable function:

   (**Lebesgue Convergence Theorem** or **Dominated Convergence Theorem**)

   Suppose g is an integrable function on E such that $|f_n| \leq g$ on E. If $f_n \to f$ almost everywhere (or in measure), then $f_n \to f$ in L^1.

8. Even though the three modes of convergence are different, they are compatible in the following sense:

   If $f_n \to f$ along one mode of convergence, and $f_n \to g$ along another mode of convergence at the same time, then $f = g$ almost everywhere.