Suggested Solution of Assignment 6

1. Show that the uniform limit of a sequence of continuous functions is continuous: Let (f_n) be a sequence of continuous functions on X ($\subseteq \mathbb{R}$, or a topological space) such that $\forall\, \varepsilon > 0$ $\exists N \in \mathbb{N}$ satisfying

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall\, n \geq N, \text{ and } \forall\, x \in X.$$

Show that f is continuous at each $x_0 \in X$.

Solution. Let $\varepsilon > 0$ be given. Choose $N \in \mathbb{N}$ such that

$$|f_N(x) - f(x)| < \varepsilon/3 \qquad \forall\, x \in X.$$

Since f_N is continuous at x_0, there exists $\delta > 0$ such that

$$|f_N(x) - f_N(x_0)| < \varepsilon/3, \qquad \text{whenever } x \in X \text{ and } |x - x_0| < \delta.$$

Now, if $x \in X$ and $|x - x_0| < \delta$, we have

$$\begin{aligned} |f(x) - f(x_0)| &\leq |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Hence f is continuous at x_0. ◀

2. Let $F = \dot{\bigcup}_{n=1}^{N} F_n$, a disjoint union of closed sets $F_1, \ldots, F_N$. Let $f : F \to \mathbb{R}$ be such that $f\big|_{F_n}$ is continuous, $\forall\, n$. Show that f is continuous.

Solution. Let $c \in F$. Without loss of generality, we assume that $c \in F_1$. We shall show that f is continuous at c using sequential criterion. Let (x_n) be a sequence in F that converges to c. Then $x_n \in F_1$ for all but finitely many $n \in \mathbb{N}$. For otherwise, (x_n) has a subsequence (x_{n_k}) that is contained in F_j, $j \neq 1$. Now, since F_j is closed, we have $c \in F_j$, contradicting the fact that $F_1, \ldots, F_N$ are disjoint. Therefore, by the continuity of $f\big|_{F_1}$, $\lim\,(f(x_n)) = \lim\left(f\big|_{F_1}(x_n)\right) = f\big|_{F_1}(c) = f(c)$. ◀

3. Let $F_n \subseteq (n, n+1]$ be closed ($\mathbb{R} \setminus F_n$ open) $\forall\, n \in \mathbb{N}$, and let $F = \dot{\bigcup}_{n \in \mathbb{N}} F_n$. Show that $f : F \to \mathbb{R}$ is continuous if each $f\big|_{F_n}$ is continuous. (Can the condition $F_n \subseteq (n, n+1]$ be weakened to $F_n \subseteq \mathbb{R}$?)

Solution. Suppose $c \in F_{n-1} \subseteq (n-1, n]$ for some $n \geq 2$. Since F_n is closed and bounded, hence compact, there exists $\delta_n \in (0, 1)$ such that

$$x - n \geq \delta_n \qquad \text{for all } x \in F_n.$$

Hence, $|x - c| \geq \delta_n$ for all $x \in F \setminus F_{n-1}$. Now any sequence (x_k) in F that converges to c must be contained in F_{n-1} eventually. It then follows from the continuity of $f\big|_{F_{n-1}}$ that

$$\lim\,(f(x_k)) = \lim\left(f\big|_{F_{n-1}}(x_k)\right) = f\big|_{F_{n-1}}(c) = f(c).$$

Therefore, f is continuous at c.

The result above is not true if the condition is weakened. For example, let $\{p_n/q_n : n \in \mathbb{N}\}$ be an enumeration of $\mathbb{Q}^+$, where $p_n, q_n \in \mathbb{N}$ are relatively prime and define $F_n = \{p_n/q_n\}$, $f\big|_{F_n}(x) = (-1)^{q_n}$. Then clearly $F = \bigcup_{n \in \mathbb{N}} F_n$ is a disjoint union of closed sets and each $f\big|_{F_n}$ is continuous. However, f is discontinuous everywhere on F. ◀

4. Let $G = \dot{\bigcup}_{n=1}^{\infty} I_n$, countable disjoint open intervals I_n, and let $F := \mathbb{R} \setminus G$. Let $x < y < z$ with $x, z \in F$ and $y \in I_n := (a_n, b_n)$. Show that $a_n \in F, b_n \in F, x \le a_n$ and $b_n \le z$.

   Solution. Since I_m's are disjoint open intervals and $a_n, b_n \notin I_n$, we have $a_n, b_n \notin G = \bigcup_{m=1}^{\infty} I_m$. Hence $a_n, b_n \in \mathbb{R} \setminus G = F$.

   Suppose $x > a_n$. Then $a_n < x < y < b_n$, so that $x \in I_n \subseteq G$, contradicting $x \in F = \mathbb{R} \setminus G$. Therefore $x \le a_n$. Similarly, one can show $b_n \le z$. ◀

5. Let G, I_n, F be as in Q4, and let $f : \mathbb{R} \to \mathbb{R}$ be such that $f\big|_F$ and $f\big|_{\overline{I}_n}$ are continuous, $\forall\, n$ ($\overline{I}_n$ denotes the closure of I_n). Suppose further that the graph of $f\big|_{\overline{I}_n}$ is a line-segment. Show that f is continuous. (By symmetry, need only show that f is right-continuous at each $x_0 \in \mathbb{R}$: $\lim_{x \to x_0+} f(x) = f(x_0)$, i.e. $\forall\, \varepsilon > 0\ \exists\, \delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ $\forall\, x \in (x_0, x_0 + \delta)$. This is evident if $x_0 \in G$ (so $\exists n \in \mathbb{N}$ such that $x_0 \in I_n$). We may hence assume that $x_0 \in F$, and there are three cases to consider:

   (a) $\exists\, \delta > 0$ such that $(x_0, x_0 + \delta) \subseteq F$ (so $[x_0, x_0 + \delta] \subseteq F$)

   (b) $\exists\, \delta > 0$ such that $(x_0, x_0 + \delta) \subseteq G$ (so $(x_0, x_0 + \delta) \subseteq I_n$ for some n)

   (c) $(x_0, x_0 + \delta)$ intersects F and G, $\forall\, \delta > 0$.)

Hint: For case (a), you use the continuity of $f\big|_F$.

For case (b), you use the continuity of $f\big|_{[x_0, x_0+\delta]}$.

For case (c), let $\varepsilon > 0$, $\exists\, \delta_0 > 0$ such that $|f(x) - f(x_0)| < \varepsilon\ \forall\, x \in F \cap [x_0, x_0 + \delta]$ as $f\big|_F$ is continuous at x_0. By the assumption in case (c) and consider smaller $\delta_0 > 0$ if necessary, we may assume that $x_0 + \delta \in F$. Show that if $x \in G \cap (x_0, x_0 + \delta)$, then $\exists!\, n \in \mathbb{N}$ with $x \in (a_n, b_n)$. Since $x_0, x_0 + \delta_0 \in F$, one has (?) $x_0 \le a_n < x < b_n \le x_0 + \delta_0$ and $a_n, b_n \in F$, $|f(\cdot) - f(x_0)| < \varepsilon$ at a_n, b_n and so at x.

Solution. (a) Since $[x_0, x_0 + \delta] \subseteq F$, it follows from the continuity of $f\big|_F$ that

$$\lim_{x \to x_0+} f(x) = \lim_{x \to x_0+} f\big|_F(x) = f\big|_F(x_0) = f(x_0).$$

(b) $(x_0, x_0 + \delta) \subseteq I_n$ implies that $[x_0, x_0 + \delta] \subseteq \overline{I}_n$. It follows from the continuity of $f\big|_{\overline{I}_n}$ that

$$\lim_{x \to x_0+} f(x) = \lim_{x \to x_0+} f\big|_{\overline{I}_n}(x) = f\big|_{\overline{I}_n}(x_0) = f(x_0).$$

(c) Let $\varepsilon > 0$ and choose $\delta_0 > 0$ as in the hint and assume that $x_0 + \delta \in F$.
Suppose $x \in G \cap (x_0, x_0 + \delta)$. Since $G = \bigcup_{n=1}^{\infty} I_n$ is a disjoint union, there exists a unique $n \in \mathbb{N}$ such that $x \in I_n := (a_n, b_n)$. As $x_0, x_0 + \delta_0 \in F$, Q4 implies that

$$x_0 \le a_n < x < b_n \le x_0 + \delta_0 \text{ and } a_n, b_n \in F.$$

Since the graph of $f\big|_{[a_n, b_n]}$ is a line segment, it follows that

$$|f(x) - f(x_0)| \le \max\{|f(a_n) - f(x_0)|, |f(b_n) - f(x_0)|\} < \varepsilon.$$

Combining the estimates, we have

$$|f(x) - f(x_0)| < \varepsilon \qquad \text{whenever } x \in (x_0, x_0 + \delta).$$

◀