Suggested Solution of Assignment 4

1.* (Montone Convergence Lemmas for measures) Let $m : \widetilde{\mathcal{M}} \to [0, +\infty]$ be a measure, where $\widetilde{\mathcal{M}}$ is an arbitrary σ-algebra. Show that

(a) if $E_n \uparrow E$ (with each $E_n \in \widetilde{\mathcal{M}}\ \forall n$), i.e. $E = \bigcup_{n\in\mathbb{N}} E_n$ and $E_n \subseteq E_{n+1}\ \forall n$, then $m(E_n) \uparrow m(E)$;

(b) if $E_n \downarrow E$ (with each $E_n \in \widetilde{\mathcal{M}}\ \forall n$), i.e. $E = \bigcap_{n\in\mathbb{N}} E_n$ and $E_n \supseteq E_{n+1}\ \forall n$, then $m(E_n) \downarrow m(E)$ provided that $m(E_0) < +\infty$ for some n_0. Provide a counter-example if the added condition is dropped.

Solution. (a) Clearly $\{m(E_n)\}_{n=1}^{\infty}$ is increasing. Let $B_1 = E_1$, $B_n = E_n \backslash E_{n-1}$ for $n \geq 2$. Then $\{B_n\}_{n=1}^{\infty}$ is a collection of pairwise disjoint sets in $\widetilde{\mathcal{M}}$ such that $\bigcup_{n=1}^{N} B_n = E_N$ for all N. Hence, by the countable additivity of m,

$$
\begin{aligned}
m(E) &= m(\bigcup_{n=1}^{\infty} E_n) = m(\bigcup_{n=1}^{\infty} B_n) \\
&= \sum_{n=1}^{\infty} m(B_n) = \lim_{N\to\infty} \sum_{n=1}^{N} m(B_n) \\
&= \lim_{N\to\infty} m(E_N).
\end{aligned}
$$

(b) Clearly $\{m(E_n)\}_{n=1}^{\infty}$ is decreasing. Without loss of generality, we may assume that $m(E_1) < +\infty$. Define $A_n = E_1 \setminus E_n$. Then $A_n \subseteq A_{n+1}$ and $\bigcup_{n=1}^{\infty} A_n = E_1 \setminus (\bigcap_{n=1}^{\infty} E_n) = E_1 \setminus E$. By (a),

$$
m(E_1 \setminus E) = \lim_{n\to\infty} m(A_n) = \lim_{n\to\infty} m(E_1 \setminus E_n). \tag{1}
$$

Since $m(E_1) < +\infty$, we have $m(E), m(E_n) < +\infty$, and hence

$$
m(E_1 \setminus E) = m(E_1) - m(E) \quad \text{and} \quad m(E_1 \setminus E_n) = m(E_1) - m(E_n).
$$

This together with (1) implies that

$$
m(E_1) - m(E) = m(E_1) - \lim_{n\to\infty} m(E_n),
$$

and hence $\lim_{n\to\infty} m(E_n) = m(E)$.

A counter-example is given by taking $(\mathcal{M}, m)$ as the Lebesgue measure and $E_n := [n, \infty) \in \mathcal{M}$. Then clearly $E_n \downarrow \emptyset$ and $m(\emptyset) = 0$ while $m(E_n) = \infty$ for all n.

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2. Let $\varphi := \sum_{i=1}^{m} a_i \chi_{E_i}$ (each $a_i \in \mathbb{R}$ and $E_i \in \mathcal{M}$) be a "simple function".

(a) Show by MI that range(φ) is a finite set (and so one can list all its non-zero values $b_1, \ldots, b_N$ for some N, unless φ is the zero-function); show that

$$
\varphi = \sum_{j=1}^{N} b_j \chi_{\varphi^{-1}(b_j)} \quad \text{where} \quad \varphi^{-1}(b_j) := \{x : \varphi(x) = b_j\}
$$

and

$$\varphi = \sum_{j=0}^{N} b_j \chi_{\varphi^{-1}(b_j)} \quad \text{where} \quad b_0 = 0.$$

(Both representations can be referred as $\begin{cases} \text{canonical} \\ \text{normal} \\ \text{standard} \end{cases}$ representation of φ.)

(b) Define $\int \varphi := \sum_{j=1}^{N} b_j m(\varphi^{-1}(b_j)) = \sum_{j=0}^{N} b_j m(\varphi^{-1}(b_j))$ (with the convention that $0\cdot\infty = 0$) whenever $\varphi \in \mathcal{S}_0$ or $\varphi \in \mathcal{S}^+$ (meaning that the simple function φ is supported by a set of finite measure or $\varphi \geq 0$). Show that if $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ with each $a_i \in \mathbb{R}$ and $m(E_i) < +\infty$, then $\int \varphi = \sum_{i=1}^{n} a_i m(E_i)$ provided that E_i's are pairwise disjoint (the added condition can be dropped; see Q3 below).

Solution. (a) We prove by induction on m that $\varphi = \sum_{i=1}^{m} a_i \chi_{E_i}$ has $\#(\text{range}(\varphi)) < \infty$. Here $\#(A)$ denotes the cardinality of A.

For $m = 1$, $\varphi = a_1 \chi_{E_1}$, and hence $\#(\text{range}(\varphi)) \leq 2 < \infty$. Suppose the statement is true for $m = k$. For $m = k + 1$, let $\psi = \sum_{i=1}^{k} a_i \chi_{E_i}$. By induction assumption, $\text{range}(\psi) = \{b_1, \ldots, b_N\}$ for some $N < \infty$. Since $\varphi = \psi + a_{k+1} \chi_{E_{k+1}}$, we have

$$\text{range}(\varphi) \subseteq \{b_1, \ldots, b_N, b_1 + a_{k+1}, \ldots b_N + a_{k+1}\},$$

so that $\#(\text{range}(\varphi)) \leq 2N < \infty$. Thus the statement is true by MI.

Since $\text{range}(\varphi)$ is a finite set, we can list all its non-zero values $b_1, \ldots, b_N$ for some N, and write $b_0 = 0$. Then clearly

$$\varphi = \sum_{j=0}^{N} b_j \chi_{\varphi^{-1}(b_j)}.$$

(b) Let $\varphi = \sum_{i=1}^{n} a_i \chi_{E_i}$ with $a_i \in \mathbb{R}$, E_i pairwise disjoint and $m(E_i) < +\infty$ for each i. Without loss of generality, we may assume that $a_i \neq 0 \ \forall i$. Let $\varphi = \sum_{j=1}^{N} b_j \chi_{\varphi^{-1}(b_j)}$ be its canonical representation. Then

$$\bigcup_{i: a_i = b_j} E_i = \varphi^{-1}(b_j) \quad \text{for } j = 1, \ldots, N,$$

and hence

$$\begin{aligned} \int \varphi : &= \sum_{j=1}^{N} b_j m(\varphi^{-1}(b_j)) = \sum_{j=1}^{N} b_j \sum_{i: a_i = b_j} m(E_i) \\ &= \sum_{j=1}^{N} \sum_{i: a_i = b_j} a_i m(E_i) = \sum_{i=1}^{n} a_i m(E_i). \end{aligned}$$

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3. Let $\varphi, \psi \in \mathcal{S}_0$ with their canonical presentations $\varphi = \sum_{j=0}^{N} b_j \chi_{\varphi^{-1}(b_j)}$ and $\psi = \sum_{k=0}^{M} c_k \chi_{\psi^{-1}(c_k)}$. Show that $\int (\varphi + \psi) = \int \varphi + \int \psi$.

Solution. Let $B_j := \varphi^{-1}(b_j)$ and $C_k := \psi^{-1}(c_k)$. Since $\{C_0, \dots C_M\}$ is a partition of $\mathbb{R}$, we have $\sum_{k=0}^{M} \chi_{C_k} = 1$. Then

$$\varphi = \sum_{j=0}^{N} b_j \chi_{B_j} = \sum_{j=0}^{N} b_j (\sum_{k=0}^{M} \chi_{C_k} \cdot \chi_{B_j}) = \sum_{(j,k)} b_j \chi_{B_j \cap C_k} = \sum_{(j,k) \neq (0,0)} b_j \chi_{B_j \cap C_k}.$$

Similarly

$$\psi = \sum_{k=0}^{N} c_k \chi_{C_k} = \sum_{k=0}^{M} c_k (\sum_{j=0}^{N} \chi_{B_j} \cdot \chi_{C_k}) = \sum_{(j,k) \neq (0,0)} c_k \chi_{B_j \cap C_k}.$$

Note that $\{B_j \cap C_k : (j,k) \neq (0,0)\}$ are pairwise disjoint and $m(B_j \cap C_k) < +\infty$. In particular $\varphi + \psi = \sum_{(j,k) \neq (0,0)} (b_j + c_k) \chi_{B_j \cap C_k} \in \mathcal{S}_0$. Using Q2 twice, we have

$$\begin{aligned}
\int (\varphi + \psi) &= \sum_{(j,k) \neq (0,0)} (b_j + c_k) m(B_j \cap C_k) \\
&= \sum_{(j,k) \neq (0,0)} b_j m(B_j \cap C_k) + \sum_{(j,k) \neq (0,0)} c_k m(B_j \cap C_k) \\
&= \int \varphi + \int \psi.
\end{aligned}$$

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4.* (3rd: P.70, Q21)

(a) Let D and E be measurable sets and f a function with domain $D \cup E$. Show that f is measurable if and only if its restrictions to D and E are measurable.

(b) Let f be a function with measurable domain D. Show that f is measurable if and only if the function g defined by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

Solution. (a) ($\Rightarrow$): Suppose f is measurable. Then $\{x \in D \cup E : f(x) > \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$. Fix $\alpha \in \mathbb{R}$. Then

$$\{x \in D : f\big|_D(x) > \alpha\} = \{x \in D \cup E : f(x) > \alpha\} \cap D$$

which is measurable. Similarly $\{x \in E : f\big|_E(x) > \alpha\}$ is measurable. Since $\alpha \in \mathbb{R}$ is arbitrary, we have that $f\big|_D$ and $f\big|_E$ are measurable.
($\Leftarrow$): It follows immediately from the following equation:

$$\{x \in D \cup E : f(x) > \alpha\} = \{x \in D : f\big|_D(x) > \alpha\} \cup \{x \in E : f\big|_E(x) > \alpha\}.$$

(b) ($\Rightarrow$): Suppose f is measurable. Then

$$\{x : g(x) > \alpha\} = \begin{cases} \{x \in D : f(x) > \alpha\} & \text{if } \alpha \geq 0, \\ \{x \in D : f(x) > \alpha\} \cup D^c & \text{if } \alpha < 0, \end{cases}$$

which is measurable in either cases. Hence g is measurable.
($\Leftarrow$): The converse follows immediately from (a) since $f = g\big|_D$ and D is measurable.

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5.* (3rd: P.71, Q22)

(a) Let f be an extended real-valued function with measurable domain D, and let $D_1 = \{x : f(x) = \infty\}$, $D_2 = \{x : f(x) = -\infty\}$. Then f is measurable if and only if D_1 and D_2 are measurable and the restriction of f to $D \setminus (D_1 \cup D_2)$ is measurable.

(b) Prove that the product of two measurable extended real-valued function is measurable.

(c) If f and g are measurable extended real-valued functions and α is a fixed number, then $f + g$ is measurable if we define $f + g$ to be α whenever it is of the form $\infty - \infty$ or $-\infty + \infty$.

(d) Let f and g be measurable extended real-valued functions that are finite almost everywhere. Then $f + g$ is measurable no matter how it is defined at points where it has the form $\infty - \infty$.

Solution. (a) ($\Rightarrow$): Suppose f is measurable. Then D_1 and D_2 are measurable as usual. Hence $D \setminus (D_1 \cup D_2)$ is measurable, and so is $f\big|_{D\setminus(D_1\cup D_2)}$ by 4(a).

($\Leftarrow$): Suppose D_1 and D_2 are measurable and $f\big|_{D\setminus(D_1\cup D_2)}$ is measurable. Then, for $\alpha \in \mathbb{R}$,

$$\{x : f(x) > \alpha\} = D_1 \cup \{x : f\big|_{D\setminus(D_1\cup D_2)} > \alpha\}$$

which is measurable. Thus f is measurable.

(b) Let f and g be measurable extended real-valued functions defined on D. Let $D_1 = \{fg = \infty\}$ and $D_2 = \{fg = -\infty\}$. Then

$$D_1 = \{f = \infty, g > 0\} \cup \{f = -\infty, g < 0\} \cup \{f > 0, g = \infty\} \cup \{f < 0, g = -\infty\},$$

which is measurable. Similarly, D_2 is measurable. By (a), it suffices to show that $h := fg\big|_{D\setminus(D_1\cup D_2)}$ is measurable. Let $\alpha \in \mathbb{R}$. If $\alpha \geq 0$, then

$$\{x : h(x) > \alpha\} = \{x : f\big|_{D\setminus\{f=\pm\infty\}} \cdot g\big|_{D\setminus\{g=\pm\infty\}} > \alpha\},$$

which is measurable; if $\alpha < 0$, then

$$\{x : h(x) > \alpha\} = \{x : f(x) = 0\} \cup \{x : g(x) = 0\} \cup \{x : f\big|_{D\setminus\{f=\pm\infty\}} \cdot g\big|_{D\setminus\{g=\pm\infty\}} > \alpha\},$$

which is also measurable. Thus h is measurable.

Therefore fg is measurable.

(c) Let f and g are measurable extended real-valued functions and α is a fixed number. Define $f + g$ to be α whenever it is of the form $\infty - \infty$ or $-\infty + \infty$. Then

$$D_1 := \{f + g = \infty\} = \{f \in \mathbb{R}, g = \infty\} \cup \{f = g = \infty\} \cup \{f = \infty, g \in \mathbb{R}\}$$

is measurable, and so is $D_2 := \{f + g = -\infty\}$. By (a), it suffices to show that $h := (f + g)\big|_{D\setminus(D_1\cup D_2)}$ is measurable. Let $\beta \in \mathbb{R}$. If $\beta \geq \alpha$, then

$$\{x : h(x) > \beta\} = \{x : f\big|_{D\setminus\{f=\pm\infty\}} + g\big|_{D\setminus\{g=\pm\infty\}} > \beta\},$$

which is measurable; if $\beta < \alpha$, then

$$\begin{aligned}\{x : h(x) > \beta\} = \{f = \infty, g = -\infty\} &\cup \{f = -\infty, g = +\infty\} \\ &\cup \{x : f\big|_{D\setminus\{f=\pm\infty\}} + g\big|_{D\setminus\{g=\pm\infty\}} > \beta\},\end{aligned}$$

which is also measurable. Thus h is measurable.

Therefore $f + g$ is measurable.

(d) Let f and g be measurable extended real-valued functions that are finite a.e. Then the sets D_1, D_2, $\{x : h(x) > \beta\}$ can be written as unions of sets as in (c), possibly with an additional set of measure zero. Thus these sets are measurable and $f+g$ is measurable.

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6.* Let $\varphi \in \mathcal{S}_0$ and $E \in \mathcal{M}$. Define $\int_E \varphi := \int \varphi \cdot \chi_E$. Show that $\varphi \mapsto \int_E \varphi$ is linear:

$$\int_E (\alpha\varphi + \beta\psi) = \alpha \int_E \varphi + \beta \int_E \psi, \quad \forall\, \varphi, \psi \in \mathcal{S}_0 \text{ and } \alpha, \beta \in \mathbb{R}.$$

Solution. Clearly, if $\varphi \in \mathcal{S}_0$ and $E \in \mathcal{M}$, then $\varphi \cdot \chi_E \in \mathcal{S}_0$. Using the result in Q3, we have

$$\int_E (\varphi + \psi) = \int (\varphi \cdot \chi_E + \psi \cdot \chi_E) = \int \varphi \cdot \chi_E + \int \psi \cdot \chi_E = \int_E \varphi + \int_E \psi, \quad \forall\, \varphi, \psi \in \mathcal{S}_0.$$

It remains to show that

$$\int \alpha\varphi = \alpha \int \varphi, \quad \forall\, \varphi \in \mathcal{S}_0 \text{ and } \alpha \in \mathbb{R}.$$

Let $\alpha \in \mathbb{R}$ and $\varphi \in \mathcal{S}_0$ with canonical representation $\varphi = \sum_{j=0}^N b_j \chi_{\varphi^{-1}(b_j)}$. Then $\alpha\varphi = \sum_{j=0}^N \alpha b_j \chi_{B_j}$, and hence, by Q2(b),

$$\int \alpha\varphi = \sum_{j=0}^N \alpha b_j m(B_j) = \alpha \sum_{j=0}^N b_j m(B_j) = \alpha \int \varphi.$$

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7.* Let $\varphi \in \mathcal{S}_0$. Show that $A \mapsto \int_A \varphi$ is a linear "signed" measure on $\mathcal{M}$.

Solution. Clearly $\int_\emptyset \varphi = 0$. It remains to show that if $\{A_n\}_{n=1}^\infty$ is a sequence of pairwise disjoint, measurable sets, then

$$\int_{\bigcup_{n=1}^\infty A_n} \varphi = \sum_{n=1}^\infty \int_{A_n} \varphi.$$

Let $\varphi \in \mathcal{S}_0$ have a canonical representation $\varphi = \sum_{j=1}^N b_j \chi_{B_j}$, where $B_j := \varphi^{-1}(b_j)$ satisfies $m(B_j) < +\infty$. Then $\varphi \cdot \chi_{\bigcup_{n=1}^\infty A_n} = \sum_{j=1}^N b_j \chi_{(B_j \cap \bigcup_{n=1}^\infty A_n)}$ is also a simple function in $\mathcal{S}_0$ with

$$m(B_j \cap \bigcup_{n=1}^\infty A_n) \le m(B_j) < +\infty, \qquad j = 1, \ldots, N.$$

Since $\{A_n\}_{n=1}^\infty$ are pairwise disjoint, it follows from Q2(b) that

$$\begin{aligned}
\int_{\bigcup_{n=1}^\infty A_n} \varphi &= \int \varphi \cdot \chi_{\bigcup_{n=1}^\infty A_n} = \sum_{j=1}^N b_j m(B_j \cap \bigcup_{n=1}^\infty A_n) \\
&= \sum_{j=1}^N b_j \sum_{n=1}^\infty m(B_j \cap A_n) = \sum_{n=1}^\infty \sum_{j=1}^N b_j m(B_j \cap A_n) \\
&= \sum_{n=1}^\infty \int \varphi \cdot \chi_{A_n} = \sum_{n=1}^\infty \int_{A_n} \varphi.
\end{aligned}$$

Note that we can interchange the summation on the second line since

$$\sum_{n=1}^{\infty} m(B_j \cap A_n) = m(B_j \cap \bigcup_{n=1}^{\infty} A_n) < +\infty.$$

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