Suggested Solution of Assignment 7

1. Using the Structure Theorem for Open Sets (and check by epsilon-delta terminology) show that each continuous function f on a closed set F in $\mathbb{R}$ can be continuously extended to be on the whole of $\mathbb{R}$ (this is known as the Tietze extension Theorem).

   Solution. By the Structure Theorem for Open Sets, the open set $G := \mathbb{R} \setminus F$ can be expressed as $G = \dot{\bigcup}_{n=1}^{\infty} I_n$, where I_n's are countable disjoint open intervals. Now let $g : \mathbb{R} \to \mathbb{R}$ be linear on each $\overline{I}_n$ and $g(x) = f(x)$ for all $x \in F$. By HW6 Q5, g is a continuous function and $g\big|_F = f$. ◀

2. Let f be a measurable real-valued function on a set E of finite measure. Show that there exists a sequence of continuous functions convergent to f almost everywhere on E. Hence, for any $r > 0$, there exists a closed set F contained in E with $m(E \setminus F) < r$ such that the above convergence is uniform on F and the restriction of f to F is continuous.

   Solution. By Littlewood's Second Principle, for each $n \in \mathbb{N}$, there exists a continuous function $f_n : \mathbb{R} \to \mathbb{R}$ and $A_n \subseteq E$ such that $m(A_n) < 2^{-n}$ and

   $$|f_n(x) - f(x)| < 2^{-n} \quad \text{for all } x \in E \setminus A_n.$$

   Let $A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$. Then $m(A) = \lim_n m(\bigcup_{k=n}^{\infty} A_k) = 0$ (since $m(\bigcup_{k=n}^{\infty} A_k) \leq 2^{-n+1}$), and for all $x \in E \setminus A$, there exists $N_x \in \mathbb{N}$ such that

   $$|f_n(x) - f(x)| < 2^{-n} \quad \text{for all } n \geq N_x.$$

   Hence (f_n) converges to f a.e. on E.

   Let $r > 0$. By Egoroff's Theorem, there exists a measurable set $B \subseteq E$ such that $m(E \setminus B) < r/2$ and

   $$f_n \to f \text{ uniformly on } B.$$

   By Littlewood's First Principle, we can find a closed set $F \subseteq B$ such that $m(B \setminus F) < r/2$. Now F is closed, $F \subseteq E$, $m(E \setminus F) < r/2 + r/2 = r$ and $f_n \to f$ uniformly on F. As the uniform limit of a sequence of continuous functions, $f\big|_F$ is also continuous.

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3. Let f be a ~~continuous~~ measurable real-valued function on a measurable set E of possibly infinite measure, and let $r > 0$. Apply Q2 to get a corresponding closed set F_n contained in the intersection of E with $(n, n+1]$ for each integer n. Show that the union F of F_n is closed and that the restriction of f to F is continuous. Moreover we can arrange in such a way that $m(E \setminus F) < r$.

   Solution. Fix $n \in \mathbb{Z}$. By Q2, we can find a closed set $F_n \subseteq E_n := E \cap (n, n+1]$ such that $m(E_n \setminus F_n) < r \cdot 2^{-|n|-2}$ and $f\big|_{F_n}$ is continuous. Let $F = \bigcup_{n \in \mathbb{Z}} F_n$. Then it follows from HW6 Q3 that $f\big|_F$ is continuous. Moreover,

   $$m(E \setminus F) = m\left(\bigcup_{n \in \mathbb{Z}} (E_n \setminus F_n)\right) \leq \sum_{n \in \mathbb{Z}} m(E_n \setminus F_n) < r \sum_{n \in \mathbb{Z}} 2^{-|n|-2} < r.$$

It remains to check that F is closed. Indeed, since each F_n is closed and $F_n \subseteq (n, n+1]$, $F_n \subseteq [n+\delta_n, n+1]$ for some $\delta_n > 0$. Hence $F \cap \bigcup_{n\in\mathbb{Z}}(n, n+\delta_n) = \emptyset$. Thus

$$\mathbb{R} \setminus F = \bigcup_{n\in\mathbb{Z}} ((n, n+1] \setminus F_n) = \bigcup_{n\in\mathbb{Z}} ((n, n+1+\delta_{n+1}) \setminus F_n),$$

which is open. ◀

4. Let f be a non-negative extended real function on a measurable set E. Show that the sequence (f_n) of simple functions monotonically increases and converges point-wisely to f, where

$$f_n := \sum_{k=1}^{n\cdot 2^n} \frac{k-1}{2^n}\chi_{B_{n,k}} + n\chi_{A_n},$$
$$A_n = \{x \in E \cap [-n, n] : n \le f(x)\},$$
$$B_{n,k} = \left\{x \in E \cap [-n, n] : \frac{k-1}{2^n} \le f(x) < \frac{k}{2^n}\right\}, \quad k = 1, \ldots, n\cdot 2^n.$$

Solution. Since

$$B_{n,k} \subseteq B_{n+1,2k-1} \cup B_{n+1,2k} \qquad \text{for } n \in \mathbb{N},\ k = 1, \ldots, n\cdot 2^n$$

and

$$A_n \subseteq A_{n+1} \cup \bigcup_{k=(n+1)\cdot 2^n+1}^{(n+1)\cdot 2^{n+1}} B_{n+1,k} \qquad \text{for } n \in \mathbb{N},$$

the sequence (f_n) is clearly monotonically increasing.

If $f(x) < +\infty$, choose $N \in \mathbb{N}$ such that $0 \le f(x) < N$. Now, for $n \ge N$, we have $0 \le f(x) - f_n(x) \le \frac{1}{2^n}$, so that $\lim_n f_n(x) = f(x)$.

If $f(x) = +\infty$, then $f_n(x) = n \to +\infty = f(x)$, as $n \to \infty$.

Hence the sequence (f_n) converges point-wisely to f. ◀

5. Let $m(E) \le +\infty$, and $\mathcal{L}(E)$ consist of all measurable functions f on E such that $\int_E |f| < +\infty$. Let $f_n, g_n, f, g \in \mathcal{L}(E)$ be such that $f_n \to f, g_n \to g$ and $|f_n| \le g_n \,\forall\, n$ (all are pointwise or a.e. on E). Suppose further that $\lim_n \int_E g_n = \int_E g$. Show that $\lim_n \int_E |f_n - f| = 0$.

Solution. By Triangle inequality,

$$|f_n - f| \le |f_n| + |f| \le g_n + |f| \quad \text{a.e. on } E,$$

so that $g_n + |f| - |f_n - f| \ge 0$ a.e. on E. By the assumptions, $\lim_n (g_n + |f| - |f_n - f|) = g + |f|$. Using Fatou's lemma, we have

$$\begin{aligned}\int_E g + \int_E |f| &= \int_E (g + |f|) \\ &\le \liminf_n \int_E (g_n + |f| - |f_n - f|) \\ &= \int_E g + \int_E |f| - \limsup_n \int_E |f_n - f|.\end{aligned}$$

As $\displaystyle\int_E |f|, \int_E g < +\infty$, we have $\displaystyle\limsup_n \int_E |f_n - f| \le 0$, and thus $\displaystyle\lim_n \int_E |f_n - f| = 0$. ◀

6. Let $f_n, f \in \mathcal{L}(E)$ and $f_n \to f$ a.e. on E. Suppose $\displaystyle\lim_n \int_E |f_n| = \int_E |f|$. Show that $\displaystyle\lim_n \int_E f_n = \int_E f$.

   Solution. Take $g_n := |f_n|$ and $g := |f|$. Then clearly $g_n \to g$ a.e. on E and $|f_n| \le g_n$ on E for all n. It follows from Q5 that $\displaystyle\lim_n \int_E |f_n - f| = 0$. Now $\displaystyle\lim_n \int_E f_n = \int_E f$ since

$$\left| \int_E f_n - \int_E f \right| \le \int_E |f_n - f| \,.$$

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