Suggested Solution of Assignment 3

1. Show that $\inf X \geq \inf Y$ whenever $X \subseteq Y (\subseteq \mathbb{R})$ and hence that $m^*(A) \uparrow$ (i.e. $m^*(A) \leq m^*(B)$ if $A \subseteq B(\subseteq \mathbb{R})$).

   Solution. Let $x \in X$. Then $x \in Y$, and hence by the definition of infimum, $x \geq \inf Y$. Since $x \in X$ is arbitrary, we have $\inf X \geq \inf Y$. The last statement follows immediately from the definition

$$m^*(A) := \inf\{\sum_{k=1}^{\infty} \ell(I_k) : \{I_k\}_{k=1}^{\infty} \text{ is a countable open-interval cover of } A\},$$

   and the fact that if $A \subseteq B \subseteq \mathbb{R}$, then any countable interval cover of B is also a countable interval cover of A. ◀

2. Let $\mathcal{A}$ be an algebra of subsets of X. Show that $\mathcal{A}$ is a σ-algebra if (and only if) $\mathcal{A}$ is stable with respect to countable disjoint unions:

$$\bigcup_{n=1}^{\infty} A_n \in \mathcal{A} \quad \text{whenever } A_n \in \mathcal{A}\ \forall\, n \in \mathbb{N} \text{ and } A_m \cap A_n = \emptyset\ \forall\, m \neq n.$$

   Solution. Suppose $\mathcal{A}$ is an algebra of subset of X that is stable with respect to countable disjoint unions. To show that $\mathcal{A}$ is a σ-algebra, it suffices to show that $\mathcal{A}$ is stable with respect to countable (but not necessarily disjoint) union. Let $B_n \in \mathcal{A}$ for $n \in \mathbb{N}$. Define

$$C_1 := B_1 \quad \text{and} \quad C_n := B_n \setminus \bigcup_{k=1}^{n-1} B_k \ \text{ for } n \geq 2.$$

   Clearly the collection $\{C_n\}_{n=1}^{\infty}$ is pairwise disjoint, and each $C_n \in \mathcal{A}$ since $\mathcal{A}$ is an algebra. Moreover,

$$\begin{aligned} C_1 \cup C_2 &= B_1 \cup (B_2 \setminus B_1) = B_1 \cup B_2, \\ C_1 \cup C_2 \cup C_3 &= B_1 \cup B_2 \cup (B_3 \setminus (B_1 \cup B_2)) = B_1 \cup B_2 \cup B_3, \\ &\vdots \end{aligned}$$

   and so on. Hence $\bigcup_{n=1}^{\infty} B_n = \bigcup_{n=1}^{\infty} C_n \in \mathcal{A}$. ◀

3. Suppose $[a, b]\,(\subseteq \mathbb{R})$ is covered by a finite family $\mathcal{C}$ of open intervals. Show that $b - a \leq$ sum of lengths of intervals in $\mathcal{C}$ (by MI to $n := \#(\mathcal{C})$, the number of elements of $\mathcal{C}$).

   Solution. Let $P(n)$ be the statement: if $[a, b]$ is a closed bounded interval that is covered by a finite family $\mathcal{C}$ of open intervals with $\#(\mathcal{C}) = n$, then $b - a \leq$ sum of lengths of intervals in $\mathcal{C}$.

   Suppose $\#(\mathcal{C}) = 1$ and $\mathcal{C} = \{[c, d]\}$. Then clearly $b - a \leq d - c$. Hence $P(1)$ is true.

   Assume that $P(k)$ is true. Suppose $[a, b]$ is a closed bounded interval that is covered by a finite family $\mathcal{C} = \{(c_i, d_i)\}_{i=1}^{k+1}$ of open intervals. Without loss of generality, we may assume that $a \in (c_1, d_1)$.

   If $b \leq d_1$, then $b - a \leq d_1 - c_1 \leq \sum_{i=1}^{k+1} |c_i - d_i|$, and we are done.

On the other hand, suppose $b > d_1$. Then $[d_1, b]$ is a closed bounded interval covered by $\{(c_i, d_i)\}_{i=2}^{k+1}$. Now the induction assumption implies that

$$b - d_1 \leq \sum_{i=2}^{k+1} |c_i - d_i|,$$

and hence

$$b - a = (d_1 - a) + (b - d_1) \leq |c_1 - d_1| + \sum_{i=2}^{k+1} |c_i - d_i| = \sum_{i=1}^{k+1} |c_i - d_i|.$$

So $P(k+1)$ is true.

By MI, $P(n)$ is true for all $n \in \mathbb{N}$. ◀

4. (cf. Royden 3rd, p.52, Q51) Upper/Lower Envelopes of $f : [a, b] \to \mathbb{R}$.

Define $h, g : [a, b] \to [-\infty, \infty]$ by

$$h(y) := \inf\{h_\delta(y) : \delta > 0\} \quad \text{for all } y \in [a, b],$$

where $h_\delta(y) := \sup\{f(x) : x \in [a, b], |x - y| < \delta\}$; and

$$g(y) := \sup\{g_\delta(y) : \delta > 0\} \quad \text{for all } y \in [a, b],$$

where $g_\delta(y) := \inf\{f(x) : x \in [a, b], |x - y| < \delta\}$. Prove the following:

(a) $g \leq f \leq h$ pointwisely on $[a, b]$, and for all $x \in [a, b]$, $g(x) = f(x)$ if and only if f is lower semicontinuous (l.s.c) at x ($f(x) = h(x)$ if and only if f is upper semicontinuous (u.s.c) at x), so $g(x) = h(x)$ if and only if f is continuous at x.

(b) If f is bounded (so g, h are real-valued), then g is l.s.c and h is u.s.c.

(c) If ϕ is a l.s.c function on $[a, b]$ such that $\phi \leq f$ (pointwise) on $[a, b]$, then $\phi \leq g$. State and show the corresponding result for h.

(d) Let $C_n := \{x \in [a, b] : h(x) - g(x) < \frac{1}{n}\}$ for all $n \in \mathbb{N}$. Then $C := \bigcap_{n=1}^{\infty} C_n$ is exactly the set of all continuity points of f and is a G_δ-set.

Note: More suggestive notations for g, h are $\underline{f}, \overline{f}$.

Solution. (a) Clearly $g_\delta(x) \leq f(x) \leq h_\delta(x)$ for all $x \in [a, b]$ and $\delta > 0$. Hence $g \leq f \leq h$ pointwisely on $[a, b]$.

Suppose f is l.s.c at x, that is, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $f(x) - \varepsilon < f(y)$ whenever $y \in [a, b]$ and $|y - x| < \delta$. Then $f(x) - \varepsilon \leq g_\delta(x) \leq g(x)$. Since $\varepsilon > 0$ is arbitrary, we have $f(x) \leq g(x)$, and hence $f(x) = g(x)$.

On the other hand, suppose $f(x) = g(x)$. Then, by the definition of g and g_δ, given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$f(x) - \varepsilon = g(x) - \varepsilon < g_\delta(x) \leq f(y) \qquad \text{whenever } y \in [a, b] \text{ and } |y - x| < \delta.$$

Thus f is l.s.c at x.

Similarly, one can show that $f(x) = h(x)$ if and only if f is u.s.c at x.

The last assertion now follows immediately from above and the simple fact that f is continuous at x if and only if it is both l.s.c and u.s.c at x.

(b) Let $x \in [a,b]$ and $\varepsilon > 0$. Since g is real-valued, we can find $\delta > 0$ such that $g(x) < g_\delta(x) + \varepsilon$. Note that $(y - \delta/2, y + \delta/2) \subseteq (x - \delta, x + \delta)$ if $|x - y| < \delta/2$. It follows from the definition of g and g_δ that whenever $y \in [a,b]$ with $|y - x| < \delta/2$, we have

$$g(x) - \varepsilon < g_\delta(x) \le g_{\delta/2}(y) \le g(y).$$

Therefore g is l.s.c at x and hence on $[a,b]$.
Similarly one can show that h is u.s.c on $[a,b]$.

(c) It suffices to prove that if ϕ is l.s.c at x and $\phi \le f$ on $[a,b]$, then $\phi(x) \le g(x)$. From the definition,

$$\underline{\phi}(x) := \sup_{\delta>0} \inf_{|y-x|<\delta} \phi(y) \le \sup_{\delta>0} \inf_{|y-x|<\delta} f(y) = g(x).$$

Since ϕ is l.s.c at x, we have $\underline{\phi}(x) = \phi(x)$ by (a), and the result follows.
Similarly, one can prove the corresponding result for h: if ψ is a u.s.c function on $[a,b]$ such that $f \le \psi$ on $[a,b]$, then $h \le \psi$.

(d) By (a), we have

$$\begin{aligned}\{x \in [a,b] : f \text{ continuous at } x\} &= \{x \in [a,b] : g(x) = h(x)\} \\ &= \bigcap_{n=1}^{\infty} \{x \in [a,b] : h(x) - g(x) < 1/n\} \\ &= \bigcap_{n=1}^{\infty} C_n = C.\end{aligned}$$

Note that $k := h - g$ is u.s.c on $[a,b] \setminus \{x : h(x) = +\infty \text{ or } g(x) = -\infty\}$.
To see that C is a G_δ-set (in $[a,b]$), it suffices to show that, given any $\lambda \in \mathbb{R}$, $A := \{x \in [a,b] : k(x) < \lambda\}$ is an open set in $[a,b]$.
Let $x \in A$. Then $k(x) \ne +\infty$. Set $\varepsilon_0 := (\lambda - k(x))/2 > 0$. Since k is u.s.c at x, there exists $\delta > 0$ such that if $y \in [a,b]$ and $|y - x| < \delta$, then

$$k(y) < k(x) + \varepsilon_0 = k(x) + \frac{\lambda - k(x)}{2} = \frac{\lambda + k(x)}{2} < \lambda.$$

Thus $B_\delta(x) \cap [a,b] \subseteq A$. Hence A is open in $[a,b]$.

◀

5. Let $f : [a,b] \to [m,M]$. For each $P \in \operatorname{Par}[a,b]$, let $u(f;P)$ and $U(f;P)$ denote the lower/upper Riemann-sum functions. Let $\{P_n : n \in \mathbb{N}\}$ be a sequence of partitions such that $P_n \subseteq P_{n+1}\ \forall\, n$ and $\|P_n\| \to 0$ ($\|P\|$ is the max subinterval length of P). Show that, $\forall\, x \in [a,b] \setminus A$

$$\lim_n \left(u(f;P_n)\right)(x) = \underline{f}(x) \quad \text{and} \quad \lim_n \left(U(f;P_n)\right)(x) = \overline{f}(x),$$

where A denotes the union of all end-points of $P_n\ \forall\, n$.

Solution. Let P be the partition $a = t_0 < t_1 < \cdots < t_k = b$. Then the lower and upper Riemann-sum functions can be defined as follow:

$$u(f,P) := \sum_{i=1}^{k} \inf_{x \in (t_{i-1}, t_i]} f(x) \chi_{(t_{i-1}, t_i]}, \qquad U(f,P) := \sum_{i=1}^{k} \sup_{x \in (t_{i-1}, t_i]} f(x) \chi_{(t_{i-1}, t_i]}.$$

Let $\{P_n\}$ be a sequence of partitions such that $P_n \subseteq P_{n+1}\ \forall\, n$ and $\|P_n\| \to 0$. Then $u(f;P_n)$ is an increasing sequence of functions, so that $\lim_n u(f;P_n)$ exists. Since $u(f;P_n)$ is bounded above by f on $(a,b]$ and is l.s.c at $x \in [a,b] \setminus A$, it follows from (the proof of) 4(c) that

$$u(f;P_n)(x) \le \underline{f}(x), \quad \text{for all } x \in [a,b] \setminus A,\ n \in \mathbb{N}. \tag{1}$$

Fix $x \in [a,b] \setminus A$. Let $\varepsilon > 0$. Since $\underline{f}$ is l.s.c at x, there exists $\delta > 0$ such that

$$\underline{f}(x) - \varepsilon < \underline{f}(y) \le f(y) \quad \text{whenever } y \in [a,b] \text{ and } |y - x| < \delta.$$

Choose N so large such that $\|P_N\| < \delta$. Suppose $a = t_0 < t_1 < \cdots < t_k = b$ are the end-points of P_N. Then

$$\underline{f}(x) - \varepsilon \le \sum_{i=1}^{k} \inf_{y \in (t_{i-1}, t_i]} f(y) \chi_{(t_{i-1}, t_i]}(x) = u(f;P_N)(x). \tag{2}$$

Combining (1) and (2), we have

$$\underline{f}(x) - \varepsilon \le u(f;P_N)(x) \le u(f;P_n)(x) \le \underline{f}(x) \quad \text{for } n \ge N,$$

and hence $\lim_n u(f;P_n)(x) = \underline{f}(x)$.

Similarly we can show that $\lim_n U(f;P_n)(x) = \overline{f}(x)$ for $x \in [a,b] \setminus A$.

◀