Suggested Solution of Assignment 2

In this assignment, $\{x_n\}$ and $\{y_n\}$ are sequences of real numbers. E is a subset of $\mathbb{R}$.

Recall that the limit superior of $\{x_n\}$ is defined by

$$\limsup x_n := \inf_n \sup_{k\geq n} x_k.$$

Clearly $z_n := \sup_{k\geq n} x_k$ is monotone decreasing, and hence

$$\lim_n z_n = \inf_n z_n = \limsup x_n, \tag{1}$$

where the limit is taken in the extended real number. Similarly the limit inferior of $\{x_n\}$ is given by

$$\liminf x_n := \sup_n \inf_{k\geq n} x_k = \lim_n \inf_{k\geq n} x_k. \tag{2}$$

1.* (3rd: P.39, Q12)

Show that $x = \lim x_n$ if and only if every subsequence of $\{x_n\}$ has in turn a subsequence that converges to x. How about $x \in \{-\infty, \infty\}$?

Solution. ($\Longrightarrow$) Suppose $\lim x_n = x$. Then every subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converges to x. Therefore $\{x_{n_k}\}$ has itself as a further subsequence that converges to x.

($\Longleftarrow$) Suppose on the contrary that $\{x_n\}$ does not converge to x. Then there exists $\varepsilon_0 > 0$ such that for all $N \in \mathbb{N}$, there is $n > N$ such that

$$|x_n - x| \geq \varepsilon_0.$$

Take $N = 1$, then we can find $n_1 > 1$ such that $|x_{n_1} - x| \geq \varepsilon_0$. Take $N = n_1$, we can find $n_2 > n_1$ such that $|x_{n_2} - x| \geq \varepsilon_0$. Continue in this way, we can find a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$|x_{n_k} - x| \geq \varepsilon_0 \quad \text{for all } k \in \mathbb{N}.$$

Now $\{x_{n_k}\}$ has no further subsequence that converges to x.

Similar results hold if $x = -\infty$ or ∞. ◀

2. (3rd: P.39, Q13)

Show that the real number l is the limit superior of the sequence $\{x_n\}$ if and only if (i) given $\varepsilon > 0$, $\exists\, n$ such that $x_k < l + \varepsilon$ for all $k \geq n$, and (ii) given $\varepsilon > 0$ and n, $\exists\, k \geq n$ such that $x_k > l - \varepsilon$.

Solution. We show that

(a) $\limsup x_n \leq l$ if and only if (i) holds ; and

(b) $\limsup x_n \geq l$ if and only if (ii) holds.

(a): By the definition of supremum and infinmum,

$$\limsup x_n \leq l \implies (\forall\, \varepsilon > 0)(\limsup x_n < l + \varepsilon) \implies (\forall\, \varepsilon > 0)(\inf_n \sup_{k\geq n} x_k < l + \varepsilon)$$
$$\implies (\forall\, \varepsilon > 0)(\exists\, n)(\sup_{k\geq n} x_k < l + \varepsilon) \implies (\forall\, \varepsilon > 0)(\exists\, n)(\forall\, k \geq n)(x_k < l + \varepsilon);$$

while on the other hand,

$$(\forall\,\varepsilon>0)(\exists\,n)(\forall\,k\geq n)(x_k<l+\varepsilon) \implies (\forall\,\varepsilon>0)(\exists\,n)(\sup_{k\geq n}x_k \leq l+\varepsilon)$$
$$\implies (\forall\,\varepsilon>0)(\inf_n\sup_{k\geq n}x_k \leq l+\varepsilon) \implies (\forall\,\varepsilon>0)(\limsup x_k \leq l+\varepsilon) \implies \limsup x_n\leq l.$$

(b): Similarly,

$$\limsup x_n\geq l \implies (\forall\,\varepsilon>0)(\limsup x_n>l-\varepsilon) \implies (\forall\,\varepsilon>0)(\inf_n\sup_{k\geq n}x_k>l-\varepsilon)$$
$$\implies (\forall\,\varepsilon>0)(\forall\,n)(\sup_{k\geq n}x_k>l-\varepsilon) \implies (\forall\,\varepsilon>0)(\forall\,n)(\exists\,k\geq n)(x_k>l-\varepsilon);$$

while on the other hand,

$$(\forall\,\varepsilon>0)(\forall\,n)(\exists\,k\geq n)(x_k>l-\varepsilon) \implies (\forall\,\varepsilon>0)(\forall\,n)(\sup_{k\geq n}x_k>l-\varepsilon)$$
$$\implies (\forall\,\varepsilon>0)(\inf_n\sup_{k\geq n}x_k \geq l-\varepsilon) \implies (\forall\,\varepsilon>0)(\limsup x_n \geq l-\varepsilon) \implies \limsup x_n\geq l.$$

Now the desired statement follows from (a) and (b) immediately.

Similarly, one can show that

(c) $\liminf x_n\geq l$ if and only if $\forall\,\varepsilon>0$, $\exists\,n$ such that $x_k>l-\varepsilon$ for all $k\geq n$; and

(c) $\liminf x_n\leq l$ if and only if $\forall\varepsilon>0$, $\forall\,n$, $\exists\,k\geq n$ such that $x_k<l+\varepsilon$.

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3.* (3rd: P.39, Q14)

Show that $\limsup x_n=\infty$ if and only if given Δ and n, $\exists\,k\geq n$ such that $x_k>\Delta$.

Solution. The statement follows immediately from (b) in question 2 and the fact that $x=\infty$ if and only if $x>\Delta$ for any $\Delta\in\mathbb{R}$. Indeed,

$$\limsup x_n=\infty \implies (\forall\,\Delta\in\mathbb{R})(\limsup x_n>\Delta) \implies (\forall\,\Delta\in\mathbb{R})(\forall\,n\in\mathbb{N})(\sup_{k\geq n}x_k>\Delta)$$
$$\implies (\forall\,\Delta\in\mathbb{R})(\forall\,n\in\mathbb{N})(\exists\,k\geq n)(x_k>\Delta).$$

while on the other hand,

$$(\forall\,\Delta\in\mathbb{R})(\forall\,n\in\mathbb{N})(\exists\,k\geq n)(x_k>\Delta) \implies (\forall\,\Delta\in\mathbb{R})(\forall\,n\in\mathbb{N})(\sup_{k\geq n}x_k>\Delta)$$
$$\implies (\forall\,\Delta\in\mathbb{R})(\limsup x_n \geq \Delta) \implies \limsup x_n=\infty.$$

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4. (3rd: P.39, Q15)

Show that $\liminf x_n \le \limsup x_n$ and $\liminf x_n = \limsup x_n = l$ if and only if $l = \lim x_n$.

Solution. Clearly

$$\inf_{k\ge n} x_k \le x_n \le \sup_{k\ge n} x_k \qquad \text{for all } n \ge 1. \tag{3}$$

Hence, by (1) and (2), and letting $n\to\infty$, we have

$$\liminf x_n = \lim_n \inf_{k\ge n} x_k \le \lim_n \sup_{k\ge n} x_k = \limsup x_n.$$

Suppose $\liminf x_n = \limsup x_n = l$. Then it follows from (3) and the Squeeze Theorem that $\lim x_n = l$.

Conversely, if $l = \lim x_n$, then for any $\varepsilon > 0$, there exists $n \in \mathbb{N}$ such that $l-\varepsilon < x_k < l+\varepsilon$ for all $k \ge n$, so that

$$l-\varepsilon \le \inf_{k\ge n} x_k \le x_k \le \sup_{k\ge n} x_k \le l+\varepsilon \qquad \text{for all } k \ge n.$$

Letting $n\to\infty$, we have $l-\varepsilon \le \liminf x_n \le \limsup x_n \le l+\varepsilon$. As ε is arbitrary, we have $\liminf x_n = \limsup x_n = l$.

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5.* (3rd: P.39, Q16)

Prove that

$$\limsup x_n + \liminf y_n \le \limsup (x_n + y_n) \le \limsup x_n + \limsup y_n,$$

provided the right and left sides are not of the form $\infty - \infty$.

Solution. For all $n \ge 1$,

$$x_k + \inf_{j\ge n} y_j \le x_k + y_k \quad \text{whenever } k \ge n,$$

so that

$$\sup_{k\ge n} x_k + \inf_{j\ge n} y_j \le \sup_{k\ge n}(x_k + y_k).$$

By (1) and (2), we can let $n\to\infty$ on both sides and obtain

$$\limsup x_n + \liminf y_n \le \limsup (x_n + y_n),$$

provided the left side is not of the form $\infty - \infty$.

On the other hand, for all $n \ge 1$,

$$x_j + y_j \le \sup_{k\ge n} x_k + \sup_{k\ge n} y_k \qquad \text{whenever } j \ge n,$$

so that

$$\sup_{k\ge n}(x_k + y_k) \le \sup_{k\ge n} x_k + \sup_{k\ge n} y_k.$$

Again letting $n\to\infty$, we obtain

$$\limsup (x_n + y_n) \le \limsup x_n + \limsup y_n,$$

provided the right side is not of the form $\infty - \infty$.

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6. (3rd: P.39, Q17)

   Prove that if $x_n > 0$ and $y_n \geq 0$, then
   $$\limsup(x_n y_n) \leq (\limsup x_n)(\limsup y_n),$$
   provided the product on the right is not of the form $0 \cdot \infty$.

   Solution. For all $n \geq 1$,
   $$0 \leq x_k \leq \sup_{j \geq n} x_j \quad \text{and} \quad 0 \leq y_k \leq \sup_{j \geq n} y_j \quad \text{whenever } k \geq n,$$
   so that
   $$0 \leq x_k y_k \leq (\sup_{j \geq n} x_j)(\sup_{j \geq n} y_j) \quad \text{whenever } k \geq n.$$
   Thus, for all $n \geq 1$,
   $$\sup_{k \geq n}(x_k y_k) \leq (\sup_{k \geq n} x_k)(\sup_{k \geq n} y_k).$$
   Using (1) and (2), and letting $n \to \infty$, we have
   $$\limsup(x_n y_n) \leq (\limsup x_n)(\limsup y_n),$$
   provided the right side is not of the form $0 \cdot \infty$. ◀

7. (3rd: P.46, Q27)

   $x \in \mathbb{R}$ is called a *point of closure* of E if each neighbourhood of x intersects E. Show that x is a point of closure of E if and only if there is a sequence $\{y_n\}$ with $y_n \in E$ and $x = \lim y_n$.

   Solution. Suppose x is a point of closure of E. Then the open ball $B(x, 1/n)$, which is centred at x and of radius $1/n$, intersects E for all $n \geq 1$. Pick $y_n \in E \cap B(x, 1/n)$ for each n. Then $\{y_n\}$ is a sequence in E such that $\lim y_n = x$, since $|y_n - x| < 1/n$ for all n.

   On the other hand, suppose $\{y_n\}$ is a sequence in E such that $x = \lim y_n$. Let U be a neighbourhood of x. Then $y_n \to x$ implies that $y_n \in U$ for all sufficiently large n. In particular, $U \cap E \neq \emptyset$. ◀

8. (3rd: P.46, Q28; 4th: P.20, Q30(i))

   A number x is called an *accumulation point* of a set E if it is a point of closure of $E \setminus \{x\}$. Show that the set E' of accumulation points of E is a closed set.

   Solution. We would like to show that the complement of E' is open. Let $x \in (E')^c$. Then x is not a point of closure of $E \setminus \{x\}$. Hence, by definition, there is an open neighbourhood U of x such that $U \cap (E \setminus \{x\}) = \emptyset$. We claim that every $y \in U$ is not an accumulation point of E, so that $x \in U \subseteq (E')^c$, and hence $(E')^c$ is open.

   Let $y \in U \setminus \{x\}$. Since $U \setminus \{x\}$ is open, there is a neighbourhood V of y such that $V \subseteq U \setminus \{x\}$. Hence
   $$V \cap (E \setminus \{y\}) \subseteq (U \setminus \{x\}) \cap E = \emptyset.$$
   Thus y is not a point of closure of $E \setminus \{y\}$, that is, y is not an accumulation point of E.

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9. (3rd: P.46, Q29; 4th: P.20, Q30(ii))

   Show that $\overline{E} = E \cup E'$.

   Solution. Recall that $\overline{E}$ is the set of all point of closure of E. From the definitions, it is clear that $E \cup E' \subseteq \overline{E}$. On the other hand, if $x \in \overline{E} \setminus E$, then for every neighbourhood U of x,
$$U \cap (E \setminus \{x\}) = U \cap E \neq \emptyset.$$
   Hence $x \in E'$. Therefore $\overline{E} \subseteq E \cup E'$. ◀

10. (3rd: P.46, Q30; 4th: P.20, Q31)

    A set E is called *isolated* if $E \cap E' = \emptyset$. Show that every isolated set of real numbers is countable.

    Solution. Suppose E is isolated. Then no point in E is an accumulation point of E, whence, for all $x \in E$, there is $r_x > 0$ such that $(x - r_x, x + r_x) \cap (E \setminus \{x\}) = \emptyset$. Let $I_x = (x - r_x/2, x + r_x/2)$. Then $\{I_x : x \in E\}$ is a collection of open intervals such that
$$I_x \cap I_y = \emptyset \quad \text{if } x, y \in E,\ x \neq y.$$
    For otherwise, $u \in I_x \cap I_y \implies |x - y| \leq |x - u| + |u - y| < r_x/2 + r_y/2 \leq \max\{r_x, r_y\}$, contradicting $x \notin I_y$ and $y \notin I_x$.

    By the density of $\mathbb{Q}$, for every $x \in E$, we can find $\varphi(x) \in \mathbb{Q}$ such that $\varphi(x) \in I_x$. Now $\varphi : E \to \mathbb{Q}$ is an injection since $\{I_x : x \in E\}$ are pairwise disjoint. Therefore E is countable.

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